Austrian Journal of Statistics
January 2023, Volume 52, 1–22.
http://www.ajs.or.at/
doi:10.17713/ajs.v52i1.1322

A Review of Goodness-of-Fit Tests for the Rayleigh Distribution

Shawn C. Liebenberg
North-West University

James S. Allison
North-West University

Abstract

The Rayleigh distribution has recently become popular as a model for a range of phenomena. As a result, a number of goodness-of-fit tests have been developed for this distribution. In this paper, we provide the first overview of goodness-of-fit tests for the Rayleigh distribution and compare these tests in a Monte-Carlo study to identify the tests that provide the highest powers against a wide range of alternatives. Our findings suggest that two recently developed tests as well as a test based on the Laplace transform and a test based on the Hellinger distance are the better performing tests.

Keywords: Rayleigh distribution, goodness-of-fit, Monte Carlo.

1. Introduction

The Rayleigh distribution naturally arises in a two dimensional setting when the resultant of two independently normally distributed vectors are considered. As a result, this distribution has applications in numerous research disciplines such as astronomy, see Bovaird and Lineweaver (2017), environmental sciences, see Morgan, Lackner, Vogel, and Baise (2011); Casas-Prat and Holthuijsen (2010) and medicine, see Belaid and Boukerroui (2018). A primary concern for researchers in the aforementioned fields is often if data originated from a Rayleigh distribution. Our goal in this paper is therefore to investigate the existing goodness-of-fit tests with a Monte Carlo simulation in a comparative study to assess the performance of each of these tests. To proceed, we formally introduce the Rayleigh distribution and fix notation. Let $Z_1, Z_2 \ldots Z_n$ be i.i.d $N(0, \theta^2)$ random variables. If $D(Z_1, Z_2 \ldots Z_n) = \sqrt{\sum_{i=1}^n Z_i^2}$, then $D(Z_1, Z_2 \ldots Z_n)$ has probability density function (pdf),

$$h(x, n, \theta) = \frac{2x^{n-1} \exp\left(\frac{-x^2}{2\theta^2}\right)}{(2\theta^2)^{n/2}\Gamma(n/2)}, \tag{1}$$

where $x > 0$, $\theta > 0$ and $\Gamma(\cdot)$ is the gamma function. Note that the geometric interpretation of $D(Z_1, Z_2 \ldots Z_n)$ is the Euclidean distance between a random point in $\mathbb{R}^n$ and the origin. In the case where $n = 2$, i.e., the point (Z_1, Z_2), in (1) reduces to the Rayleigh distribution with pdf given by

$$g(x, \theta) = \frac{x}{\theta^2} \exp\left(\frac{-x^2}{2\theta^2}\right), x \geq 0 \tag{2}$$

and cumulative distribution function (cdf),

$$G(x,\theta) = 1 - \exp\left(\frac{-x^2}{2\theta^2}\right), x \geq 0. \tag{3}$$

The pdf and cdf contain the population parameter θ, which has to be estimated when inference is performed. In considering the estimation of the parameter θ from a random sample $X_1, X_2, \ldots, X_n$, we have access to the maximum likelihood estimate (MLE),

$$\widehat{\theta}_n^{ML} = \sqrt{(2n)^{-1}\sum_{j=1}^{n} X_j^2},$$

and the methods of moment estimate (MME),

$$\widehat{\theta}_n^{MM} = \sqrt{\frac{2}{\pi}} n^{-1}\sum_{j=1}^{n} X_j.$$

It can easily be shown that $\widehat{\theta}_n^{MM}$ is an unbiased estimator of θ, while $\widehat{\theta}_n^{ML}$ is biased. However, $\widehat{\theta}_n^{ML}$ is asymptotically unbiased. Further properties of the univariate Rayleigh distribution and its relationship to other distributions are discussed in, e.g., Siddiqui (1962) and Johnson, Kotz, and Balakrishnan (1994).

The Rayleigh distribution also has inherent connections with other distributions. It is well known that if a random variable X has a Rayleigh distribution with parameter θ, then X^2 is exponentially distributed with parameter $2\theta^2$. The Rice distribution is also closely related to the Rayleigh distribution which can be seen when the parameter υ of the Rice distribution, the distance between a reference point and the centre of the bivariate distribution, is set to zero. The popularity of the Rayleigh distribution and its various applications also sparked interest in generalizations or modification of this distribution. See Balakrishnan and Kocherlakota (1985), Vodă (1976), Merovci (2013), Roy (2004), Simon and Alouini (1998) and Jensen (1970) for more on this.

Now, let $X, X_1, X_2, \ldots, X_n$ be independent and identically distributed nonnegative random variables. $X \sim Ral(\theta)$ will denote that the random variable X follows a Rayleigh distribution with pdf given in (2). The composite goodness-of-fit hypothesis to be tested is

$$H_0 : \text{the distribution of } X \text{ is } Ral(\theta), \tag{4}$$

for some $\theta > 0$, against general alternatives. The majority of test statistics that we will consider in this paper are based on the scaled values $Y_j = X_j/\widehat{\theta}_n$, where $\widehat{\theta}_n$ is a consistent estimator for θ (either $\widehat{\theta}_n^{ML}$ or $\widehat{\theta}_n^{MM}$). The use of scaled values is motivated from the invariance property of the Rayleigh distribution with respect to scale transformations. Therefore we can also write $X/\theta \sim Ral(1)$. Denote by $X_{(j)}$ the order statistics, i.e., $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$.

The remainder of the paper outlines as follows. Section 2 provides an overview of existing tests for the Rayleigh distribution. Section 3 provides the details of the simulation setup wherein the various tests are compared to each other and Section 4 discusses the results of this Monte Carlo study. An example based on observed data is given in Section 5 and the paper concludes in Section 6.

2. Goodness-of-fit tests for the Rayleigh distribution

In this section we present some of the existing goodness-of-fit tests for the Rayleigh distribution. These tests are arranged according to the property of the Rayleigh distribution that the tests are based on. Unless stated otherwise, the test under discussion is scale invariant and Monte Carlo critical values can be calculated by simulating from a $Ral(1)$ distribution.

2.1. Classical tests based on the empirical distribution function

The empirical cumulative distribution function (ecdf) is given by

$$G_n(x) = \frac{1}{n}\sum_{j=1}^{n} I(Y_j \le x).$$

where $I(\cdot)$ is the indicator function. There are various tests based on the deviation of the ecdf and the cdf with estimated parameter $\widehat{\theta}_n$ specified under the null hypothesis. One such test is the Kolmogorov-Smirnov test which relies upon the maximum deviation between $G_n(x)$ and the hypothesized distribution $G(x, \widehat{\theta}_n)$. The Kolmogorov-Smirnov test statistic has a closed-form,

$$D_n = \max(D_n^+, D_n^-),$$

where $D_n^+ = \max_{1\le j\le n}[j/n - G_0(Y_{(j)})]$ and $D_n^- = \max_{1\le j\le n}[G_0(Y_{(j)}) - (j-1)/n]$, with $G_0(Y_{(j)}) = 1 - \exp(-Y_{(j)}^2/2)$ and rejects the null hypothesis for large values of D_n. A test that utilizes the L^2-norm and aforementioned deviation is the Cramér-von Mises test with closed-form,

$$W_n = \sum_{j=1}^{n}\left[G_0(Y_{(j)}) - \frac{2j-1}{2n}\right]^2 + \frac{1}{12n}.$$

The Anderson-Darling test is similar to that of the Cramér-von Mises test but with an incorporated weight function that gives it the closed-form,

$$A_n = -n - \frac{1}{n}\sum_{j=1}^{n} 2j-1\left[\log G_0(Y_{(j)}) + \log\{1 - G_0(Y_{(n-j+1)})\}\right].$$

The Watson test incorporates the Cramér-von Mises test and is given by the closed form,

$$V_n = W_n - n(\bar{G}_0 - 1/2)^2,$$

where $\bar{G}_0 = 1/n\sum_{j=1}^{n} G_0(Y_{(j)})$. The aforementioned tests all reject the null hypothesis for large values. See, Watson (1962) and D'Agostino (1986) for a more thorough treatment of these classical tests.

2.2. Tests based on integral transforms

Below we consider tests based on integral transforms of an observed sample. We consider a test based on the Laplace transform before considering tests based on the Mellin transform.

A test based on the Laplace transform by Meintanis and Iliopoulos (2003):

The Laplace transform of a random variable X with distribution function $F(x)$ is defined as

$$L(t) = \mathsf{E}\left[e^{-tx}\right] = \int_0^\infty e^{-tx}\,\mathrm{d}F(x), \tag{5}$$

where t is a real number. Furthermore, let

$$L_n(t) = \frac{1}{n}\sum_{j=1}^{n}\exp(-tX_j),$$

be the empirical Laplace transform, which is an estimate of (5). For more details on the Laplace transform refer to Schiff (1999). Now, consider the following characterisation of the Rayleigh distribution based on the Laplace transform.

Corollary 2.1. *The Laplace transform of the standard Rayleigh distribution is given by*

$$\ell(t) = 1 - \frac{\sqrt{\pi}}{2} t e^{t^2/4} \ \mathrm{erfc}\left(\frac{t}{2}\right),$$

with the complement error function $\mathrm{erfc}(z) = 2/\sqrt{\pi}\int_z^\infty e^{-u^2} du$. $\ell(t)$ *is the unique solution to the differential equation*

$$t\ell'(t) - [1 + \frac{t^2}{2}]\ell(t) + 1 = 0 \tag{6}$$

subject to $\lim_{t\to\infty} \ell(t) = 0$.

