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RICHARDSON EXTRAPOLATION OF KANTOROVICH AND DEGENERATE KERNEL METHODS FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. We propose two methods based on projections for approximating the solution of Fredholm integral equations of the second kind. The projection is either the orthogonal projection or an interpolatory projection onto a space of piecewise polynomials of any degree $\leq r-1$. We show that the two methods have asymptotic series expansions and that the orders of convergence can be further improved by multi-step Richardson extrapolation, where the calculation is repeated with each subinterval halved. These orders of convergence are preserved in the corresponding discrete methods obtained by calculating the integrals with a numerical quadrature formula. Numerical examples are given to validate the theoretical estimates.

1. INTRODUCTION

Let us consider the *Fredholm* integral equation defined on $\mathcal{X} = \mathcal{L}^\infty[0,1]$ by

$$u(s) - \int_0^1 \kappa(s,t)u(t)dt = f(s), \quad 0 \leq s \leq 1, \tag{1.1}$$

where κ denotes a smooth kernel, f is a real continuous function, and u is an unknown function. Classical methods for the numerical solution of (1.1) include the Galerkin method based on the orthogonal projection onto a finite-dimensional subspace of $\mathcal{X}$ and the collocation method based on an interpolatory projection (see [4]). The iterated Galerkin solution is obtained by one step of iteration and was proposed by Sloan [13]. The authors in [2] developed a new type of

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superconvergent Nyström and degenerate kernel methods for eigenvalue problems. The Kantorovich method, based on "*Kantorovich* regularization" (Kantorovich, 1948) was discussed by Schock [11]. It is shown that if the right-hand side f of the operator equation is less smooth than the kernel of the integral operator, then the Kantorovich solution has a higher order of convergence than the Galerkin solution. Sloan [14] also introduced the iterated Kantorovich method and established that it has a faster convergence than the Galerkin, iterated Galerkin, and Kantorovich methods. Recently, this method was investigated in [1] in the nonlinear case.

One is often interested in improving the orders of convergence of the approximate solutions. Asymptotic error analysis of approximate solutions is a classical numerical analysis topic. If the error expansions for numerical solutions are established, then the Richardson extrapolation can then be used to obtain approximate solutions of higher order. Asymptotic series expansions for the iterated Galerkin and iterated collocation solutions were proved by Mclean [10]. Then for the degenerate and Nyström methods, they were proved in [3]. Asymptotic series expansions for the iterated collocation method were also obtained in [9]. This method was already used for iterated modified projection solutions in [7] and for the corresponding eigenvalue problems in [8].

The main aim of this paper is to give an asymptotic error expansion of the iterated Kantorovich method. Thus the Richardson extrapolation can be performed on the solution, and this will increase the accuracy of the numerical solution greatly. We also give an asymptotic expansion for a degenerate kernel solution, and we show that the extrapolated solution converges as rapidly as the corresponding one in the Kantorovich method. We show that the obtained orders of convergence are still valid after taking into account the errors introduced by the numerical quadrature formula.

Now, we give a summary of the paper. In Section 2, notation is set, the numerical methods are described, and some relevant results are recalled. Asymptotic series expansion for the iterated Kantorovich method with both the orthogonal projection and the interpolatory projection at Gauss points is obtained in Section 3. The degenerate kernel method using an interpolatory projection is analyzed in Section 4. Section 5 is devoted to the discrete version of the proposed methods. Numerical results are given in Section 6.

2. Preliminaries and notations

For any positive integer n, let

$$\Delta_n : \quad 0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1 \tag{2.1}$$

be the uniform partition of $[0,1]$, with knots $\{s_i = \frac{i}{n},\ i = 0,\dots,n\}$ and meshlength $h = \frac{1}{n}$. For a fixed $r \geq 1$, we denote by Π_r the space of polynomials of degree $\leq r-1$. Let

$$\mathbb{X}_n := \left\{ v : [0,1] \longrightarrow \mathbb{R} : v|_{[s_{i-1},s_i]} \in \Pi_r,\ 1 \leq i \leq n \right\}$$

be the set of functions that are polynomials of degree $\leq r-1$ on each subinterval $[s_{i-1}, s_i]$. The functions in $\mathbb{X}_n$ need not be continuous at the node points s_i.

We consider two types of projections from $\mathcal{X}$ to $\mathbb{X}_n$.

• The map π_n is the restriction to $\mathcal{X}$ of the orthogonal projection from $\mathcal{L}^2[0,1]$ to $\mathbb{X}_n$. Then

$$(\pi_n u)(s) := \sum_{i=0}^{nr} \langle u, \varphi_i \rangle \varphi_i(s), \tag{2.2}$$

where $\{\varphi_1, \varphi_2, \ldots, \varphi_{nr}\}$ is an orthogonal basis for $\mathbb{X}_n$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{L}^2[0,1]$.

