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# HÖLDER ESTIMATES AND ASYMPTOTIC BEHAVIOR FOR DEGENERATE ELLIPTIC EQUATIONS IN THE HALF SPACE

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Abstract. In this article we investigate the asymptotic behavior at infinity of viscosity solutions to degenerate elliptic equations. We obtain Hölder estimates, up to the flat boundary, by using the rescaling method. Also as a byproduct we obtain a Liouville type result on Baouendi-Grushin type operators.

## 1. Introduction

In this article we study the asymptotic behavior at infinity of viscosity solutions to the degenerate non-divergence elliptic equation

$$Lu = x_n^{2\alpha}\sum_{i,j=1}^{n-1} a_{ij}(x)D_{ij}u(x) + 2x_n^{\alpha}\sum_{i=1}^{n-1} a_{in}(x)D_{in}u(x) + D_{nn}u(x) = 0 \tag{1.1}$$

in $\mathbb{R}^n_+\backslash\overline{B}_1^+$, where $n\geq 2$, $\alpha>0$, $\mathbb{R}^n_+=\mathbb{R}^n\cap\{x_n>0\}$, $B_1^+=\mathbb{R}^n_+\cap\{|x|<1\}$.

To ensure the ellipticity of operator $L$, we assume that $a_{ij}(x), a_{in}(x)\in C(\mathbb{R}^n_+)$ $(i,j=1,\dots,n-1)$ and that there exist constants $0<\lambda\leq\Lambda<\infty$ such that for each $\xi\in\mathbb{R}^{n-1}$,

$$\lambda|\xi|^2\leq \xi^T\sum_{i,j=1}^{n-1} a_{ij}(x)\xi\leq\Lambda|\xi|^2,\quad \forall x\in\mathbb{R}^n_+, \tag{1.2}$$

and for some $0<\delta<1$,

$$1-\lambda^{-1}\sum_{i=1}^{n-1}\|a_{in}\|^2_{L^\infty(\mathbb{R}^n_+)}>\delta. \tag{1.3}$$

In this article, solutions always indicate viscosity solutions (see [3] for definition). For $\alpha=0$, by (1.2) and (1.3), $L$ is uniformly elliptic. The asymptotic behavior at infinity was considered in [7]. Note that the crucial key to obtain the asymptotic behavior is the boundary Hölder estimates, which is classical for uniformly elliptic equations (see [3, 5]).

For $\alpha>0$, $a_{ij}\equiv 1$ and $a_{in}\equiv 0$ $(i,\, j\leq n-1)$, $L$ is a Baouendi-Grushin type operator,

$$\mathfrak{L}u := x_n^{2\alpha}\sum_{i=1}^{n-1} D_{ii}u(x) + D_{nn}u(x) = 0, \tag{1.4}$$

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which was introduced in [1, 6]. There have been extensive works on the studies of the Baouendi-Grushin type operators (see [2, 4, 8, 10] and references therein). For $\alpha > 0$ and $a_{ij}$ satisfies (1.2), Le and Savin [9] obtained the boundary Schauder estimates for solutions of the degenerate elliptic equation

$$x_n^\alpha \sum_{i,j=1}^{n-1} a_{ij}(x) D_{ij} u(x) + D_{nn} u(x) = x_n^\alpha f(x) \quad \text{in } B_1^+.$$

In this article, we study the asymptotic behavior at infinity of solutions of (1.1) with the coefficients satisfying (1.2) and (1.3).

By rescaling method similar to the one in [9], we establish the Hölder estimates up to the flat boundary of solutions of (1.1).

**Theorem 1.1.** *Let $u \in C(\overline{B}_1^+)$ be a solution of*

$$\begin{gathered} Lu(x) = 0 \quad \text{in } B_1^+ \\ u(x) = 0 \quad \text{on } B_1 \cap \{x_n = 0\}, \end{gathered} \tag{1.5}$$

*where $L$ is given by* (1.1) *with the coefficients satisfying* (1.2) *and* (1.3)*. Then $u \in C^{\frac{1}{1+\alpha}}(\overline{B}_{1/2}^+)$.*

Theorem 1.1, Harnack inequalities, and the comparison principle yield our main theorem as follows.