The test statistic is based on the characterisation in Corollary 2.1 and utilizes the differential equation given in (6) to set up the test

$$MI_{n,\varphi} = n \int_0^\infty D_n^2(t) w(t) \ \mathrm{d}t,$$

where $D_n(t) = t\ell_n'(t) - [1 + (t^2/2)]\ell_n(t) + 1$, $\ell_n(t) = n^{-1}\sum_{j=1}^n \exp(tY_j)$ and $w(t) = \exp(-\varphi t)$ with $\varphi > 0$ being a chosen tuning parameter. The test statistic rejects for large values of $MI_{n,\varphi}$. A closed-form expression of $MI_{n,\varphi}$, adapted for our parametrisation, is given by

$$\begin{aligned} MI_{n,\varphi} =& \frac{n}{\varphi} + \frac{\sqrt{2}}{n}\sum_{j=1}^n\sum_{k=1}^n \left\{ \frac{1}{(Y_j+Y_k+\varphi)} + \frac{Y_j+Y_k}{(Y_j+Y_k+\varphi)^2} + \frac{2\,(Y_jY_k+2)}{(Y_j+Y_k+\varphi)^3} + \frac{6\,(Y_j+Y_k)}{(Y_j+Y_k+\varphi)^4} \right. \\ & \left. + \frac{24}{(Y_j+Y_k+\varphi)^5} \right\} - 2\sqrt{2}\sum_{j=1}^n \left\{ \frac{1}{(Y_j+\varphi)} + \frac{Y_j}{(Y_j+\varphi)^2} + \frac{2}{(Y_j+\varphi)^3} \right\}. \end{aligned}$$

Meintanis and Iliopoulos (2003) derived the null distribution and proved the consistency of the test and further provided insightful theoretical properties of the test statistic when the MLE and MME are used. From a power study, Meintanis and Iliopoulos (2003) concluded that the Laplace transform based test with $\varphi = 2$ leads to highly competitive results against the existing tests considered in their study.

A test based on the Mellin transform by Liebenberg and Allison (2019):

Liebenberg and Allison (2019) considered the Mellin transform of a random variable X,

$$M_X(t) = \mathsf{E}[X^t] = \int_0^\infty x^t \ \mathrm{d}F(x), \tag{7}$$

with $t = s - 1 > 0$ taken to be real valued and proposed the differential equation,

$$2M'(t) - \left\{\log(2) - 2\log(\theta) + \psi\left(1 + \frac{t}{2}\right)\right\} M(t) = 0, \tag{8}$$

with digamma function $\psi(\cdot)$ and boundary condition $M(0) = 1$ as a starting point for the test. The differential equation and subsequent test is motivated by the fact that the Mellin transform $m(t) = 2^{t/2}\Gamma(1 + t/2)$ of the standard Rayleigh distribution is the unique solution to the differential equation $D(t) = 2m'(t) - \left\{\log(2) + \psi\left(1 + \frac{t}{2}\right)\right\} m(t) = 0$. In estimating $D(t)$ by its empirical counterpart $D_n(t) = 2m_n'(t) - \left\{\log(2) + \psi\left(1 + \frac{t}{2}\right)\right\} m_n(t)$, Liebenberg and Allison (2019) suggested a test statistic of the form

$$LA_{n,\varphi} = n \int_0^\infty D_n^2(t) w(t) \ \mathrm{d}\widehat{G}_n(t),$$

where $w(t)$ is an appropriate nonnegative weight function, $\widehat{G}_n(t)$ is any consistent estimator of G and $m_n(t) = n^{-1}\sum_{j=1}^{n} Y_j^t$ is the empirical Mellin transform which is an estimate for (7). For a closed form of the test statistic the MLE, the weight function $w(t) = \exp(-\varphi t^2)$ with tuning parameter $\varphi > 0$ and $\widehat{G}_n(t)$ as the ecdf were implemented to give

$$LA_{n,\varphi} = \sum_{k=1}^{n}\left[2\sum_{j=1}^{n} Y_j^{Y_k}\log(Y_j) - \left\{\log(2)+\psi\left(1+\frac{Y_k}{2}\right)\right\}\sum_{j=1}^{n} Y_j^{Y_k}\right]^2 \exp(-\varphi Y_k^2)$$

It is also known that if the Mellin transform exists, it has the following connection to the Laplace transform, $M_X(t) = L_Z(t) = \mathsf{E}[e^{-tZ}]$, where $Z = -\log(X)$. The authors commented that the test is consistent and also concluded from a simulation study that the test performed well compared to its counterparts.

2.3. Tests based on entropy

Below we consider tests based on the entropy of an observed sample. We consider tests based on cumulative residual entropy, the Kullback–Leibler divergence, the Hellinger distance and the quantile function.

A cumulative residual entropy test by Baratpour and Khodadadi (2012):

The differential entropy of a random variable X is defined as

$$H(f) = -\int f(x)\ln f(x)\ \mathrm{d}x,$$

where $f(x)$ is the pdf of X. The cumulative residual entropy (CRE) is the result of exchanging the density function in the definition of the well-known Shannon entropy by the survival function, $S(x) = P(X > x) = 1 - F(x)$. Rao, Chen, Vemuri, and Wang (2004) established the new CRE as a nonnegative entropy measure of the form

$$\mathrm{CRE}(X) = -\int_0^\infty S(x)\ln S(x)\ \mathrm{d}x. \tag{9}$$

By using $S(x) = \exp(-x^2/2\theta^2)$ in (9) it can be shown that the CRE of the Rayleigh distribution is $\mathrm{CRE}(X) = \mathsf{E}[X/2] = \theta\frac{\sqrt{2\pi}}{4}$. This ultimately leads to the following characterisation of the Rayleigh distribution.

Corollary 2.2. *The random variable X attains maximum CRE among all nonnegative, absolutely continuous random variables Y subject to $E[Y] = \upsilon$, $E[Y^3] = \omega$ and $\theta^2 = \frac{\omega}{3\upsilon}$ if, and only if, X has the Rayleigh distribution with parameter θ.*

The authors defined a new measure of distance between two distributions based on CRE and named it the cumulative Kullback-Leibler (CKL) divergence. If F and G are distributions of two nonnegative random variables X_1 and X_2 then the CKL is given by

$$\mathrm{CKL}(F,G) = \int_0^\infty \bar{F}(x)\log\frac{\bar{F}(x)}{\bar{G}(x)}\mathrm{d}x - \{E(X_1) - E(X_2)\},$$

where $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ are the survival functions of X_1 and X_2. By using the characterisation given in Theorem 2.2 for the Rayleigh distribution and utilizing a discrimination information statistic based on the CKL, a test statistic of the form

$$CK_n = \frac{1}{\bar{X}}\left(\sum_{i=1}^{n-1}\left(\frac{n-i}{n}\right)\left(\log\left(\frac{n-i}{n}\right)\right)(X_{(i+1)} - X_{(i)}) + \sqrt{\frac{\pi}{2}}\sqrt{\frac{\sum_{i=1}^{n} X_i^3}{3\sum_{i=1}^{n} X_i}}\right),$$

can be constructed, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Rao *et al.* (2004) proved the consistency of the CRE and Baratpour and Khodadadi (2012) extended the proof to the test statistic CK_n.

A Kullback–Leibler divergence test by Alizadeh Noughabi, Alizadeh Noughabi, and Behabadi (2012):

This test is based on an estimator of the well-known Kullback–Leibler divergence function given by

$$KL(g||g_0) = \int_{-\infty}^{\infty} g(x) \log\left(\frac{g(x)}{g_0(x)}\right) \mathrm{d}x,$$

where $g_0(x)$ is the density under the null hypothesis (i.e., the Rayleigh density function) and $g(x)$ is the density function of a random variable X. The estimator is formed by first noting that $KL(g||g_0)$ reduces to

$$KL(g||g_0) = -H(g) - \int_0^{\infty} g(x) \log\{g_0(x)\}\ \mathrm{d}x,$$

where $H(g) = \mathsf{E}[-\log g(X)]$ is the entropy of a random variable X. The sample estimate of $KL(g||g_0)$ is then given by

$$KL_{n,m} = -H_{n,m} + 2\log(\widehat{\theta}_n) - \frac{1}{n}\sum_{i=1}^{n} \log(X_i) + 1,$$

where $H_{n,m} = (1/n)\sum_{i=1}^{n} \log\left\{(n/2m)(X_{(i+m)} - X_{(i-m)})\right\}$ is the sample-entropy estimator introduced by Vasicek (1976) and m is a window width restricted to $m \leq n/2$. The choice of $\widehat{\theta}_n$ is restricted to the maximum likelihood estimate in $KL_{n,m}$. The consistency and standard normal asymptotic distribution under the null hypothesis was proven in the original paper.

A Hellinger distance test by Jahanshahi, Habibi Rad, and Fakoor (2016):

This entropy-based statistic utilizes the Hellinger distance,

$$D_{g,g_0} = \frac{1}{2}\int_0^{\infty} \left\{\sqrt{g(x)} - \sqrt{g_0(x)}\right\}^2 \mathrm{d}x, \tag{10}$$

instead of the traditionally used Kullback–Leibler divergence which experiences difficulties when the probability density function is zero. The Hellinger distance evaluates the deviation of a density $g(x)$ from the hypothesized density $g_0(x)$. By setting the distribution function $G(x) = p$, (10) can be rewritten as

$$D_{g,g_0} = \frac{1}{2}\int_0^{1} \left(\sqrt{\left(\frac{d}{dp}G^{-1}(p)\right)^{-1}} - \sqrt{\frac{G^{-1}(p)\exp(-(G^{-1}(p))^2/2\theta^2)}{\theta^2}}\right)^2 \frac{d}{dp}G^{-1}(p)\ \mathrm{d}p.$$

Using the approximation $(d/dp\ G^{-1}(p))^{-1} \cong \left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\}^{-1}$ leads to the following test statistic

$$DH_{n,m} = \frac{1}{2n}\sum_{i=1}^{n} \frac{\left[\sqrt{\left(\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right)^{-1}} - \sqrt{\left(X_{(i)}e^{\left(-X_{(i)}^2/2\widehat{\theta}_n^2\right)}/\hat{\theta}_n^2\right)}\ \right]^2}{\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\}^{-1}},$$

where $X_{(i)} = X_{(1)}$ for $i < 1$, $X_{(i)} = X_{(n)}$ for $i > n$ and m is a window width subject to $m \leq n/2$. Jahanshahi *et al.* (2016) provides a proof for the consistency of the test.