• For $u \in \mathcal{C}[0,1]$, let $\pi_n u$ denote the unique piecewise polynomial of degree $r-1$ that satisfies

$$(\pi_n u)(t_{ij}) = u(t_{ij}), \tag{2.3}$$

where the collocation points are

$$t_{ij} := (i - 1 + \tau_j)h, \quad 1 \le i \le n, \quad 1 \le j \le r,$$

and $\{\tau_1, \ldots, \tau_r\}$ are the r Gauss points in $[0,1]$. This map, if necessary, is extended to $\mathcal{X}$ as in [5], and then π_n is a projection. In both cases, π_n converges pointwise to the identity operator. Moreover, the projection π_n is uniformly bounded with respect to n, that is,

$$p = \sup_n \|\pi_n\|_{\mathcal{X} \to \mathcal{X}} < \infty.$$

Let $u, v \in \mathcal{C}^r[0,1]$. If π_n is the restriction of the orthogonal projection to $\mathcal{L}^\infty[0,1]$, then it follows that

$$\begin{aligned} \left| \int_0^1 u(t)(\mathcal{I} - \pi_n)v(t)dt \right| &= |\langle u - \pi_n u, v - \pi_n v \rangle| \\ &\le (c_1)^2 \|u^{(r)}\|_\infty \|v^{(r)}\|_\infty h^{2r}. \end{aligned} \tag{2.4}$$

Let α be a positive integer. For $u \in \mathcal{C}^\alpha[0,1]$, we set

$$\|u\|_{\alpha,\infty} := \sum_{i=0}^{\alpha} \|u^{(i)}\|_\infty.$$

If π_n is the interpolatory projection at r *Gauss* points, then for $u \in \mathcal{C}^r[0,1]$ and $v \in \mathcal{C}^{2r}[0,1]$ (see [6]) it holds

$$\left| \int_0^1 u(t)(\mathcal{I} - \pi_n)v(t)dt \right| \le c_2 \|u\|_{r,\infty} \|v\|_{2r,\infty} h^{2r}, \tag{2.5}$$

where c_2 is a constant independent of n.

Let $\mathcal{K}$ be the integral operator defined by

$$(\mathcal{K}u)(s) := \int_0^1 \kappa(s,t)u(t)dt, \quad s \in [0,1]. \tag{2.6}$$

Then (1.1) can be written in the operator form as

$$(\mathcal{I} - \mathcal{K})u = f. \tag{2.7}$$

For our convenience, we let

$$z := \mathcal{K}u. \tag{2.8}$$

Then writing the solution to (2.7) as $u = z + f$, we have

$$z = \mathcal{K}z + \mathcal{K}f. \tag{2.9}$$

The Kantorovich method is obtained by applying the classical projection method to the "regularized" equation (2.9). Thus the approximate solution is

$$u_n = z_n + f, \tag{2.10}$$

where z_n satisfies

$$(\mathcal{I} - \pi_n\mathcal{K})z_n = \pi_n\mathcal{K}f. \tag{2.11}$$

Note that the above equations are equivalent to a single equation for u_n

$$(\mathcal{I} - \pi_n\mathcal{K})u_n = f. \tag{2.12}$$

Throughout this paper, this method will be, respectively, called a Kantorovich-Galerkin or Kantorovich-collocation method when the orthogonal projection or the interpolatory projection is used.

Finally, the iterated Kantorovich approximation is defined by

$$\widetilde{u}_n = \mathcal{K}u_n + f = \widetilde{z}_n + f, \tag{2.13}$$

where

$$\widetilde{z}_n = \mathcal{K}z_n + \mathcal{K}f. \tag{2.14}$$

From (2.11) and (2.13), we observe that $z_n = \pi_n\widetilde{z}_n$, and hence

$$(\mathcal{I} - \mathcal{K}\pi_n)\widetilde{z}_n = \mathcal{K}f. \tag{2.15}$$

Let π_n be the interpolatory projection given by (2.3). Interpolation is a simple way to obtain degenerate kernel approximations. In fact, we consider the degenerate kernel

$$h_n(s,t) := \pi_n\kappa(s,t), \quad s,t \in [0,1],$$

obtained by interpolating the kernel with respect to the variable t. Then the associated degenerate kernel operator is defined by

$$(\mathcal{K}_n u)(s) := \int_0^1 h_n(s,t)u(t)dt. \tag{2.16}$$

The corresponding approximation of (1.1) is

$$(\mathcal{I} - \mathcal{K}_n)u_n^D = f. \tag{2.17}$$

Using the expression of π_n, equations (2.15) and (2.17) can be reduced to linear systems of equations of size nr.

Let $B_0(t) = 1$ and for $j \geq 1$, let $B_j(\tau)$ denote the *Bernoulli* polynomial of degree j. Let $\overline{B}_j$ be the periodic *Bernoulli* function defined by

$$\overline{B}_j(t) = B_j(t), \quad 0 \leq t < 1, \qquad \overline{B}_j(t+1) = \overline{B}_j(t), \quad t \in \mathbb{R}.$$

We give the following analysis of *Euler–Maclaurin* series expansion (*Steffensen* [9]). Let $f : [0,1] \to \mathbb{R}$ be ℓ times differentiable on $[0,1]$ and let $0 \leq \tau \leq 1$. Then

$$h\sum_{i=1}^{n} f[(i-1+\tau)h] = \int_0^1 f(t)dt + \sum_{j=1}^{\ell} \frac{B_j(\tau)}{j!}\left[f^{(j-1)}(t)\right]_{t=0}^{1} h^j + E_\ell, \tag{2.18}$$

where

$$E_\ell := -h^\ell \int_0^1 \frac{B_\ell(\tau - nt)}{\ell!} f^{(\ell)}(t)dt.$$

Henceforth, we shall always assume that the kernel function κ belongs to $\mathcal{C}^{2m+2}[0,1]^2$ for some integer $m \geq 0$. This implies that $\mathcal{K} : \mathcal{C}^\ell[0,1] \to \mathcal{C}^{2m+2}[0,1]$ is compact for $0 \leq \ell \leq 2m+2$. Furthermore, it is assumed that -1 is not an eigenvalue of $\mathcal{K}$. These assumptions ensure that $(\mathcal{I} - \mathcal{K})^{-1} : \mathcal{C}^\ell[0,1] \to \mathcal{C}^{2m+2}[0,1]$ exists and is uniformly bounded.
In the next section, we establish an asymptotic error expansions for the iterated kantorovich method.