**Theorem 1.2.** *Let $u \in C^1(\overline{\mathbb{R}}_+^n \backslash B_1^+)$ be a solution of*

$$\begin{gathered} Lu = 0 \quad \text{in } \mathbb{R}_+^n \backslash \overline{B}_1^+, \\ u = 0 \quad \text{on } \{x_n = 0, |x| \geq 1\}, \end{gathered} \tag{1.6}$$

*where $L$ is given by* (1.1) *with the coefficients satisfying* (1.2) *and* (1.3)*; and for some $s > 0$,*

$$|a_{ij}(x) - \delta_{ij}| + |a_{in}(x)| \leq \big(|x'| + x_n^{1+\alpha}\big)^{-s} \quad \text{in } \mathbb{R}_+^n \backslash \overline{B}_1^+, \quad i,j < n. \tag{1.7}$$

*Assume that $|u| \leq 1$ on $\partial B_1 \cap \{x_n > 0\}$, $|Du| \leq 1$ in $\overline{\mathbb{R}}_+^n \backslash B_1^+$ and $|Du| \to 0$ as $|x| \to \infty$. Then*

$$|u(x)| \leq \frac{C x_n}{\big(|x'|^2 + \frac{1}{(1+\alpha)^2} x_n^{2+2\alpha}\big)^{\frac{n-1}{2} + \frac{1}{2(1+\alpha)}}} \quad \text{in } \overline{\mathbb{R}}_+^n \backslash B_R^+, \tag{1.8}$$

*where $C > 0$ and $R \geq 1$ depend only on $\alpha$, $\delta$, $s$ and $n$.*

**Remark 1.3.** When $\alpha = 0$, Theorem 1.2 still holds (see [7]).

By Theorem 1.2 and the comparison principle, we have the following Liouville type theorem.

**Theorem 1.4.** *Let $u \in C^1(\mathbb{R}_+^n)$ be a solution of*

$$\begin{gathered} \mathfrak{L} u = 0 \quad \text{in } \mathbb{R}_+^n, \\ u = 0 \quad \text{on } \{x_n = 0\}, \end{gathered} \tag{1.9}$$

*where $\mathfrak{L}$ is as in* (1.4)*. If $|Du| \to 0$ as $|x| \to \infty$. Then $u(x)$ must be zero.*

The rescaling method is a classical one to show the boundary Schauder/Hölder estimates in (degenerate) linear elliptic equations (see [9]). Similarly, one can also show the boundary Hölder estimates

$$x_n^{2\alpha}\sum_{i,j=1}^{n-1} a_{ij}(x)D_{ij}u(x) + 2x_n^{\alpha}\sum_{i=1}^{n-1} a_{in}(x)D_{in}u(x) + D_{nn}u(x) = x_n^{2\alpha}f(x).$$

The asymptotic result (Theorem 1.2) may push forward the study on asymptotic behavior of solutions of the following degenerate Monge-Ampère equation

$$\det D^2u = f(x)x_n^{2\alpha} \quad \text{on } \{x_n > 0\},$$

where $\alpha > 0$, and $f(x)$ is positive and continuous.

This article is organized as follows. In *Section 2*, we show the boundary Hölder estimates, which can be approached by the interior Hölder estimates via rescaling. In *Section 3*, a supersolution is constructed according to the fundamental solution of one Baouendi-Grushin type operator in the half space. Then it together with the Hölder estimates up to the flat boundary implies that Theorem 1.2 holds.

## 2. Proof of Theorem 1.1

First, we show that (1.2) and (1.3) ensure the ellipticity of $L$.

**Lemma 2.1.** *Let the coefficients of $L$ in* (1.1) *satisfy* (1.2) *and* (1.3)*. Then $L$ is elliptic in $\overline{B}_1^+$. Furthermore, for each fixed $\varepsilon_0 > 0$, $L$ is uniformly elliptic in $\overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$.*

The proof of the above lemma is standard, and is shown in the Appendix. To show the Hölder estimates up to the flat boundary, we need to give some notion (see [9]).