A quantile function based test by Ahrari, Baratpour, Habibirad, and Fakoor (2019):

Let $Q(\cdot)$ be the quantile function of a random variable X and let $Q_0(x,\theta) = \theta\left\{-2\log(1-p)\right\}^{1/2}$ be the quantile function of the Rayleigh distribution. With this as starting point, Ahrari *et al.* (2019) proposed three new distance measures between the quantile functions of two distributions P and Q. The measures resemble the Kulback-Leibler divergence measures and Tsallis generalized entropy measure, see Tsallis (1998). Conforming to the notation of Ahrari *et al.* (2019), let Q_1 and Q_2 be the respective quantile functions of two nonnegative random variables X and Y. The three new distance measure are given by

$$\begin{aligned}
D_{KL_1}(Q_1\|Q_2) &= \int_0^1 Q_1(x)\log\frac{Q_1(x)}{Q_2(x)}\mathrm{d}x - \int_0^1 Q_1(x)\mathrm{d}x\log\frac{\int_0^1 Q_1(x)\mathrm{d}x}{\int_0^1 Q_2(x)\mathrm{d}x},\\
D_{KL_2}(Q_1\|Q_2) &= \int_0^1 Q_1(x)\log\frac{Q_1(x)}{Q_2(x)}\mathrm{d}x - \int_0^1 Q_1(x)\mathrm{d}x + \int_0^1 Q_2(x)\mathrm{d}x,\\
D_T(Q_1\|Q_2) &= \frac{1}{(\alpha-1)}\Big\{\int_0^1 Q_1^{\alpha}(x)Q_2^{1-\alpha}(x)\mathrm{d}x - \alpha\int_0^1 Q_1(x)\mathrm{d}x\\
&\quad -(1-\alpha)\int_0^1 Q_2(x)\mathrm{d}x\Big\},
\end{aligned}$$

with $0 < \alpha < 1$. The authors prove that the divergence measures are larger or equal to zero and are zero if, and only, if $Q_1 = Q_2$. The test statistics based on the aforementioned divergence measures are then

$$\begin{aligned}
Q_{KL_1} &= \frac{D_{KL_1}\left(Q_n\|Q_0(\cdot;\widehat{\theta}_n)\right)}{\bar{X}_n}\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{\bar{X}_n}\log\frac{X_i}{\bar{X}_n} - \frac{1}{2}\sum_{i=1}^{n}\frac{X_{(i)}}{\bar{X}_n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\log(-2\log(1-x))\mathrm{d}x + \log\left(\sqrt{\frac{\pi}{2}}\right),
\end{aligned}$$

$$\begin{aligned}
Q_{KL_2} &= \frac{D_{KL_2}\left(Q_n\|Q_0(\cdot;\widehat{\theta}_n)\right)}{\bar{X}_n}\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{\bar{X}_n}\log X_i\widehat{\theta}_n - 1 + \frac{\hat{\theta}_n}{\bar{X}_n}\sqrt{\frac{\pi}{2}} - \frac{1}{2}\sum_{i=1}^{n}\frac{X_{(i)}}{\bar{X}_n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\log(-2\log(1-x))\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
Q_T &= D_T\left(Q_n\|Q_0(\cdot;\widehat{\theta}_n)\right)\\
&= \frac{1}{\alpha-1}\left\{\sum_{i=1}^{n}\left(\frac{X_{(i)}^{\alpha}\widehat{\theta}_n^{(1-\alpha)}}{\bar{X}_n}\int_{\frac{i-1}{n}}^{\frac{1}{n}}(-2\log(1-x))^{\frac{1}{2}(1-\alpha)}\mathrm{d}x\right) - \alpha\right\} + \frac{\widehat{\theta}_n}{\bar{X}_n}\sqrt{\frac{\pi}{2}},
\end{aligned}$$

where $\widehat{\theta}_n$ is the MLE of θ, $Q_n(t) = X_{(r)}$, $\frac{r-1}{n} < t < \frac{r}{n}$, is the r-th order statistic and the empirical counterpart of Q_1. The authors prove the test to be consistent.

2.4. Tests based on the Phi-divergence measure

Below we consider tests based on the the Phi-divergence measure.

A Phi-divergence test by Zamanzade and Mahdizadeh (2017):

In the paper by Zamanzade and Mahdizadeh (2017), several test statistics are based on the Phi-divergence measure

$$D_{\phi}(P_1||P_2) = \int_{\Omega}\phi\left(\frac{\mathrm{d}P_1}{\mathrm{d}P_2}\right)\mathrm{d}P_2,$$

where P_1 and P_2 are probability measures and $\phi(\cdot)$ is a convex function such that $\phi(1) = 0$ and second derivative $\phi''(1) > 0$. Let $D_n(g||\widehat{g}_h)$ be a sample estimate of $D_\phi(\cdot)$ written in the form

$$D_n(g||\widehat{g}_h) = \frac{1}{n}\sum_{i=1}^{n}\phi\left(\frac{g(X_i,\widehat{\theta}_n)}{\widehat{g}_h(X_i)}\right),$$

where $\widehat{g}_h(x) = (nh)^{-1}\sum_{i=1}^{n} k((X_i - x)/h)$, k is a kernel function and h is a suitably chosen bandwidth. The new test statistics are constructed by choosing appropriate functions for $\phi(\cdot)$. Zamanzade and Mahdizadeh (2017) specifically studied the following selection of functions and resultant test statistics:

- $\phi(t) = -\log(t)$ resulting in the Kullback-Leibler distance with test statistic

$$PKL_n = \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\widehat{g}_h(X_i)}{g(X_i,\widehat{\theta}_n)}\right).$$

- $\phi(t) = \frac{1}{2}(1-\sqrt{t})^2$ resulting in the Hellinger distance with test statistic

$$PH_n = \frac{1}{2n}\sum_{i=1}^{n}\left(1-\left(\frac{g(X_i,\widehat{\theta}_n)}{\widehat{g}_h(X_i)}\right)^{1/2}\right)^2.$$

- $\phi(t) = (t-1)\log(t)$ resulting in the Jeffreys distance with test statistic

$$PJ_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{g(X_i,\widehat{\theta}_n)}{\widehat{g}_h(X_i)} - 1\right)\log\left(\frac{g(X_i,\widehat{\theta}_n)}{\widehat{g}_h(X_i)}\right).$$

- $\phi(t) = |t-1|$ resulting in the total variation distance with test statistic

$$PTV_n = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{g(X_i,\widehat{\theta}_n)}{\widehat{g}_h(X_i)} - 1\right|.$$

- $\phi(t) = \frac{1}{2}(1-t)^2$ resulting in the chi-square distance with test statistic

$$PC_n = \frac{1}{2n}\sum_{i=1}^{n}\left(1-\frac{g(X_i,\widehat{\theta}_n)}{\widehat{g}_h(X_i)}\right)^2.$$

The authors found that among the proposed tests, the Jeffreys and Hellinger distance tests performed the best.

A test based on a new proximity measure by Torabi, Montazeri, and Grané (2016):

In this paper we adapt the test of Torabi *et al.* (2016) specifically for the Rayleigh distribution. Torabi *et al.* (2016) suggested a new proximity measure which was inspired by the Phi-divergence approach. This measure is used to develop a test for the location-scale family of distribution and specifically implemented to test for the normal distribution. The discrepancy measure between the hypothesized null distribution F_0 (in this case the normal distribution with unknown mean, μ, and variance, σ^2) and the unknown distribution F of the data, is defined as

$$D(F_0||F) = \int_{-\infty}^{\infty}\Psi\left(\frac{1+F_0(x)}{1+F(x)}\right)\mathrm{d}F(x),$$

where $\Psi(\cdot) : (0,\infty) \to \mathbb{R}^+$ is continuous, decreasing on $(0,1)$ and increasing on $(1,\infty)$ with $\Psi(1) = 0$. Now, estimating F by the ecdf F_n, leads to the easily calculable test statistic

$$H_n = n^{-1} \sum_{i=1}^{n} \Psi\left(\frac{1 + F_0(Z_{(i)})}{1 + i/n}\right),$$

where $Z_{(i)} = (X_{(i)} - \widehat{\mu})/\widehat{\sigma}$ is the scaled residuals for the location-scale families with consistent estimators $\widehat{\mu}$ and $\widehat{\sigma}$. Torabi *et al.* (2016) discussed possible options for the function $\Psi(\cdot)$ and suggest choosing $\Psi(x) = ((x-1)/(x+1))^2$ as it exhibited the highest powers for testing normality in their simulation study. The authors showed the test to be invariant under location-scale transformations and proved the test to be consistent.

In testing for the Rayleigh distribution the test statistic maintains the form

$$C_n = n^{-1} \sum_{i=1}^{n} \Psi\left(\frac{1 + G_0(Y_{(i)})}{1 + i/n}\right),$$

with the only change being the scaled observations are now $Y_{(i)} = X_{(i)}/\widehat{\theta}_n$, and $G_0(Y_{(i)}) = 1 - \exp(-Y_{(i)}^2/2)$.

2.5. Tests adapted for the Rayleigh distribution

Below we consider a transformation to uniformity test that we adapt for the Rayleigh distribution.

A transformation to uniformity test by Meintanis (2009):

The test by Meintanis (2009) states that for a suitable transformation $G_\theta(x) = U_\theta(x)$, where $G_\theta(x)$ is the hypothesized distribution under the null hypothesis with unknown parameter θ, a test statistic can be constructed between the empirical characteristic function of $U_j = U_\theta(X_j)$ and the characteristic function of the standard uniform distribution, $\phi_U(t)$. More formally, the test statistic can be written as

$$M_n = \int_{-\infty}^{\infty} |\phi_n(t) - \phi_U(t)|^2 \, w(t) \, \mathrm{d}t, \tag{11}$$

where $w(t)$ is a suitable chosen weight function, $\phi_n(t) = n^{-1}\sum_{i=1}^{n} \exp(itU_j)$ is the characteristic function of the (unknown) U_j and $\phi_U(t) = t^{-1}\{\sin t + i(1 - \cos t)\}$ is the characteristic function of a uniform random variable on $(0,1)$. If we now estimate θ by $\widehat{\theta}_n$ (in the case of testing for the Rayleigh distribution) we obtain

$$\widehat{U}_j = G_{\widehat{\theta}_n}(X_j) = 1 - \exp\left(\frac{-X_j}{2\widehat{\theta}_n^2}\right).$$

Meintanis (2009) shows that different closed forms for (11) can be obtained for different choices of the weight function. Specifically, by choosing $w(t) = \exp(-\varphi|t|)$ with tuning parameter $\varphi > 0$, and with U_j replaced by $\widehat{U}_j$, M_n becomes

$$\begin{aligned} M1_{n,\varphi} &= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{2\varphi}{\widehat{U}_{jk}^2 + \varphi^2} + 2n\left[2\tan^{-1}\left(\frac{1}{\varphi}\right) - \varphi \log\left(1 + \frac{1}{\varphi^2}\right)\right] \\ &- 4\sum_{j=1}^{n}\left[\tan^{-1}\left(\frac{\widehat{U}_j}{\varphi}\right) + \tan^{-1}\left(\frac{1 - \widehat{U}_j}{\varphi}\right)\right], \end{aligned} \tag{12}$$

where $\widehat{U}_{jk}^2 = (\widehat{U}_j - \widehat{U}_k)^2$. Instead of using the probability integral transforms as is the case in (12), we adapt the approach for the Rayleigh distribution by considering the following

transformation for exponentiality given by Alzaid and Al-Osh (1991):
Let X_1 and X_2 be two independent observations from a distribution F, then $\frac{X_1}{X_1+X_2}$ is distributed standard uniform $U(0,1)$ if, and only if, F is exponential. The transformation holds true for the Rayleigh distribution by noting that, if $X \sim Ral(\theta)$, then $X^2/2\theta^2$ follows a standard exponential distribution. This result is now formally stated in Corollary 2.3.