3. Kantorovich method

Throughout this paper, we assume that the sum $\sum_{n_1}^{n_2}$ equals zero when $n_1 > n_2$. Let $\eta_1, \eta_2, \ldots,$ be the sequence of orthonormal polynomials in with respect to the inner product considered above, that is, η_p is a polynomial of degree $p-1$, and

$$\langle \eta_p, \eta_q \rangle = \delta_{pq} \qquad \text{for all } p, q \geq 1.$$

Define

$$\Lambda_r(\sigma, \tau) := \sum_{p=1}^{r} \eta_p(\sigma)\eta_q(\tau).$$

The following result was proved by Mclean [10].

Theorem 3.1. *Let π_n be the projection operator defined by (2.2) or (2.3). Assume that $u \in \mathcal{C}^{2m+2}[0,1]$. Then*

$$(\mathcal{K}\pi_n u)(s) = (\mathcal{K}u)(s) + \sum_{p=r}^{m} h^{2p}(R_{2p}u)(s) + \mathcal{O}(h^{2m+2}), \tag{3.1}$$

where, for the orthogonal projection, we have

$$(R_p u)(s) = c_{pp}(\mathcal{K}u^{(p)})(s) + \sum_{q=1}^{p-1} c_{pq}\left[\left(\frac{\partial}{\partial t}\right)^{p-q-1} \kappa(s,t)u^{(q)}(t)\right]_{t=0}^{1},$$

and

$$c_{pq} := \int_0^1 \int_0^1 \Lambda_r(\sigma,\tau)\frac{B_{p-q}(\tau)}{(p-q)!}\frac{(\sigma-\tau)^q}{q!}d\sigma d\tau,$$

while, for the interpolatory projection, it holds that

$$(R_p u)(s) = \bar{c}_{pp}(\mathcal{K}u^{(p)})(s) + \sum_{q=r}^{p-1} \bar{c}_{pq}\left[\left(\frac{\partial}{\partial t}\right)^{p-q-1} \kappa(s,t)u^{(q)}(t)\right]_{t=0}^{1}, \tag{3.2}$$

$$\bar{c}_{pq} := -\int_0^1 \Phi_q(\tau)\frac{B_{p-q}(\tau)}{(p-q)!}\omega_r(\tau)d\tau, \tag{3.3}$$

$$\Phi_q(\tau) := \int_0^1 \frac{(\sigma-\tau)^{q-r}}{(q-r)!}\frac{[\tau_1,\ldots,\tau_r,\tau](\bullet-\sigma)_+^{r-1}}{(r-1)!}d\sigma, \tag{3.4}$$

and $\omega_r(\tau) := \prod_{i=1}^{r}(\tau - \tau_i)$.

The proof of the proposition below is similar to the proof of [10, Theorem 2.1].

Proposition 3.2. *For n large enough, when restricted to $\mathcal{C}^{2m+2}[0,1]$, we have*

$$(\mathcal{K}\pi_n - \mathcal{I})^{-1} = (\mathcal{K}-\mathcal{I})^{-1} + \sum_{p=r}^{m} h^{2p}\mathcal{S}_p + \mathcal{O}(h^{2m+2}), \tag{3.5}$$

where

$$\mathcal{S}_p = -(\mathcal{K}-\mathcal{I})^{-1}R_{2p}(\mathcal{K}-\mathcal{I})^{-1}, \quad (r \le p \le 2r-1),$$
$$\mathcal{S}_p = -(\mathcal{K}-\mathcal{I})^{-1}\left(R_{2p}(\mathcal{K}-\mathcal{I})^{-1} + \sum_{i=r}^{p-r} R_{2p-2i}\mathcal{S}_i\right) \quad (2r \le p \le m).$$

Proof. Note first that R_{2p} maps $\mathcal{C}^{2m+2}[0,1]$ on itself. On the other hand,

$$(\mathcal{K}\pi_n - \mathcal{I}) = (\mathcal{K}-\mathcal{I}) + \sum_{p=r}^{m} h^{2p}R_{2p} + \overline{R}_{2m+2}, \tag{3.6}$$

where $\overline{R}_{2m+2}$ maps $\mathcal{C}^{2m+2}[0,1]$ on itself and $\|\overline{R}_{2m+2}\| = \mathcal{O}(h^{2m+2})$. Using (3.6) for the integers m and $m-p$, we obtain successively

$$(\mathcal{K}\pi_n - \mathcal{I})(\mathcal{K}-\mathcal{I})^{-1} = \mathcal{I} + \sum_{p=r}^{m} h^{2p}R_{2p}(\mathcal{K}-\mathcal{I})^{-1} + \overline{R}_{2m+2}(\mathcal{K}-\mathcal{I})^{-1},$$

$$\begin{aligned}\sum_{p=r}^{m} h^{2p}(\mathcal{K}\pi_n - \mathcal{I})\mathcal{S}_p &= \sum_{p=r}^{m} h^{2p}\left((\mathcal{K}-\mathcal{I})\mathcal{S}_p + \sum_{i=r}^{m-p} h^{2i}R_{2i}\mathcal{S}_p + \overline{R}_{2m-2p+2}\mathcal{S}_p\right)\\ &= \sum_{p=r}^{m} h^{2p}\left((\mathcal{K}-\mathcal{I})\mathcal{S}_p + \sum_{i=r}^{p-r} R_{2p-2i}\mathcal{S}_i\right) + h^{2p}\sum_{p=r}^{m}\overline{R}_{2m-2p+2}\mathcal{S}_p.\end{aligned}$$