**Definition 2.2.** *We define a distance $d_\alpha$ between point $y$ and point $z$ by*

$$d_\alpha(y,z) := |y' - z'| + \left|y_n^{1+\alpha} - z_n^{1+\alpha}\right|.$$

Observe that the relation between $d_\alpha$ and the Euclidean distance,

$$c|y-z|^{1+\alpha} \leq d_\alpha(y,z) \leq C|y-z|, \tag{2.1}$$

$$d_\alpha(y,z) \sim |y-z| \quad \text{if } y,z \in \overline{B}_1^+ \cap \big\{x_n \geq \frac{1}{8}\big\}. \tag{2.2}$$

For each $h > 0$ and each $\widetilde{x} \in \mathbb{R}^n$, we denote

$$E_h(\widetilde{x}) = \big\{x \in \mathbb{R}^n : |x' - \widetilde{x}'|^2 + |x_n - \widetilde{x}_n|^{2(1+\alpha)} < h\big\}, \tag{2.3}$$

and $F_h = \operatorname{diag}\Big(h^{\frac12}, h^{\frac12}, \dots, h^{\frac12}, h^{\frac{1}{2(1+\alpha)}}\Big)$. For simplicity, we denote

$$E_h = E_h(0) = \big\{x \in \mathbb{R}^n : |x'|^2 + |x_n|^{2(1+\alpha)} < h\big\}; \quad E_h^+ = E_h \cap \{x_n > 0\}.$$

A simple calculation gives

$$F_hE_{\alpha'}\Big(\frac12 e_n\Big) = E_{\alpha' h}\Big(\frac12 h^{\frac{1}{2(1+\alpha)}} e_n\Big), \quad F_hE_1^+ = E_h^+, \tag{2.4}$$

where $e_n = (0,\dots,0,1)$, $\alpha' = 4^{-2(1+\alpha)}$.

Note that (1.1) and $d_\alpha$ keep their forms under the transformation $x \to F_hx$. Precisely, let

$$\widetilde{u}(x) = u(F_hx), \quad x \in E_1\,. \tag{2.5}$$

Then it solves

$$\widetilde{L}\widetilde{u} = x_n^{2\alpha}\sum_{i,j=1}^{n-1}\widetilde{a}_{ij}(x)D_{ij}\widetilde{u}(x) + x_n^{\alpha}\sum_{i=1}^{n-1}2\widetilde{a}_{in}(x)D_{in}\widetilde{u}(x) + D_{nn}\widetilde{u}(x) = 0 \tag{2.6}$$

with

$$\widetilde{a}_{ij}(x) = a_{ij}(F_h x), \quad \widetilde{a}_{in}(x) = a_{in}(F_h x), \quad i,j \le n-1, \tag{2.7}$$

$$d_\alpha(y,z) = h^{-1/2} d_\alpha(F_h y, F_h z). \tag{2.8}$$

If function $w$ is $\gamma$-Hölder continuous in $\Omega \subset \overline{B}_1^+$ with respect to $d_\alpha$, we write $w \in C_\alpha^\gamma(\overline{\Omega})$ and define

$$[w]_{C_\alpha^\gamma(\overline{\Omega})} = \sup_{y,z\in\overline{\Omega}, y\neq z}\frac{|w(y)-w(z)|}{(d_\alpha(y,z))^\gamma}, \quad \|w\|_{C_\alpha^\gamma(\overline{\Omega})} = \|w\|_{L^\infty(\overline{\Omega})} + [w]_{C_\alpha^\gamma(\overline{\Omega})}.$$

*Prrof of Theorem 1.1.* We divided this proof into two cases.

**Case 1.** $u \in C^{\frac{1}{1+\alpha}}\big(\overline{B}_{1/2}^+ \cap \{x_n > \frac{1}{8}\}\big)$. By Lemma 2.1, $L$ is uniformly elliptic in $\overline{B}_{1/2}^+ \cap \{x_n > 1/8\}$. Applying the classical Hölder estimates to $u$, there exists $C > 0$, depending only on $\lambda$, $\Lambda$, $\alpha$, $\delta$, $n$ and $\|u\|_{L^\infty}$, such that

$$[u]_{C^{\frac{1}{1+\alpha}}\left(\overline{E_{\alpha'}(\frac{1}{2}e_n)}\right)} \le C\|u\|_{L^\infty} \le C.$$

**Case 2.** $u \in C^{\frac{1}{1+\alpha}}\big(\overline{B}_{1/2}^+ \cap \{x_n \le 1/8\}\big)$. We show this case by four steps.