Corollary 2.3. *Let X_1 and X_2 be two independent observations from a distribution $G(x)$, then $\frac{X_1^2}{X_1^2+X_2^2}$ follows a standard uniform distribution $U(0,1)$ if, and only if, $G(x)$ is the Rayleigh distribution with parameter θ (i.e., $G(x) = 1 - \exp(-x^2/2\theta^2)$).*

Proof. If $X_1 \sim Ral(\theta)$, then $X_1^2/2\theta^2$ follows a standard exponential distribution. The same holds for X_2. Thus we have that

$$\frac{\frac{X_1^2}{2\theta^2}}{\frac{X_1^2}{2\theta^2} + \frac{X_2^2}{2\theta^2}} = \frac{X_1^2}{X_1^2 + X_2^2},$$

follows a standard uniform distribution if, and only if, X_1^2 and X_2^2 are exponentially distributed, or then if, and only if, X_1 and X_2 follows a Rayleigh distribution. □

Now, let $\widehat{Z}_{ij} = X_{(i)}^2/(X_{(i)}^2 + X_{(j)}^2)$, $i,j = 1,\ldots,n$, $i \neq j$, then the test statistic in (11), which is now based on this new transformation to uniformity, becomes

$$\begin{aligned} M2_{n,\varphi} &= \frac{1}{n(n-1)} \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k\neq l}}^{n} \sum_{l=1}^{n} \frac{2\varphi}{(\widehat{Z}_{ij} - \widehat{Z}_{kl})^2 + \varphi^2} \\ &+ 2n\left[2\tan^{-1}\left(\frac{1}{\varphi}\right) - \varphi \log\left(1 + \frac{1}{\varphi^2}\right)\right] \\ &- 4\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\tan^{-1}\left(\frac{\widehat{Z}_{ij}}{\varphi}\right) + \tan^{-1}\left(\frac{1-\widehat{Z}_{ij}}{\varphi}\right)\right], i,j,k,l = 1,2,\ldots n. \end{aligned}$$

This test statistic is based on a four-fold sum and is thus more computer intensive.

2.6. Other tests for the Rayleigh distribution

Below we consider tests based on a conditional expectation characterisation, the empirical likelihood ratio, moments, as well as the transformation of the data.

A test based on a conditional expectation characterisation by Liebenberg, Ngatchou-Wandji, and Allison (2020):

Liebenberg *et al.* (2020) considered the characterisation by Ahsanullah and Shakil (2013) and proposed an analogous statement in Corollary 2.4 that served as the basis of their goodness-of-fit test.

Corollary 2.4. *Let X be a nonnegative random variable with absolutely continuous cumulative distribution function F satisfying $F(0) = 0$, $F(x) > 0$ for all $x > 0$, and with finite $E(X^{2k})$, for some fixed $k \geq 1$. Then X has a Rayleigh distribution with parameter θ, if, and only if, for all $t > 0$,*

$$E[X^{2k}I(X > t)] = S(t)\nu_{k,\theta}(t),$$

where $\nu_{k,\theta}(t) := \sum_{l=0}^{k} 2^l\theta^{2l}k^{(l)}t^{2(k-l)}$, with $k^{(l)} = k(k-1)\ldots(k-l+1)$ and $k^{(0)} = 1$.

The approach for the test rests on the fact that $E[X^{2k}I(X > t)] - S(t)\nu_{k,\theta}(t) = 0$ if, and only if, for all $t > 0$ the random variable X has a Rayleigh distribution. By taking a normalized empirical version of the above statement,

$$\mathcal{T}_n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} I(X_i > t)[X_i^{2k} - \nu_{k,\widehat{\theta}_n}(t)],$$

and by considering the Kolmogorv-Smirnov (KS) and Cramér-von Mises (CM) distance measures, new test statistics can be defined as

$$KS_n = \sup_{t\in\Theta} |\mathcal{T}_n(t)| \quad \text{and} \quad CM_n = \int_{\Theta} \mathcal{T}_n^2(t)w(t)\,\mathrm{d}\widetilde{F}_n(t),$$

where w is a suitable weight function and $\widetilde{F}_n$ is any consistent estimator of F. Although the authors based their test on X_j, we will implement it on $Y_j = X_j/\widehat{\theta}_n$, which renders the test scale invariant. The authors proved the consistency of the test CM_n and derived the limiting null distribution.

A test based on the empirical likelihood ratio by Safavinejad, Jomhoori, and Alizadeh Noughabi (2015):

When testing for the Rayleigh distribution, the likelihood ratio tests statistic takes the form

$$R = \frac{\prod_{i=1}^{n} f_{H_1}(X_i)}{\prod_{i=1}^{n} f_{H_0}(X_i)} = \frac{\prod_{i=1}^{n} f_{H_1}(X_i)}{(\prod_{i=1}^{n} X_i/\theta^{2n})\exp(-\sum_{i=1}^{n} X_i/2\theta^2)},$$

where f_{H_1} is the density under the alternative hypothesis and f_{H_0} is the density under H_0. A density-based empirical likelihood technique is employed by Safavinejad *et al.* (2015) to estimate $\prod_{i=1}^{n} f_{H_1}(X_i)$. Given $X_1, X_2, \ldots, X_n$ i.i.d. from a random sample Vexler and Gurevich (2010) and Safavinejad *et al.* (2015) states the empirical likelihood function to be $L_p = \prod_{i=1}^{n} p_i$ where p_i , $i = 1, 2, \ldots, n$ are components that maximize the function L_p. The density based likelihood function under H_1 is then

$$L_f = \prod_{i=1}^{n} f(X_{(i)}) := \prod_{i=1}^{n} f_i \ .$$

The approach rests on finding values for f_i that maximizes L_f subject to empirical constraints dependent on H_1 that are exemplified in Vexler and Gurevich (2010) and Safavinejad *et al.* (2015). The authors conclude that using a Lagrange multiplier method to maximize $\log(f_i)$ yields a usable expression to estimate $f_{H_1}(X_i)$ in the form

$$f_j = \frac{2m}{n\left(X_{(j+m)} - X_{(j-m)}\right)},$$

where $X_{(j)} = X_{(1)}$, if $j \leq n$ and $X_{(j)} = X_{(n)}$, if $j \geq 1$ and m is a window width. Noting the test statistic is dependent on the parameter m, Safavinejad *et al.* (2015) adopted the modification suggested by Vexler and Gurevich (2010) to consider choices of m in the range $(1, n^{1-\delta})$, $0 < \delta < 1$, which then leads to the tests statistic

$$\widehat{R}_n = \frac{\min_{1\leq m<n^{1-\delta}} \prod_{i=1}^{n}\left\{2m/n\left(X_{(i+m)} - X_{(i-m)}\right)\right\}}{\left(\prod_{i=1}^{n} X_i/\widehat{\theta}_n^{2n}\right)\exp\left(-\sum_{i=1}^{n} X_i^2/2\widehat{\theta}_n^2\right)}.$$

A proof of the consistency of the test is provided in the paper by Safavinejad *et al.* (2015).

A test based on transformation of the data by Gulati (2011):

Gulati (2011) also offers a method based on transforming the data to exponentiality under the null hypothesis and suggests using a test statistic based on the work by Brain and Shapiro (1983). The first step in the process of Gulati (2011) is to transform the data to exponentiality by using $\widetilde{Y}_{(i)} = X_{(i)}^2/2\widehat{\theta}_n^2$. The subsequent test statistic is the sum of two components where each component is a test statistic in itself proposed by Brain and Shapiro (1983). The test utilizes weighted spacing which can be obtained by calculating

$$Z_i = (n-i+1)(\widetilde{Y}_{(i)} - \widetilde{Y}_{(i-1)}) \ , \ i = 1, 2, \dots, n.$$

and $\widetilde{Y}_{(0)} = 0$. The test statistics is then given by $V = V_1^2 + V_2^2$ where

$$V_1 = \sqrt{12(n-1)}\left(\bar{u} - \frac{1}{2}\right),$$

with $\bar{u} = (n-1)^{-1}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i} Z_j / \sum_{j=1}^{n} Z_j\right)$, and

$$V_2 = \sqrt{\frac{5(n-1)}{(n+1)(n-2)}}(n-2+6\bar{u}) - 12\sum_{i=1}^{n-1}\left(\frac{\sum_{j=1}^{i} Z_j}{\sum_{j=1}^{n} Z_j}\right)\left(\frac{i}{n-1}\right).$$

The author notes that V_1 and V_2 have standard normal distributions, therefore V follows a χ_2^2 distribution.

3. Simulation study

In this section, we compare the finite-sample performance of selected test statistics against general alternatives with a Monte Carlo study.

3.1. Simulation setting

As all the tests considered in this section are scale invariant, critical values were calculated based on 50 000 independent samples from a standard Rayleigh distribution at a $\alpha = 5\%$ significance level. Note that all of the test statistics in the study reject the null hypothesis in (4) for large values. A number of tests rely on a tuning parameter $\varphi > 0$ or window-width $m > 0$ that has to be chosen. In this study, the power estimates were obtained over a wide range of tuning parameter and window-width values. Only the tuning parameter and window-width values that exhibited the highest power estimates are reported in Table 2 and Table 4. For comparability, an attempt was made to keep the tuning parameter and window-width values as similar as possible across the tests and consistently the same over tables. The power estimates in the aforementioned tables and throughout the rest of the simulation study including the real data example are obtained by using MLE. The tests considered in the simulation study are listed below,

- Classical tests: Kolmogorov-Smirnov test (D_n), Watson test (V_n), Cramér-von Mises test (W_n) and Anderson-Darling test (A_n).
- Kullback–Leibler divergence test ($KL_{n,m}$),
- Hellinger distance test ($DH_{n,m}$),
- test based on the Laplace transform ($MI_{n,\varphi}$),
- adapted tests ($M1_{n,\varphi}$,$M2_{n,\varphi}$),
- test based on a new proximity measure ($C1_n$,$C2_n$),
- cumulative residual entropy test (CK_n),
- Phi-divergence tests (PH_n,PJ_n,PTV_n,PC_n),
- tests based on a conditional expectation characterisation ($CM_{n,\varphi}$,KS_n),

- test based on the Mellin transform ($LA_{n,\varphi}$).