Set $B_n := \mathcal{I} - (\mathcal{K}\pi_n - \mathcal{I})(\mathcal{K}-\mathcal{I})^{-1} - \sum_{p=r}^{m} h^{2p}(\mathcal{K}\pi_n - \mathcal{I})\mathcal{S}_p$. It follows from the definition of $\mathcal{S}_p$ that

$$\begin{aligned}B_n &= -\sum_{p=r}^{m} h^{2p}\left(R_{2p}(\mathcal{K}-\mathcal{I})^{-1} + (\mathcal{K}-\mathcal{I})\mathcal{S}_p + \sum_{i=r}^{p-r} R_{2p-2i}\mathcal{S}_i\right)\\ &= -\left(\overline{R}_{2m+2}(\mathcal{K}-\mathcal{I})^{-1} + h^{2p}\sum_{p=r}^{m}\overline{R}_{2m-2p+2}\mathcal{S}_p\right)\\ &= -\left(\overline{R}_{2m+2}(\mathcal{K}-\mathcal{I})^{-1} + h^{2p}\sum_{p=r}^{m}\overline{R}_{2m-2p+2}\mathcal{S}_p\right) = \mathcal{O}(h^{2m+2}).\end{aligned}$$

Since the operators π_n converge to identity operator pointwise and $\mathcal{K}$ is compact, then for n large enough, the operators $(\mathcal{K}\pi_n - \mathcal{I})^{-1}$ exist and are uniformly bounded. Therefore, it suffices to multiply B_n by $(\mathcal{K}\pi_n - \mathcal{I})^{-1}$ to obtain the desired result. $\square$

For $\widetilde{u}_n$ satisfying (2.13), define

$$u_{n,0} := \widetilde{u}_n$$

and

$$u_{n,\ell} := \frac{4^{\ell+r-1} u_{2n,\ell-1} - u_{n,\ell-1}}{4^{\ell+r-1} - 1}, \quad \ell = 1, 2, \ldots, m-r. \tag{3.7}$$

Now, we are ready to state and prove the main result of this section.

Theorem 3.3. *Let the right-hand side f of (1.1) be in $\mathcal{C}[0,1]$. Then*

$$u_{n,\ell} = u + \sum_{p=\ell}^{m} h^{2r+2p} A_{\ell,p} + \mathcal{O}(h^{2m+2}), \quad \ell = 0, 1, \ldots, m-r, \tag{3.8}$$

where the functions $A_{\ell,p}$ are independent of h.

Proof. From (2.7) and (2.13), we have $\widetilde{u}_n - u = \widetilde{z}_n - z$, and by using (2.9) and (2.15), we can write

$$\widetilde{z}_n - z = (\mathcal{K} - \mathcal{I})^{-1} \mathcal{K} f - (\mathcal{K}\pi_n - \mathcal{I})^{-1} \mathcal{K} f. \tag{3.9}$$

Thus since $\mathcal{K} f \in \mathcal{C}^{2m+2}[0,1]$, (3.5) gives

$$\widetilde{u}_n - u = -\sum_{p=r}^{m} h^{2p} \mathcal{S}_p(\mathcal{K} f) + \mathcal{O}(h^{2m+2}). \tag{3.10}$$

From the above asymptotic error expansion, it follows that the function $u_{n,0}$ approximates u with accuracy of order $\mathcal{O}(h^{2r})$. For a general integer $\ell \geq 1$, the ℓth step Richardson extrapolation $u_{n,\ell}$ approximates u with accuracy of order $\mathcal{O}(h^{2r+2\ell})$. $\square$

4. Degenerate kernel method

Let π_n be the interpolatory projection defined by (2.3) and let $[t_{i1}, \ldots, t_{ir}, t]u$ denote the divided difference of u based on $\{t_{i1}, \ldots, t_{ir}, t\}$. Then we have the following result.

Proposition 4.1. *Let Φ_q be the function defined by (3.4) and assume that $u \in \mathcal{C}^{2m+2}[0,1]$. Then for any $t = (i-1+\tau)h \in [s_{i-1}, s_i]$, where $0 \leq \tau \leq 1$ and $1 \leq i \leq n$, we have*

$$(u - \pi_n u)(t) = \sum_{q=r}^{2m+1} h^q u^{(q)}(t) \Phi_q(\tau) \omega_r(\tau) + \mathcal{O}(h^{2m+2}). \tag{4.1}$$

Proof. For $i = 1, \ldots, n$, the Newton remainder for polynomial interpolation is

$$(u - \pi_n u)(t) = [t_{i1}, \ldots, t_{ir}, t]u \prod_{j=1}^{r} (t - t_{ij}), \quad t \in [s_{i-1}, s_i], \tag{4.2}$$

and the Peano representation of divided differences (see [15]) gives

$$[t_{i1},\dots,t_{ir},t]u = \int_{s_{i-1}}^{s_i} \frac{[t_{i1},\dots,t_{ir},t](\bullet - v)_+^{r-1}}{(r-1)!} u^{(r)}(v)dv,$$
$$= \int_0^1 \frac{[\tau_1,\dots,\tau_r,\tau](\bullet - \sigma)_+^{r-1}}{(r-1)!} u^{(r)}[(i-1+\sigma)h]d\sigma. \tag{4.3}$$

Then using Taylor's theorem,

$$u^{(r)}[(i-1+\sigma)h] = \sum_{q=r}^{2m+1} \frac{u^{(q)}[(i-1+\tau)h]}{(q-r)!}(\sigma-\tau)^{q-r}h^{q-r} + \mathcal{O}(h^{2m+2-r}). \tag{4.4}$$

Noting that $t - t_{ij} = (\tau - \tau_j)h$, we have

$$\prod_{j=1}^{r}(t - t_{ij}) = \omega_r(\tau)h^r. \tag{4.5}$$

Now the result follows immediately by combining (4.2)–(4.5). $\square$

In the next theorem, we provide the asymptotic expansion for the error $(\mathcal{K}_n - \mathcal{K})u$.