**Step 1.** There exists $C > 0$, depending only on $\lambda$, $\Lambda$, $\alpha$, $\delta$, $n$ and $\|u\|_{L^\infty}$, such that

$$|u(x)| \le Cx_n \quad \text{in } B_{\frac{3}{4}}^+. \tag{2.9}$$

We only need to show that for each $x_0 \in \{x_n = 0, |x'| < \frac{3}{4}\}$,

$$|u(x_0, x_n)| \le Cx_n.$$

Let

$$\overline{u}(x) = Cx_n + B|x' - x_0'|^2 - \frac{C}{2}x_n^{2+\alpha}$$

with $B = 16\|u\|_{L^\infty}$. One can choose $C > 0$, depending only on $\Lambda$, $\alpha$, $n$, and $\|u\|_{L^\infty}$, such that

$$L\overline{u} \le 0 \text{ in } B_1^+, \quad \overline{u} \ge \|u\|_{L^\infty} \ge u \text{ on } \partial B_1^+, \tag{2.10}$$

by taking

$$2(n-1)\Lambda B - (2+\alpha)(1+\alpha)C/2 \le 0, \quad \frac{C}{2}x_n + B|x' - x_0'|^2 > \|u\|_{L^\infty} \quad \text{on } \partial B_1^+.$$

Therefore, (2.10) and the comparison principe (see [11, Theorem 6]) yield (2.9).

**Step 2.** For any fixed $h \in (0,1]$,

$$[u]_{C_\alpha^{\frac{1}{1+\alpha}}\left(\overline{E_{\alpha' h}\left(\frac{1}{2}h^{\frac{1}{2(1+\alpha)}}e_n\right)}\right)} \le C. \tag{2.11}$$

In fact, let $\widetilde{u}$ be as in (2.5), and then $\widetilde{u}$ solves (2.6) in $B_1^+$. By (2.5) and (2.9),

$$\widetilde{u} \le Ch^{\frac{1}{2(1+\alpha)}} \quad \text{in } B_1^+. \tag{2.12}$$

Similar to ***Case 1***, applying the Hölder estimates to $\widetilde{u}$ in $E_{\frac{1}{4}}\left(\frac{1}{2}e_n\right)$, we have

$$[\widetilde{u}]_{C^{\frac{1}{1+\alpha}}\left(\overline{E_{\alpha'}(\frac{1}{2}e_n)}\right)} \le C\|\widetilde{u}\|_{L^\infty(B_1^+)} \le Ch^{\frac{1}{2(1+\alpha)}}.$$

By (2.2), we see

$$[\widetilde{u}]_{C_\alpha^{\frac{1}{1+\alpha}}\left(\overline{E_{\alpha'}\left(\frac{1}{2}e_n\right)}\right)} \le Ch^{\frac{1}{2(1+\alpha)}}.$$

This together with (2.4), (2.5), and (2.8) yields (2.11), since

$$\frac{|\widetilde{u}(y)-\widetilde{u}(z)|}{(d_\alpha(y,z))^{\frac{1}{1+\alpha}}} = \frac{|u(F_h y)-u(F_h z)|}{h^{-\frac{1}{2(1+\alpha)}}(d_\alpha(F_h y, F_h z))^{\frac{1}{1+\alpha}}}.$$

**Step 3.** We prove that $u \in C_\alpha^{\frac{1}{1+\alpha}}$ at 0 along $e_n$ direction, that is,

$$\sup_{0<h<1} \frac{\left|u\left(\frac{1}{2}h^{\frac{1}{2(1+\alpha)}}e_n\right)-u(0)\right|}{\left(\left(\frac{1}{2}h^{\frac{1}{2(1+\alpha)}}\right)^{1+\alpha}\right)^{\frac{1}{1+\alpha}}} \le C$$

for some $C>0$ depending only on $\lambda$, $\Lambda$, $\alpha$ and $n$. It suffices to prove that

$$\left|u\Big(\frac{1}{2}h^{\frac{1}{2(1+\alpha)}}e_n\Big)-u(0)\right| \le Ch^{\frac{1}{2(1+\alpha)}},$$

where $C>0$ independents on $h$.