The tests of Torabi *et al.* (2016), adapted for the Rayleigh distribution, are implemented for functions $\psi(x) = [(x-1)/(x+1)]^2$ in $C1_n$ and $\psi(x) = x\log(x) - x + 1$ in $C2_n$. For ease of reference, Table 1 lists the parametrization of the various alternative distributions used in the study. These distributions are commonly used as alternatives for the Rayleigh distribution and include distributions with constant hazard rates (CHR), decreasing hazard rates (DHR), increasing hazard rates (IHR) and non-monotone hazard rates (NMHR). Power estimates are obtained with 10 000 independent Monte Carlo replications for each of the considered tests and displayed in Table 2 for sample sizes $n = 20$ and in Table 4 for sample sizes $n = 30$.

Table 1: Probability density functions for choices of the alternative distributions

Alternative	$f(x)$	Notation
Gamma	$\frac{1}{\Gamma(\theta)}x^{\theta-1}e^{-x}$, $x>0$	$\Gamma(\theta)$
Weibull	$\theta x^{\theta-1}\exp(-x^{\theta})$, $x>0$	$\mathrm{W}(\theta)$
Power	$\frac{1}{\theta}x^{(1-\theta)/\theta}$, $0<x<1$	$\mathrm{PW}(\theta)$
Dhillon	$\frac{\theta+1}{x+1}\exp\left\{-(\log(x+1))^{\theta+1}\right\}(\log(x+1))^{\theta}$, $x>0$	$\mathrm{DL}(\theta)$
Chen	$2\theta x^{\theta-1}\exp\left\{x^{\theta}+2\left(1-\exp\left(x^{\theta}\right)\right)\right\}$, $x>0$	$\mathrm{CH}(\theta)$
Linear failure rate	$(1+\theta x)\exp(-x-\theta x^2/2)$, $x>0$	$\mathrm{LF}(\theta)$
Extreme value	$\frac{1}{\theta}\exp\left(x+\frac{1-e^{x}}{\theta}\right)$, $x>0$	$\mathrm{EV}(\theta)$
Lognormal	$\exp\left\{-\frac{1}{2}(\log(x)/\theta)^2\right\}/\left\{\theta x\sqrt{2\pi}\right\}$, $x>0$	$\mathrm{LN}(\theta)$
Inverse Gaussian	$\left[\frac{\theta}{2\pi x^3}\right]^{1/2}\exp\left\{\frac{-\theta(x-1)^2}{2x}\right\}$, $x>0$	$\mathrm{IG}(\theta)$
Gompertz	$\exp[-\theta x]\exp\left[-\left(\frac{1}{\theta}\right)(\exp(\theta x)-1)\right]$, $x\geq 0$	$\mathrm{GO}(\theta)$
Exponential	$\theta\exp(-\theta x)$, $x\geq 0$	$\mathrm{EXP}(\theta)$
Beta exponential	$\theta e^{-x}(1-e^{-x})^{\theta-1}$, $x>0$	$\mathrm{BEX}(\theta)$
Exponential logarithmic	$\frac{1}{-\ln\theta}\frac{(1-\theta)e^{-x}}{1-(1-\theta)e^{-x}}$, $x\geq 0$	$\mathrm{EL}(\theta)$
Exponential Nadarajah Haghighi	$\frac{\theta(1+x)^{-0.5}e^{1-(1+x)^{0.5}}}{2\left[1-e^{1-(1+x)^{0.5}}\right]^{1-\theta}}$, $x>0$	$\mathrm{ENH1}(\theta)$
Exponential geometric	$\frac{(1-\theta)e^{-x}}{(1-\theta e^{-x})^2}$, $x>0$	$\mathrm{EG}(\theta)$
Beta	$\frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)}x^{\theta_1-1}(1-x)^{\theta_2-1}$, $0<x<1$	$\mathrm{B}(\theta_1,\theta_2)$
Half normal	$\left(\frac{2}{\pi}\right)^2\exp\left(\frac{-x^2}{2}\right)$, $x\geq 0$	HN

Results for local power estimates are also displayed and are obtained as follows: A sample of size n is generated as individual values from either a $Ral(1)$-distribution with probability p, or a chosen alternative distribution with probability $1-p$. The parameter p determines the level of contamination, i.e., $p=0$ indicates a sample purely from the $Ral(1)$-distribution and $p>0$ indicates the introduction of values from the chosen alternative distribution. Table 6 shows power estimates in the case where the $\Gamma(1.5)$ distribution is used. All calculations and simulations were performed using R software, R Core Team (2013).

4. Simulation results

If we observe the classical tests (D_n, V_n, W_n, A_n) in isolation, it is clear that the Anderson-Darling test (A_n) outperforms all the other classical tests. The Cramér-von Mises (W_n) test is overall the second best performer in terms of power estimates. This trend is observed in both the $n = 20$ and $n = 30$ cases. Turning our attention to the entropy tests with window-width m ($KL_{n,m}$, $DH_{n,m}$), the $DH_{n,m}$ test performs better than the $KL_{n,m}$ test for both sample sizes considered. Interestingly, $KL_{n,m}$ outperforms its counterpart for the $PW(1)$ and $n = 30$ case. Considering entropy tests together ($KL_{n,m}$, $DH_{n,m}$, CK_n), we observe that CK_n is outperformed by its competitors. Next we turn our attention to tests based on the Phi-divergence ($C1_n$, $C2_n$, CK_n, PKL_n, PH_n, PJ_n, PTV_n, PC_n). Here we observe that PC_n either matches or performs slightly better than the PH_n, PJ_n, PTV_n tests. The aforementioned tests generally seem to outperform $C1_n$, $C2_n$, CK_n in most instances. However, for $n = 20$, the $C1_n$ test outperforms its competitors against the $DL(1.5)$, $IG(1.5)$ and $LN(0.8)$ distributions. For $n = 30$ it tends to rather match than outperform the other Phi-divergence distance tests for the specific distributions mentioned. The PKL_n test exhibit lower powers than the other tests. Focusing on the tests adapted for the Rayleigh distribution ($M1_{n,\varphi}$, $M2_{n,\varphi}$, $C1_n$, $C2_n$), we find that the transformation test $M1_{n,\varphi}$ by Meintanis (2009) tends to either match or outperform the $M2_{n,\varphi}$, $C1_n$ and $C2_n$ tests. The tuning parameter for $M1_{n,\varphi}$ and $M2_{n,\varphi}$ provides an edge in performance over the competitors that do not contain a tuning parameter. Furthermore, $M2_{n,\varphi}$ only slightly falls short of the performance of $M1_{n,\varphi}$. However, $M2_{n,\varphi}$ proves the better performer by some margin against the $G(2)$, $W(3)$, $DL(1)$, $IG(0.5)$, $IG(1.5)$ and $LN(0.8)$ distributions.

We find that there is very little difference between the powers of the tests considered against the alternatives with constant and decreasing hazard rates. The tests have extremely high powers against these alternatives, this could be attributed to the fact that the Rayleigh distribution has an increasing hazard rate. The tests, $CM_{n,\varphi}$ and $LA_{n,\varphi}$ perform very well when compared to the other tests in the study. When compared to each other in isolation, we find that $LA_{n,\varphi}$ is superior. For distributions with increasing hazard rates, we see that the $LA_{n,\varphi}$ and $MI_{n,\varphi}$ tests outperform its competitors in terms of power estimates for $n = 20$ and $n = 30$. However, the $DH_{n,m}$ test performs better than $MI_{n,\varphi}$ against the distributions $\Gamma(2)$, $GO(0.5)$ in the case where $n = 20$. For distributions with non-monotone hazard rates, we find that $LA_{n,\varphi}$ exhibits the highest powers in the majority of cases considered, closely followed by the $DH_{n,m}$ and $MI_{n,\varphi}$ tests. Overall, $LA_{n,\varphi}$ performs the best among all tests considered. The tuning parameter choices $\varphi = 2$ and $\varphi = 5$ for the tests $LA_{n,\varphi}$ and $CM_{n,\varphi}$ result in the highest powers in most cases. The tuning parameter value $\varphi = 0.5$ mostly results in the highest power for the $MI_{n,\varphi}$ test. For the $DH_{n,m}$ test, the window-width exhibiting the highest powers is $m = 6$. In general, $LA_{n,\varphi}$, $CM_{n,\varphi}$, $DH_{n,m}$ and $MI_{n,\varphi}$ prove to be the superior tests.

For the local powers we find that the $DH_{n,m}$ test with $m = 6$ attains the highest power in the case where $p = 0.05$ and $n = 20$. For $n = 30$ we find that the $LA_{n,\varphi}$ test performs the best. For the cases $p = 0.10$ to $p = 0.20$ ($n = 20$ and $n = 30$), the $LA_{n,\varphi}$ test with $\varphi = 1$ outperforms all other tests. As we move further from the Rayleigh distribution ($p = 0.25$ to 0.50), it is evident that $LA_{n,\varphi}$ still performs the best, however, for $p = 0.25$ ($n = 20$) and $p = 0.45$ ($n = 20$) the $LA_{n,\varphi}$ test is matched by $DH_{n,m}$ test with $m = 6$. The adapted test $M2_{n,\varphi}$ consistently performs better than the $M1_{n,\varphi}$ test for this mixture distribution.