Theorem 4.2. *Assume that* $f \in \mathcal{C}^{2m+2}[0,1]$. *Then*

$$(\mathcal{K}_n u)(s) = (\mathcal{K}u)(s) + \sum_{p=r}^{m} h^{2p}(T_{2p}u)(s) + \mathcal{O}(h^{2m+2}), \tag{4.6}$$

where

$$(T_p u)(s) := \bar{c}_{pp}(\mathcal{K}u)^{(p)}(s) + \sum_{q=r}^{p-1} \bar{c}_{pq}\left[\left(\frac{\partial}{\partial t}\right)^{p-q-1}\frac{\partial^q \kappa}{\partial t^q}(s,t)u(t)\right]_{t=0}^{1},$$

and the constants $\bar{c}_{pq}$ *are defined by* (3.3).

Proof. For fixed $s \in [0,1]$, we denote $\kappa_s(t) := \kappa(s,t)$. Then (4.1) implies

$$\begin{aligned}
[(\mathcal{K} - \mathcal{K}_n)u](s) &= \sum_{i=1}^{n}\int_{s_{i-1}}^{s_i} [(\mathcal{I} - \pi_n)\kappa_s(t)]u(t)dt,\\
&= \sum_{q=r}^{2m+1} h^q \sum_{i=1}^{n}\int_{s_{i-1}}^{s_i} \Phi_q(\tau)\kappa_s^{(q)}(t)u(t)dt + \mathcal{O}(h^{2m+2}),\\
&= \sum_{q=r}^{2m+1} h^q \int_0^1 \Phi_q(\tau)\left\{h\sum_{i=1}^{n}(\kappa_s^{(q)}u)[(i-1+\tau)h]\right\}\omega_r(\tau)d\tau\\
&\quad + \mathcal{O}(h^{2m+2}).
\end{aligned} \tag{4.7}$$

According to the summation formula (2.18) for the function $f = \kappa_s^{(j)} * u$, we have

$$h\sum_{i=1}^{n}(\kappa_s^{(q)}u)[(i-1+\tau)h] = \int_0^1 \frac{\partial^q\kappa}{\partial t^q}(s,t)u(t)dt + \sum_{p=1}^{2m+1-q}\frac{B_p(\tau)}{p!}\left[\left(\frac{\partial}{\partial t}\right)^{p-1}\frac{\partial^q\kappa}{\partial t^q}(s,t)u(t)\right]_{t=0}^{1} h^p + \mathcal{O}(h^{2m+2-p}).$$

Substituting the above equality into (4.7), and noting $B_0(\tau) = 1$, one finds, after rearranging,

$$[(\mathcal{K}_n - \mathcal{K})u](s) = \sum_{p=r}^{2m+1}(T_pu)(s)h^p + \mathcal{O}(h^{2m+2}).$$

Thus to complete the proof, we need only verify that $T_pu = 0$ if p is odd or if $p \leq 2r-1$. Note first that Φ_q is a polynomial of degree $q-r$ (see [12, p. 128]). Since the polynomial ω_r is orthogonal in $\mathcal{L}^2[0,1]$ to every polynomial of degree $\leq r-1$, we conclude that $T_pu = 0$ for all $p \leq 2r-1$.
On the other hand, for all $1 \leq p \leq r$, we have $\tau_{r-p+1} = 1-\tau_p$. Then using the fact that

$$[\tau_1,\dots,\tau_r,1-\tau](\bullet-\sigma)_+^{r-1} = [\tau_1,\dots,\tau_r,\tau][\bullet-(1-\sigma)]_+^{r-1}$$

and $\omega_r(1-\tau) = (-1)^r\omega_r(\tau)$, it holds

$$\Phi_q(1-\tau) = (-1)^{q-r}\Phi_q(\tau).$$

Furthermore, the *Bernoulli* polynomials satisfy

$$B_j(1-\tau) = (-1)^jB_j(\tau) \quad j \geq 0,$$

which means

$$\int_0^1 \Phi_q(\tau)B_{p-q}(\tau)\omega_r(\tau)d\tau = (-1)^j\int_0^1 \Phi_q(\tau)B_{p-q}(\tau)\omega_r(\tau)d\tau.$$

Therefore $\bar{c}_{pq} = 0$ for $r \leq q \leq p$ when p is odd. □

Let $u_{n,0} := u_n^D$ and let $u_{n,\ell}$ be the sequence defined by (3.7). The following result shows that the solution $u_{n,\ell}$ has error expansions in even powers of h, beginning with a term in $h^{2r+2\ell}$.