5. Real data application

The following popular data set that is associated with the Rayleigh distribution and is analysed in order to demonstrate the use of the existing tests in a real-world setting. The data set appears in Best, Rayner, and Thas (2010) and is populated with 30 average wind speed

Table 2: Estimated powers for the alternative distributions in Table 1 and sample size $n = 20$

	D_n	V_n	W_n	A_n	$KL_{n,3}$	$KL_{n,4}$	$KL_{n,6}$	$DH_{n,3}$	$DH_{n,4}$	$DH_{n,6}$	$MI_{n,.5}$	$MI_{n,2}$	$MI_{n,5}$	$M1_{n,1}$	$M1_{n,2}$	$M1_{n,5}$	$M2_{n,1}$
CHR																	
Exp(1)	86	77	89	95	84	84	84	94	95	96	96	97	95	93	94	94	91
DHR																	
BEX(0.7)	97	94	98	**100**	97	98	97	99	**100**	**100**	**100**	**100**	99	99	99	99	98
BEX(0.9)	91	84	93	98	90	90	90	97	97	98	98	98	97	96	96	96	94
EG(0.2)	90	83	93	97	89	89	89	96	97	98	97	98	97	95	96	96	94
EG(0.5)	96	92	97	99	95	96	95	99	99	99	99	99	99	98	98	98	98
EG(0.8)	99	98	99	**100**	99	99	99	**100**	**100**	**100**	**100**	**100**	**100**	99	**100**	**100**	99
EL(0.2)	97	94	98	99	97	97	97	99	99	**100**	99	**100**	99	99	99	99	99
EL(0.5)	92	86	95	98	92	92	91	97	98	**99**	98	**99**	98	97	97	97	96
EL(0.8)	88	80	92	96	87	87	87	95	96	97	97	**98**	96	95	95	95	93
W(0.8)	97	95	98	**100**	98	98	98	99	**100**	**100**	**100**	**100**	**100**	99	99	99	98
IHR																	
Ral(5)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
CH(1)	55	47	61	78	53	55	55	73	77	80	84	83	76	78	78	78	64
CH(1.5)	7	7	7	12	7	8	7	10	11	11	**21**	13	9	13	13	13	7
EV(1.5)	23	22	25	45	27	29	30	39	43	46	63	51	38	48	49	49	26
$\Gamma(1.5)$	57	45	63	73	44	44	42	71	74	77	70	76	74	63	64	64	67
$\Gamma(2)$	32	23	36	43	19	18	15	46	49	52	33	44	45	30	30	29	41
GO(0.5)	16	17	17	36	22	23	25	30	32	35	**56**	39	26	36	38	39	17
GO(1.5)	48	40	52	71	47	48	48	67	70	73	80	77	68	72	73	73	55
LF(2)	37	31	42	61	35	36	37	56	61	65	71	67	58	60	61	61	46
LF(4)	25	21	29	46	24	25	25	42	47	50	59	53	44	46	47	47	32
PW(1)	15	26	19	39	44	47	48	41	43	39	55	34	17	33	34	35	13
W(1.4)	36	27	41	53	25	25	24	52	56	59	55	58	53	47	48	47	46
NMHR																	
B(0.5)	87	87	89	98	96	97	97	98	98	98	**99**	98	95	98	98	98	88
DL(1)	64	51	69	74	49	48	45	75	77	80	61	75	76	55	56	55	73
DL(1.5)	26	17	29	31	13	13	10	38	40	**43**	16	29	34	14	14	14	32
ENH1(2)	91	85	93	96	86	86	85	96	96	97	94	97	96	90	91	91	94
HN(1)	47	38	52	71	45	46	46	65	69	73	78	75	67	69	70	70	54
IG(0.5)	36	26	40	39	27	25	20	50	52	**53**	6	28	38	4	4	4	41
IG(1.5)	92	87	94	95	86	86	84	95	96	**97**	87	95	96	81	82	81	95
LN(0.8)	66	54	70	72	52	51	45	76	79	79	47	68	74	37	37	38	74
PW(2)	87	87	88	98	96	97	97	97	98	98	**99**	**99**	95	98	98	98	88

Table 3: *
Estimated powers for the alternative distributions in Table 1 and sample size $n = 20$ continued

	$M2_{n,2}$	$M2_{n,5}$	$C1_n$	$C2_n$	CK_n	PKL_n	PH_n	PJ_n	PTV_n	PC_n	$CM_{n,1}$	$CM_{n,2}$	$CM_{n,5}$	KS_n	$LA_{n,1}$	$LA_{n,2}$	$LA_{n,5}$
CHR																	
Exp(1)	92	91	95	94	88	90	97	97	96	97	95	97	96	88	97	**98**	**98**
DHR																	
BEX(0.7)	98	98	99	99	97	99	**100**	**100**	**100**	**100**	99	**100**	**100**	97	**100**	**100**	**100**
BEX(0.9)	94	94	97	97	92	94	**99**	**99**	98	**99**	97	98	98	92	98	**99**	**99**
EG(0.2)	94	94	97	96	92	93	98	98	98	98	97	98	98	92	98	**99**	98
EG(0.5)	98	98	99	99	96	97	99	99	99	99	99	99	99	97	99	**100**	99
EG(0.8)	**100**	99	**100**	**100**	99	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	99	**100**	**100**	**100**
EL(0.2)	99	98	99	99	98	98	**100**	**100**	**100**	**100**	99	**100**	**100**	98	**100**	**100**	**100**
EL(0.5)	96	96	98	97	94	95	**99**	**99**	98	**99**	98	**99**	98	94	**99**	**99**	**99**
EL(0.8)	93	93	96	96	90	91	**98**	**98**	97	**98**	96	**98**	97	91	97	**98**	**98**
W(0.8)	99	98	99	99	98	99	**100**	**100**	**100**	**100**	99	**100**	**100**	98	**100**	**100**	**100**
IHR																	
Ral(5)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
CH(1)	64	62	75	74	58	64	85	85	83	85	78	82	83	59	82	87	**88**
CH(1.5)	6	6	11	11	5	8	19	18	19	20	12	16	18	7	11	17	**21**
EV(1.5)	24	23	39	37	20	36	59	59	58	59	42	50	55	23	48	59	**64**
$\Gamma(1.5)$	68	68	75	74	64	54	79	79	74	78	76	78	76	63	79	**82**	77
$\Gamma(2)$	42	43	48	47	40	24	49	49	41	45	49	51	45	38	53	**54**	46
GO(0.5)	15	14	29	27	12	30	51	50	49	51	28	37	44	15	36	47	55
GO(1.5)	54	53	68	66	48	58	80	80	78	80	69	74	76	50	76	82	**83**
LF(2)	46	45	58	56	41	47	72	72	69	71	59	65	66	41	67	**74**	**74**
LF(4)	32	32	44	42	29	35	60	59	56	58	44	50	53	28	54	61	**62**
PW(1)	8	7	25	24	21	39	51	50	55	**57**	23	32	42	14	27	41	52
W(1.4)	46	46	55	54	43	34	63	62	58	61	55	59	57	41	61	**66**	62
NMHR																	
B(0.5)	84	81	95	95	84	97	**99**	**99**	**99**	**99**	96	97	98	86	98	**99**	**99**
DL(1)	74	74	78	77	72	56	78	78	70	75	79	80	74	70	81	**82**	75
DL(1.5)	34	35	38	36	36	18	35	36	24	29	37	37	29	32	**43**	40	28
ENH1(2)	95	95	96	96	93	89	97	97	95	96	97	97	97	93	97	**98**	97
HN(1)	55	53	68	66	50	57	80	80	77	79	67	73	75	50	75	81	**82**
IG(0.5)	43	44	46	45	47	22	35	36	16	25	41	39	25	42	50	42	21
IG(1.5)	95	95	96	96	94	86	95	95	93	95	96	96	95	94	96	**97**	95
LN(0.8)	74	75	77	76	73	53	72	72	60	67	77	75	68	71	**80**	77	66
PW(2)	84	81	95	94	84	97	**99**	**99**	**99**	**99**	95	97	98	85	98	**99**	**99**

Table 4: Estimated powers for the alternative distributions in Table 1 and sample size $n = 30$

	D_n	V_n	W_n	A_n	$KL_{n,3}$	$KL_{n,4}$	$KL_{n,6}$	$DH_{n,3}$	$DH_{n,4}$	$DH_{n,6}$	$MI_{n,.5}$	$MI_{n,2}$	$MI_{n,5}$	$M1_{n,1}$	$M1_{n,2}$	$M1_{n,5}$	$M2_{n,1}$
CHR																	
EXP(1)	96	92	98	99	96	96	96	98	99	99	99	100	99	99	99	99	98
DHR																	
BEX(0.7)	**100**	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
BEX(0.9)	91	84	93	98	90	90	90	97	97	98	98	98	97	96	96	96	94
EG(0.2)	98	95	99	**100**	98	98	98	99	**100**	**100**	**100**	**100**	**100**	99	99	99	99
EG(0.5)	99	99	**100**	**100**	99	99	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
EG(0.8)	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
EL(0.2)	**100**	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
EL(0.5)	99	97	99	**100**	99	99	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	99
EL(0.8)	97	94	98	99	97	97	97	99	99	99	99	**100**	99	99	99	99	99
W(0.8)	**100**	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
IHR																	
Ral(5)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
CH(1)	74	65	79	90	74	75	74	85	87	90	94	93	89	91	90	91	81
CH(1.5)	8	9	8	14	10	10	10	12	12	13	**28**	15	10	15	16	17	7
EV(1.5)	32	31	35	58	41	42	42	49	52	55	**77**	63	50	61	62	64	36
$\Gamma(1.5)$	75	64	81	88	64	65	63	82	86	89	84	89	88	80	80	81	83
$\Gamma(2)$	46	34	52	57	29	29	26	55	59	64	45	57	60	41	41	42	56
GO(0.5)	21	23	24	46	34	34	35	37	40	42	**69**	50	35	48	49	52	23
GO(1.5)	65	56	71	86	66	67	67	78	81	85	92	89	83	86	86	87	72
LF(2)	53	44	58	76	53	53	52	68	72	76	85	80	74	76	76	77	62
LF(4)	36	30	42	60	37	37	37	52	56	61	74	66	57	60	61	62	45
PW(1)	21	39	27	53	67	70	72	57	58	57	69	41	20	41	43	46	17
W(1.4)	52	41	59	69	39	40	37	62	66	71	70	72	70	63	63	65	62
NMHR																	
B(0.5)	97	97	97	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	99	**100**	**100**	**100**	97
DL(1)	81	70	85	88	69	69	66	85	88	91	78	88	89	72	71	73	87
DL(1.5)	36	24	40	42	22	22	19	45	50	54	22	39	46	19	19	19	45
ENH1(2)	98	96	99	99	96	96	96	99	99	99	99	99	99	98	98	98	99
HN(1)	64	55	70	84	64	65	64	77	81	84	90	88	83	85	84	85	72
IG(0.5)	51	39	55	54	42	42	37	61	65	**67**	9	36	51	6	5	5	57
IG(1.5)	99	97	99	99	96	97	96	99	99	99	96	99	99	94	93	94	99
LN(0.8)	82	73	86	87	71	71	68	86	88	90	62	83	88	52	52	53	88
PW(2)	97	97	97	**100**	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	99	**100**	**100**	**100**	97

Table 5: *
Estimated powers for the alternative distributions in Table 1 and sample size $n = 30$ continued

	$M2_{n,2}$	$M2_{n,5}$	$C1_n$	$C2_n$	CK_n	PKL_n	PH_n	PJ_n	PTV_n	PC_n	$CM_{n,1}$	$CM_{n,2}$	$CM_{n,5}$	KS_n	$LA_{n,1}$	$LA_{n,2}$	$LA_{n,5}$
CHR																	
EXP(1)	98	98	99	99	96	98	100	100	99	100	99	99	99	97	99	100	100
DHR																	
BEX(0.7)	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
BEX(0.9)	94	94	97	97	92	94	**99**	**99**	98	**99**	97	98	98	92	98	**99**	**99**
EG(0.2)	99	99	**100**	99	98	98	**100**	**100**	**100**	**100**	**100**	**100**	**100**	98	**100**	**100**	**100**
EG(0.5)	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
EG(0.8)	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
EL(0.2)	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
EL(0.5)	99	99	**100**	**100**	99	99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	99	**100**	**100**	**100**
EL(0.8)	99	98	99	99	97	98	**100**	**100**	**100**	**100**	99	**100**	**100**	98	**100**	**100**	**100**
W(0.8)	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
IHR																	
Ral(5)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
CH(1)	80	78	88	87	72	80	94	94	93	94	89	92	93	76	92	**95**	**95**
CH(1.5)	6	6	13	12	5	9	22	22	23	23	13	17	23	7	12	20	26
EV(1.5)	33	31	51	49	23	47	73	72	70	72	52	61	68	31	58	70	76
$\Gamma(1.5)$	84	83	89	88	79	72	92	92	87	90	88	90	89	80	91	**93**	91
$\Gamma(2)$	57	57	62	61	54	34	63	63	52	57	61	62	57	52	**68**	**68**	59
GO(0.5)	20	18	37	35	14	39	63	62	60	63	37	47	58	20	42	58	67
GO(1.5)	71	70	83	82	61	73	92	91	90	91	83	87	89	66	87	92	**93**
LF(2)	62	61	73	71	52	62	85	85	81	84	74	78	81	55	80	86	**87**
LF(4)	44	44	56	54	36	47	73	73	67	71	57	63	68	39	65	73	**75**
PW(1)	9	7	33	32	35	55	69	68	**77**	76	29	41	55	18	27	49	64
W(1.4)	61	62	71	69	56	48	77	76	70	74	70	73	73	58	75	**79**	77
NMHR																	
B(0.5)	95	92	99	99	94	**100**	**100**	**100**	**100**	**100**	99	**100**	**100**	96	**100**	**100**	**100**
DL(1)	88	88	90	90	85	72	90	90	83	87	90	90	87	85	**92**	**92**	88
DL(1.5)	46	48	48	47	47	25	45	46	29	36	47	46	37	43	**56**	53	37
ENH1(2)	99	99	99	99	98	97	**100**	**100**	99	99	**100**	**100**	99	99	**100**	**100**	**100**
HN(1)	71	70	82	81	63	72	91	90	88	90	82	86	88	66	87	**92**	**92**
IG(0.5)	57	58	60	59	60	35	49	50	22	36	56	50	35	57	65	57	29
IG(1.5)	99	99	99	99	99	96	99	99	98	99	99	99	99	99	99	**100**	99
LN(0.8)	88	89	89	89	86	70	86	87	74	81	89	88	82	86	**91**	90	81
PW(2)	95	92	99	99	94	**100**	**100**	**100**	**100**	**100**	99	**100**	**100**	96	**100**	**100**	**100**

Table 6: Estimated local powers for $n = 20$ (top row) and $n = 30$ (bottom row) for the $\Gamma(1.5) - Ral(1)$ mixture distribution

p	n	D_n	V_n	W_n	A_n	$KL_{n,3}$	$KL_{n,4}$	$KL_{n,6}$	$DH_{n,3}$	$DH_{n,4}$	$DH_{n,6}$	$MI_{n,.5}$	$MI_{n,2}$	$MI_{n,5}$	$M1_{n,1}$	$M1_{n,2}$	$M1_{n,5}$	$M2_{n,1}$
0	20	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	30	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
0.05	20	7	6	7	8	6	5	5	9	10	**11**	8	8	9	6	6	6	8
	30	8	6	8	9	7	6	5	11	12	12	8	8	9	6	5	6	9
0.10	20	9	7	10	11	7	6	5	14	15	14	10	10	11	7	6	7	11
	30	10	7	12	13	8	7	6	17	17	18	11	11	14	7	6	7	12
0.15	20	11	8	12	14	7	7	5	18	19	20	13	14	15	8	8	8	14
	30	14	9	16	17	10	9	7	22	24	25	15	16	19	9	9	9	17
0.20	20	13	10	15	18	9	8	6	22	24	24	16	17	19	10	10	10	16
	30	17	11	20	22	12	11	9	28	29	31	20	21	24	11	11	11	22
0.25	20	16	11	18	21	10	9	7	27	28	**30**	19	21	12	12	12	12	20
	30	21	14	24	27	14	13	11	32	34	37	24	27	30	14	15	16	27
0.30	20	19	12	22	25	12	10	9	31	31	34	23	26	27	14	14	15	23
	30	25	16	29	32	17	16	13	37	40	43	28	33	36	18	19	19	32
0.35	20	22	15	25	29	13	12	10	35	36	38	26	30	31	17	17	17	28
	30	29	18	33	37	20	18	16	42	44	48	33	39	42	22	23	23	36
0.40	20	24	16	28	32	15	14	12	38	40	42	29	33	36	20	19	20	31
	30	32	21	37	43	23	20	18	46	49	53	38	44	47	25	27	27	41
0.45	20	27	18	31	36	16	16	13	41	44	**48**	33	38	40	23	23	23	35
	30	36	24	42	48	26	24	21	51	53	57	43	49	52	31	31	32	45
0.50	20	29	20	34	40	19	19	15	45	48	50	36	42	44	27	27	28	38
	30	41	28	46	52	29	28	24	54	57	61	47	54	57	36	36	37	51

Table 7: *
Estimated local powers for $n = 20$ (top row) and $n = 30$ (bottom row) for the $\Gamma(1.5) - Ral(1)$ mixture distribution continued

p	n	$M2_{n,2}$	$M2_{n,5}$	$C1_n$	$C2_n$	CK_n	PKL_n	PH_n	PJ_n	PTV_n	PC_n	$CM_{n,1}$	$CM_{n,2}$	$CM_{n,5}$	KS_n	$LA_{n,1}$	$LA_{n,2}$	$LA_{n,5}$
0	20	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	30	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
0.05	20	8	9	8	8	10	7	10	10	7	8	9	8	8	8	10	10	8
	30	9	10	9	9	12	9	12	12	7	8	9	9	8	9	**13**	12	9
0.10	20	11	11	12	12	14	10	15	15	9	11	12	11	11	11	**16**	14	12
	30	13	13	14	14	18	13	17	18	9	11	13	13	12	13	**20**	17	13
0.15	20	15	15	16	15	18	12	19	19	12	14	16	16	14	14	**21**	19	15
	30	18	19	19	18	24	17	23	24	12	15	18	18	16	18	**27**	24	18
0.20	20	18	19	19	19	22	14	24	24	15	18	19	19	17	17	**26**	24	19
	30	23	24	24	23	29	20	30	31	16	20	23	23	21	22	**34**	31	23
0.25	20	22	23	23	22	25	17	28	28	18	21	23	23	21	20	**30**	29	23
	30	28	30	29	28	34	23	35	36	20	25	29	29	27	27	**40**	38	29
p0.30	20	26	26	28	27	29	19	32	33	22	25	27	28	25	24	**35**	34	28
	30	34	35	35	34	38	26	41	42	24	30	35	35	32	31	**46**	44	35
0.35	20	29	30	32	31	32	22	36	37	25	30	32	33	28	27	**39**	38	31
	30	40	40	40	39	43	30	47	47	29	35	40	41	38	36	**51**	50	40
0.40	20	32	35	36	34	36	24	41	42	30	34	36	36	33	30	**44**	43	37
	30	44	45	45	44	47	34	52	53	34	41	45	46	43	40	**56**	**56**	46
0.45	20	36	38	40	38	39	27	45	45	34	38	41	40	36	33	35	32	34
	30	48	50	51	49	50	37	57	57	40	47	50	51	48	44	**61**	60	52
0.50	20	39	41	44	42	42	29	49	50	38	42	45	45	42	37	**53**	52	45
	30	53	54	55	53	54	41	62	62	45	52	55	56	53	48	**66**	**66**	57

observations (km/h) recorded in 2007 in a suburb of Sydney, Australia. The observations are given in Table 8. A number of authors have utilized this data set in the goodness-of-fit

Table 8: Average wind speed observations, see Best *et al.* (2010)

2.7	3.2	2.1	4.8	7.6	4.7	4.2	4.0	2.9	2.9
4.6	4.8	4.3	4.6	3.7	2.4	4.9	4.0	7.7	10.0
5.2	2.6	4.2	3.6	2.5	3.3	3.1	3.7	2.8	4.0

setting for the Rayleigh distribution, see for instance Alizadeh Noughabi *et al.* (2012); Jahanshahi *et al.* (2016); Liebenberg *et al.* (2020). We calculated the test statistics using the data after the known location parameter ($\mu = 1.5$) was subtracted which is standard practice for this data set. Thereafter the p-values were calculated with the use of Monte Carlo simulation (calculated based on 10 000 samples of size 30 simulated from a $Ral(1)$ distribution) and are given in Table 9. The choice was made to use the tuning parameter or window-width that showed a satisfactory power performance in the power study. The hypothesis to be tested is that the data originated from a Rayleigh distribution. It is clear that all of the tests were

Table 9: Calculated p-values for the average wind speed data

Test statistic	D_n	V_n	W_n	A_n	$KL_{n,3}$	$KL_{n,4}$	$KL_{n,6}$	$DH_{n,3}$	$DH_{n,4}$	$DH_{n,6}$	$MI_{n,.5}$	$MI_{n,2}$	$MI_{n,5}$
p-value	0.086	0.151	0.100	0.093	0.263	0.344	0.557	0.006	0.004	0.002	0.667	0.386	0.120
Test statistic	$M1_{n,1}$	$M1_{n,2}$	$M1_{n,5}$	$M2_{n,1}$	$M2_{n,2}$	$M2_{n,5}$	$C1_n$	$C2_n$	CK_n	PKL_n	PH_n	PJ_n	PTV_n
p-value	0.972	0.989	0.998	0.076	0.053	0.045	0.104	0.111	0.005	0.074	0.157	0.118	0.901
Test statistic	PC_n	$CM_{n,1}$	$CM_{n,2}$	$CM_{n,5}$	KS_n	$LA_{n,1}$	$LA_{n,2}$	$LA_{n,5}$					
p-value	0.618	0.141	0.271	0.477	0.015	0.011	0.060	0.293					

not significant at a 5% level except for $DH_{n,m}$, CK_n and KS_n tests who were significant. It could therefore be concluded that the tests with the exception of $DH_{n,m}$, CK_n and KS_n do not reject the null hypothesis in (4) and that the data originated from a Rayleigh distribution.