Theorem 4.3. *Let the right-hand side f of (1.1) be in $\mathcal{C}^{2m+2}[0,1]$. Then*

$$u_{n,\ell} = u + \sum_{p=\ell}^{m} h^{2r+2p}C_{\ell,p} + \mathcal{O}(h^{2m+2}), \quad \ell = 0,1,\dots,m-r, \tag{4.8}$$

where the functions $C_{\ell,p}$ are independent of h.

Proof. Since $\|\mathcal{K}-\mathcal{K}_n\| \to 0$ as $n \to +\infty$, for all large n, $(\mathcal{I}-\mathcal{K}_n)$ is invertible and uniformly bounded. Following the same steps as in Proposition 4.1, it is shown that there exist bounded linear operators $\bar{\mathcal{S}}_p : \mathcal{C}^{2m+2}[0,1] \to \mathcal{C}^{2m+2}[0,1],\ r \le p \le m$, such that

$$(\mathcal{I} - \mathcal{K}_n)^{-1} = (\mathcal{I} - \mathcal{K})^{-1} + \sum_{p=r}^{m} h^{2p} \bar{\mathcal{S}}_p + \mathcal{O}(h^{2m+2}). \tag{4.9}$$

Thus from the identity

$$u - u_n^D = (\mathcal{I} - \mathcal{K})^{-1} f - (\mathcal{I} - \mathcal{K}_n^D)^{-1} f$$

we obtain

$$u - u_n^D = \sum_{p=r}^{m} h^{2p} \bar{\mathcal{S}}_p(f) + \mathcal{O}(h^{2m+2}), \tag{4.10}$$

and consequently the error expansion (4.8) is proved. □

5. Discrete method

In practice, the integrals in the definitions of the orthogonal projection π_n and the operators $\mathcal{K}$ and $\mathcal{K}_n$ involved in (2.2), (2.6), and (2.16) are not computed exactly. It is necessary to replace them with a numerical quadrature formula, giving rise to discrete methods. In this section, we provide an asymptotic error expansion for the discrete iterated Kantorovich-collocation method, and the analysis can be extended to Kantorovich–Galerkin and degenerate kernel methods. We consider a basic quadrature formula defined by

$$\mathcal{Q}(f) := \sum_{j=1}^{\rho} w_j f(\sigma_j) \simeq \int_0^1 f(t)dt, \tag{5.1}$$

with nodes $\sigma_1, \sigma_2, \ldots, \sigma_\rho \in [0,1]$, and weights are such that

$$\sum_{j=1}^{\rho} w_j = 1.$$

For $1 \le i \le n$ and $1 \le j \le \rho$, let $s_{ij} := (i - 1 + \sigma_j)h$. Then (5.1) gives rise to the composite quadrature formula

$$\mathcal{Q}_n(f) := h \sum_{i=1}^{n} \sum_{j=1}^{\rho} w_j f(s_{ij}) \simeq \int_0^1 f(t)dt. \tag{5.2}$$

Thus the Nyström approximation of the integral operator $\mathcal{K}$ is defined as

$$(\mathcal{K}_n^D u)(s) := \mathcal{Q}_n(\kappa(s,.)u(.)) = h \sum_{i=1}^{n} \sum_{j=1}^{\rho} w_j \kappa(s, s_{ij}) u(s_{ij}). \tag{5.3}$$

Suppose that the quadrature formula (5.1) is symmetric, that is,

$$\sigma_{\ell-j+1} = 1 - \sigma_j \quad \text{and} \quad w_{\ell-j+1} = w_j, \quad 1 \le j \le \rho,$$

and is exact for all polynomials of degree $\le 2r-1$, that is

$$\mathcal{Q}_n = \int_0^1 p(t)dt, \quad \forall p \in \prod\nolimits_{2r-1}.$$

The following result can be proved by using Proposition 4.1.

Theorem 5.1. *Let π_n be the interpolatory projection defined by (2.3). Assume that $u \in \mathcal{C}^{2m+2}[0,1]$. Then*

$$(\mathcal{K}_n^D \pi_n u)(s) = (\mathcal{K}_n^D u)(s) + \sum_{p=r}^{2m+1} h^{2p}(R_{2p}u)(s) + \mathcal{O}(h^{2m+2}), \tag{5.4}$$

where R_p is defined by (3.2) with

$$\bar{c}_{pq} := -\frac{\mathcal{Q}(\Phi_q B_{p-q}\omega_r)}{(p-q)!}. \tag{5.5}$$

Let $\widetilde{v}_n$ be the iterated discrete solution satisfying $\widetilde{v}_n = \widetilde{y}_n + f$, where

$$(\mathcal{I} - \mathcal{K}_n^D \pi_n)\widetilde{y}_n = \mathcal{K}_n^D f. \tag{5.6}$$

Let $u_{n,0} := v_n$ and let $u_{n,\ell}$ be given by (3.7). Then we have the following result.