6. Conclusion

The purpose of the study was to review the existing goodness-of-fit tests for the Rayleigh distribution and compare these test with a Monte Carlo study. From the Monte Carlo study it is clear that no test can outright be declared the best as no test outperforms all other tests uniformly. This is in accordance with the findings of Janssen (2000). However in the study we found that the better performing tests are $LA_{n,\varphi}$, $CM_{n,\varphi}$, $DH_{n,m}$ and $MI_{n,\varphi}$. These tests performed more favourable in terms of power estimates against general alternatives and local power estimates. The $LA_{n,\varphi}$ tests often attained or matched the highest power estimates, while the $CM_{n,\varphi}$ test proved to be the most stable across tuning parameter values. That is, the $CM_{n,\varphi}$ test achieved competitive power estimates for any $\varphi \geq 1$. For implementation of the aforementioned tests, we advise choosing the tuning parameter as $\varphi = 2$ or $\varphi = 5$ as these choices exhibited good performance in most cases. For implementation of the $DH_{n,m}$ test a choice of $m = 6$ proved to perform well. Alternatively, one can use the methods described in Allison and Santana (2015) or Tenreiro (2019) to choose the tuning parameter data dependently.

7. Acknowledgements

The authors would like to extend their gratitude to the anonymous referees, associate editor and the editor for providing useful comments that lead to improvements from earlier versions of the manuscript.

References

Ahrari V, Baratpour S, Habibirad A, Fakoor V (2019). "Goodness of Fit Tests for Rayleigh Distribution Based on Quantiles." *Communications in Statistics-Simulation and Computation*, pp. 1–17.

Ahsanullah M, Shakil M (2013). "Characterizations of Rayleigh Distribution Based on Order Statistics and Record Values." *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, **3**.

Alizadeh Noughabi R, Alizadeh Noughabi H, Behabadi AEM (2012). "An Entropy Test for the Rayleigh Distribution and Power Comparison." *Journal of Statistical Computation and Simulation*, **84**(1), 151–158. doi:10.1080/00949655.2012.698620.

Allison JS, Santana L (2015). "On a Data-dependent Choice of the Tuning Parameter Appearing in Certain Goodness-of-fit Tests." *Journal of Statistical Computation and Simulation*, **85**(16), 3276–3288.

Alzaid AA, Al-Osh MA (1991). "Characterization of Probability Distributions Based on the Relation." *Sankhyā: The Indian Journal of Statistics, Series B*, pp. 188–190.

Balakrishnan N, Kocherlakota S (1985). "On the Double Weibull Distribution: Order Statistics and Estimation." *Sankhyā: The Indian Journal of Statistics, Series B*, pp. 161–178.

Baratpour BS, Khodadadi F (2012). "A Cumulative Residual Entropy Characterization of the Rayleigh Distribution and Related Goodness-of-Fit Test." *Journal of Statistical Research of Iran*, **9**(2), 115–1294.

Belaid A, Boukerroui D (2018). "Local Maximum Likelihood Segmentation of Echocardiographic Images with Rayleigh Distribution." *Signal, Image and Video Processing*, **12**(6), 1087–1096.

Best DJ, Rayner JCW, Thas O (2010). "Easily Applied Tests of Fit for the Rayleigh Distribution." *Sankhyā: The Indian Journal of Statistics, Series B*, **72**(2), 254–263.

Bovaird T, Lineweaver CH (2017). "A Flat Inner Disc Model as an Alternative to the Kepler Dichotomy in the Q1-Q16 Planet Population." *Monthly Notices of the Royal Astronomical Society*, **468**(2), 1493–1504.

Brain CW, Shapiro SS (1983). "A Regression Test for Exponentiality: Censored and Complete Samples." *Technometrics*, **25**(1), 69–76.

Casas-Prat M, Holthuijsen LH (2010). "Short-term Statistics of Waves Observed in Deep Water." *Journal of Geophysical Research: Oceans*, **115**(C9).

D'Agostino RB (1986). *Goodness-of-Fit Techniques*, volume 68. CRC Press.

Gulati S (2011). "Goodness of Fit Test for the Rayleigh and the Laplace Distributions." *International Journal of Applied Mathematics and Statistics*, **24**, 74–85.

Jahanshahi SMA, Habibi Rad A, Fakoor V (2016). "A Goodness-of-Fit Test for Rayleigh Distribution Based on Hellinger Distance." *Annals of Data Science*, **3**(4), 401–411. ISSN 2198-5812. doi:10.1007/s40745-016-0088-6.

Janssen A (2000). "Global Power Functions of Goodness of Fit Tests." *Annals of Statistics*, pp. 239–253.

Jensen DR (1970). "A Generalization of the Multivariate Rayleigh Distribution." *Sankhyā: The Indian Journal of Statistics, Series A*, pp. 193–208.

Johnson N, Kotz S, Balakrishnan N (1994). *Continuous Univariate Distributions*. Wiley.

Liebenberg SC, Allison JS (2019). "A Goodness-of-fit Test for the Rayleigh Distribution Based on the Mellin Transform." In *Annual Proceedings of the South African Statistical Association Conference*, volume 2019, pp. 17–24. South African Statistical Association (SASA).

Liebenberg SC, Ngatchou-Wandji J, Allison JS (2020). "On a New Goodness-of-fit Test for the Rayleigh Distribution Based on a Conditional Expectation Characterization." *Communications in Statistics - Theory and Methods*, pp. 1–15. doi:10.1080/03610926.2020.1836220. URL https://doi.org/10.1080%2F03610926.2020.1836220.

Meintanis SG (2009). "Goodness-of-fit Tests and Minimum Distance Estimation via Optimal Transformation to Uniformity." *Journal of Statistical Planning and Inference*, **139**(2), 100–108.

Meintanis SG, Iliopoulos G (2003). "Tests of Fit for the Rayleigh Distribution Based on the Empirical Laplace Transform." *Annals of the Institute of Statistical Mathematics*, **55**(1), 137–151.

Merovci F (2013). "Transmuted Rayleigh Distribution." *Austrian Journal of Statistics*, **42**(1), 21–31.

Morgan EC, Lackner M, Vogel RM, Baise LG (2011). "Probability Distributions for Offshore Wind Speeds." *Energy Conversion and Management*, **52**(1), 15–26.

R Core Team (2013). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. URL http://www.R-project.org/.

Rao M, Chen Y, Vemuri BC, Wang F (2004). "Cumulative Residual Entropy: A New Measure of Information." *IEEE Transactions on Information Theory*, **50**(6), 1220–1228.

Roy D (2004). "Discrete Rayleigh Distribution." *IEEE Transactions on Reliability*, **53**(2), 255–260.

Safavinejad M, Jomhoori S, Alizadeh Noughabi H (2015). "A Density-based Empirical Likelihood Ratio Goodness-of-fit Test for the Rayleigh Distribution and Power Comparison." *Journal of Statistical Computation and Simulation*, **85**(16), 3322–3334.

Schiff JL (1999). *The Laplace Transform. Undergraduate Texts in Mathematics.* Springer-Verlag, New York.

Siddiqui MM (1962). "Some Problems Connected with Rayleigh Distributions." *Journal of Research of the National Bureau of Standards*, **66**(2), 167–174.

Simon MK, Alouini M (1998). "A Simple Single Integral Representation of the Bivariate Rayleigh Distribution." *IEEE Communications Letters*, **2**(5), 128–130.

Tenreiro C (2019). "On the Automatic Selection of the Tuning Parameter Appearing in Certain Families of Goodness-of-fit Tests." *Journal of Statistical Computation and Simulation*, **89**(10), 1780–1797.

Torabi H, Montazeri NH, Grané A (2016). "A Test for Normality Based on the Empirical Distribution Function." *SORT: Statistics and Operations Research Transactions*, **40**(1), 0055–88.

Tsallis C (1998). "Generalized Entropy-based Criterion for Consistent Testing." *Physical Review E*, **58**(2), 1442.

Vasicek O (1976). "A Test for Normality Based on Sample Entropy." *Journal of the Royal Statistical Society. Series B (Methodological)*, **38**(1), 54–59.

Vexler A, Gurevich G (2010). "Empirical Likelihood Ratios Applied to Goodness-of-fit Tests Based on Sample Entropy." *Computational Statistics & Data Analysis*, **54**(2), 531–545.

Vodă VG (1976). "Inferential Procedures on a Generalized Rayleigh Variate. I." *Aplikace Matematiky*, **21**(6), 395–412.

Watson GS (1962). "Goodness-of-fit Tests on a Circle. II." *Biometrika*, **49**(1/2), 57–63.

Zamanzade E, Mahdizadeh M (2017). "Goodness of Fit Tests for Rayleigh Distribution Based on Phi-Divergence." *Revista Colombiana de Estadística*, **40**(2), 279–290.

Affiliation:

Shawn C. Liebenberg
School of Mathematical and Statistical Science
North-West University
Potchefstroom, South Africa
E-mail: shawn.liebenberg@nwu.ac.za
URL: http://natural-sciences.nwu.ac.za/statistics/staff

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 52 *Submitted:* 2021-03-19
January 2023 *Accepted:* 2021-08-31