Theorem 5.2. *Assume that $f \in \mathcal{C}[0,1]$. Then the solution $u_{n,\ell}$ has error expansions in even powers of h, beginning with a term in $h^{2r+2\ell}$.*

Proof. According to [10], we have

$$(\mathcal{K}_n^D u)(s) = (\mathcal{K}u)(s) + \sum_{p=r}^{2m+1} (T_{2p}u)(s)h^{2p} + \mathcal{O}(h^{2m+2}), \tag{5.7}$$

where

$$(T_p u)(s) := \frac{\mathcal{Q}(B_p)}{(p)!}\left[\left(\frac{\partial}{\partial t}\right)^{p-1} \kappa(s,t)u(t)\right]_{t=0}^{1}$$

provided $u \in \mathcal{C}^{2m+2}[0,1]$. Therefore, writing

$$\mathcal{K}_n^D \pi_n - \mathcal{K} = \mathcal{K}_n^D \pi_n - \mathcal{K}_n^D + \mathcal{K}_n^D - \mathcal{K}$$

it follows from Theorem 5.1 and (5.7) that

$$\mathcal{K}_n^D \pi_n - \mathcal{K} = \sum_{p=r}^{2m+1} U_{2p}h^{2p} + \mathcal{O}(h^{2m+2}).$$

Using the same arguments as previously, we can show that

$$(\mathcal{I} - \mathcal{K}_n^D \pi_n)^{-1} - (\mathcal{I} - \mathcal{K})^{-1} = \sum_{p=r}^{2m+1} V_{2p}h^{2p} + \mathcal{O}(h^{2m+2}), \tag{5.8}$$

for some bounded linear operators $U_{2p}, V_{2p},\ p = r, \ldots, 2m+1$. Then

$$\begin{aligned}\widetilde{v}_n - u &= \widetilde{y}_n - z = (\mathcal{I} - \mathcal{K}_n^D \pi_n)^{-1}\mathcal{K}_n^D f - (\mathcal{I} - \mathcal{K})^{-1}\mathcal{K}f, \\ &= (\mathcal{I} - \mathcal{K}_n^D \pi_n)^{-1}\mathcal{K}f - (\mathcal{I} - \mathcal{K})^{-1}\mathcal{K}f + (\mathcal{I} - \mathcal{K}_n^D \pi_n)^{-1}(\mathcal{K}_n^D f - \mathcal{K}f).\end{aligned}$$

Using (5.7) and (5.8), it holds

$$\begin{aligned}(\mathcal{I}-\mathcal{K}_n^D\pi_n)^{-1}(\mathcal{K}_n^D f-\mathcal{K}f) =&(\mathcal{I}-\mathcal{K})^{-1}(\mathcal{K}_n^D f-\mathcal{K}f)\\ &+\sum_{p=r}^{2m+1} V_{2p}(\mathcal{K}_n^D f-\mathcal{K}f)h^{2p}+\mathcal{O}(h^{2m+2}),\\ =&\sum_{p=r}^{2m+1}\left\{(\mathcal{I}-\mathcal{K})^{-1}T_{2p}+V_{2p}\sum_{q=r}^{2m+1}U_{2q}h^{2q}\right\}h^{2p}\\ &+\mathcal{O}(h^{2m+2}).\end{aligned}$$

Therefore

$$\widetilde{v}_n-u=\sum_{p=r}^{2m+1}\left\{V_{2p}\mathcal{K}f+(\mathcal{I}-\mathcal{K})^{-1}T_{2p}+V_{2p}\left(\sum_{q=r}^{2m+1}U_{2q}h^{2q}\right)\right\}h^{2p}+\mathcal{O}(h^{2m+2}). \tag{5.9}$$

□

Remark 5.3. Note that when $\rho = r$ and the quadrature points used in the quadrature formula (5.1) are the r Gauss points in $[0,1]$, that is, $\sigma_i = \tau_i$, the operator $\mathcal{K}_n^D\pi_n$ coincides with the operator $\mathcal{K}_n^D$. In such a case, the discrete iterated Kantorovich-collocation method is reduced to the Nyström method applied to the regularized equation (2.9).

6. Numerical results

In this section, numerical examples are given to illustrate the theory established in the previous sections. Note that all required integrals were calculated by using the 2-points Gauss quadrature formula.

Example 6.1. We consider the following Fredholm integral equation with a degenerate kernel

$$u(s)-\int_0^1 e^{s+t}u(t)dt=f(s),\quad s\in[0,1],$$

where $f(s)$ is selected so that $u(s)=s$.

Let $\mathbb{X}_n$ be the space of piecewise constant functions ($r=1$) with respect to the uniform partition of $[0,1]$

$$0=\frac{1}{n}<\frac{2}{n}<\cdots<\frac{n}{n}=1.$$

The projection π_n is chosen to be either the interpolatory projection at the $nr=n$ midpoints

$$t_i^{(n)}:=\frac{2i-1}{2n},\qquad i=1,\dots,n,$$

or the restriction to $\mathcal{L}^\infty[0,1]$ of the orthogonal projection from $\mathcal{L}^2[0,1]$ to $\mathbb{X}_n$. Let

$$E_\ell^n:=\|u-u_{n,\ell}\|_\infty,\quad l=0,1,2,3,$$

TABLE 1. Kantorovich–Galerkin method.

n	E_0^n	α_0	E_1^n	α_1	E_2^n	α_2	E_3^n	α_3
2	8.29×10^{-2}	–	1.11×10^{-4}	–	4.65×10^{-8}	–	4.26×10^{-12}	–
4	2.06×10^{-2}	2.01	6.88×10^{-6}	4.01	7.22×10^{-10}	6.01	1.65×10^{-14}	8.03
8	5.15×10^{-3}	2.00	4.29×10^{-7}	4.00	1.13×10^{-11}	6.00	6.44×10^{-17}	8.01
16	1.29×10^{-3}	2.00	2.68×10^{-8}	4.00	1.76×10^{-13}	6.00		
32	3.22×10^{-4}	2.00	1.68×10^{-9}	4.00				
64	8.05×10^{-5}	2.00						

TABLE 2. Kantorovich-collocation method.

n	E_0^n	α_0	E_1^n	α_1	E_2^n	α_2	E_3^n	α_3
2	1.26×10^{-1}	–	6.15×10^{-4}	–	7.17×10^{-7}	–	2.09×10^{-10}	–
4	3.11×10^{-2}	2.02	3.78×10^{-5}	4.03	1.10×10^{-8}	6.03	8.03×10^{-13}	8.03
8	7.74×10^{-3}	2.01	2.35×10^{-6}	4.01	1.71×10^{-10}	6.01	3.12×10^{-15}	8.01
16	1.93×10^{-3}	2.00	1.47×10^{-7}	4.00	2.66×10^{-12}	6.01		
32	4.83×10^{-4}	2.00	9.17×10^{-9}	4.00				
64	1.21×10^{-4}	2.00						

TABLE 3. Degenerate kernel method.

n	E_0^n	α_0	E_1^n	α_1	E_2^n	α_2	E_3^n	α_3
2	5.91×10^{-2}	–	4.73×10^{-4}	–	3.86×10^{-7}	–	1.37×10^{-10}	–
4	1.44×10^{-2}	2.03	2.92×10^{-5}	4.02	5.89×10^{-9}	6.03	5.25×10^{-13}	8.02
8	3.58×10^{-3}	2.01	1.82×10^{-6}	4.00	9.16×10^{-11}	6.01	2.04×10^{-15}	8.01
16	8.95×10^{-4}	2.00	1.14×10^{-7}	4.00	1.43×10^{-12}	6.00		
32	2.23×10^{-4}	2.00	7.11×10^{-9}	4.00				
64	5.59×10^{-5}	2.00						

where $u_{n,\ell}$ is defined by (3.7) and $u_{n,0}$ is the iterated Kantorovich solution $\widetilde{u}_n$ or the degenerate kernel solution u_n^D. The numerical orders of convergence are computed as

$$\alpha_\ell = \frac{\log\left(\frac{E_\ell^n}{E_\ell^{2n}}\right)}{\log(2)}.$$

The expected orders of convergence are

$$\alpha_0 = 2, \quad \alpha_1 = 4, \quad \alpha_2 = 6, \quad \alpha_3 = 8.$$

The results are given in Tables 1–3.

It can be seen from the above tables that the computed orders of convergence match well with the theoretical ones.

Example 6.2. Consider

$$u(s) - \int_0^1 \sinh(\sqrt{2}s - 1)\cosh(t-2)u(t)dt = f(s), \quad s \in [0,1], \tag{6.1}$$

where $f \in \mathcal{C}[0,1]$ is so chosen that $u(s) = \sqrt{s}$ is the solution to (6.1). The results are given in Tables 4–6.

TABLE 4. Kantorovich–Galerkin method.

n	E_0^n	α_0	E_1^n	α_1	E_2^n	α_2	E_3^n	α_3
2	1.16×10^{-2}	–	4.26×10^{-5}	–	9.59×10^{-8}	–	3.59×10^{-11}	–
4	2.92×10^{-3}	1.98	2.75×10^{-6}	3.95	1.53×10^{-9}	5.97	1.44×10^{-13}	8.01
8	7.33×10^{-4}	1.99	1.73×10^{-7}	3.99	2.41×10^{-11}	5.99	5.66×10^{-16}	8.00
16	1.84×10^{-4}	2.00	1.09×10^{-8}	4.00	3.77×10^{-13}	6.00		
32	4.59×10^{-5}	2.00	6.79×10^{-10}	4.00				
64	1.15×10^{-5}	2.00						

TABLE 5. Kantorovich-collocation method.

n	E_0^n	α_0	E_1^n	α_1	E_2^n	α_2	E_3^n	α_3
2	1.71×10^{-2}	–	7.76×10^{-5}	–	8.28×10^{-8}	–	1.92×10^{-11}	–
4	4.33×10^{-3}	1.98	4.93×10^{-6}	3.98	1.31×10^{-9}	5.98	7.59×10^{-14}	7.98
8	1.09×10^{-3}	1.99	3.09×10^{-7}	3.99	2.06×10^{-11}	5.99	2.98×10^{-16}	8.00
16	2.72×10^{-4}	2.00	1.94×10^{-8}	4.00	3.22×10^{-13}	6.00		
32	6.80×10^{-5}	2.00	1.21×10^{-9}	4.00				
64	1.70×10^{-5}	2.00						

TABLE 6. Degenerate kernel method.

n	E_0^n	α_0	E_1^n	α_1	E_2^n	α_2	E_3^n	α_3
2	2.96×10^{-3}	–	1.04×10^{-4}	–	1.47×10^{-7}	–	6.48×10^{-11}	–
4	6.63×10^{-4}	2.16	6.67×10^{-6}	3.96	2.36×10^{-9}	5.96	2.81×10^{-13}	7.84
8	1.60×10^{-4}	2.04	4.19×10^{-7}	4.00	3.72×10^{-11}	5.98	4.15×10^{-16}	9.40
16	4.00×10^{-5}	2.01	2.62×10^{-8}	4.00	5.82×10^{-13}	6.00		
32	9.95×10^{-6}	2.00	1.64×10^{-9}	4.00				
64	2.48×10^{-6}	2.00						

Tables 4 and 5 illustrate that high accuracy is obtained by the extrapolated Kantorovich method even when the solution and the right-hand side are only continuous. However, the Richardson extrapolation in the degenerate kernel method improves the order of convergence slightly from 2 to 2.5. The full superconvergence order $2r+2\ell$ in the case when the right-hand side is not smooth can be obtained by applying the degenerate kernel method to the regularized equation (2.13) instead of (1.1).

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