Indeed, Step 2 yields that for each $k=1,2,\dots,$

$$\left|u\Big(\frac{1}{2^k}h^{\frac{1}{2(1+\alpha)}}e_n\Big)-u\Big(\frac{1}{2^{k+1}}h^{\frac{1}{2(1+\alpha)}}e_n\Big)\right| \le C2^{-k-1}h^{\frac{1}{2(1+\alpha)}},$$

This implies that

$$\begin{aligned}
\frac{\left|u\left(\frac{1}{2}h^{\frac{1}{2(1+\alpha)}}e_n\right)-u(0)\right|}{h^{\frac{1}{2(1+\alpha)}}} &\le \sum_{k=1}^{\infty} \frac{\left|u\left(\frac{1}{2^k}h^{\frac{1}{2(1+\alpha)}}e_n\right)-u\left(\frac{1}{2^{k+1}}h^{\frac{1}{2(1+\alpha)}}e_n\right)\right|}{h^{\frac{1}{2(1+\alpha)}}} \\
&\le \sum_{k=1}^{\infty} C2^{-k-1} \le C.
\end{aligned}$$

Therefore, $u \in C_\alpha^{\frac{1}{1+\alpha}}$ at 0 along $e_n$ direction.

**Step 4.** We show Case 2. Similar to Step 3, we have that $u \in C_\alpha^{\frac{1}{1+\alpha}}$ at any $x \in B_{1/2}^+ \cap \{x_n = 0\}$ along $e_n$ direction.

Let $y,z \in \overline{B}_{1/2}^+ \cap \{x_n \le \frac{1}{8}\}$ and denote by $y_n$, $z_n$ the $n^{th}$ component of $y$ and $z$, respectively. If $z \in E_{2^{-2(1+\alpha)}y_n^{2(1+\alpha)}}(y_n)$ or $y \in E_{2^{-2(1+\alpha)}z_n^{2(1+\alpha)}}(z_n)$, by (2.11), we are done. Otherwise, $z \notin E_{2^{-2(1+\alpha)}y_n^{2(1+\alpha)}}(y_n)$ and $y \notin E_{2^{-2(1+\alpha)}z_n^{2(1+\alpha)}}(z_n)$, which yields

$$|y-z|^2 \ge \max\left\{2^{-2(1+\alpha)}z_n^{2(1+\alpha)}, 2^{-2(1+\alpha)}y_n^{2(1+\alpha)}\right\}. \tag{2.13}$$

By Step 3 and the boundary value condition, we obtain

$$\begin{aligned}
|u(y)-u(z)| &\le |u(y)-u(y',0)|+|u(y',0)-u(z',0)|+|u(z',0)-u(z)| \\
&\le C|y_n|+C|z_n| \le C|y-z|^{\frac{1}{1+\alpha}} \quad \text{(by (2.13))}.
\end{aligned} \tag{2.14}$$

It follows that $u \in C^{\frac{1}{1+\alpha}}\left(\overline{B}_{1/2}^+ \cap \{x_n \le \frac{1}{8}\}\right)$. Therefore, by Case 1 and Case 2, we complete the proof of Theorem 1.1. $\square$

## 3. Proof of Theorem 1.2

In this section we divide the proof of Theorem 1.2 into two steps as the following. In fact, Subsection 3.1 gives the convergence at infinity of the solutions in Theorem 1.2, and then Subsection 3.2 shows its asymptotic behavior at infinity. Recall that the symbols $F_h$, $E_h$ and $E_h^+$ are defined in Section 2.

### 3.1. **Convergence at infinity.**

In the subsection we apply Hölder estimates up to the flat boundary to show that the solution in Theorem 1.2 converges at infinity. Hereinafter, we say a constant is universal if it depends only on $\lambda$, $\Lambda$, $\alpha$, $\delta$ and $n$. The universal constant may change from line to line if necessary. A straightforward corollary of the boundary Hölder estimates is the following result.

**Corollary 3.1.** *Let $u \in C(\overline{E_{4R}^+ \backslash E_R^+})$ be a solution of*

$$\begin{aligned} Lu = 0 \quad &\text{in } E_{4R}^+ \backslash \overline{E}_R^+, \\ u \le 1 \quad &\text{on } \partial(E_{4R}^+ \backslash \overline{E}_R^+) \cap \{x_n > 0\}, \\ u \le \frac{1}{2} \quad &\text{on } \partial(E_{4R}^+ \backslash \overline{E}_R^+) \cap \{x_n = 0\}, \end{aligned} \tag{3.1}$$

*where $L$ is given by* (1.1) *with coefficients satisfying* (1.2) *and* (1.3) *in $E_{4R}^+ \backslash \overline{E}_R^+$ for some $R > 0$. Then there exists a universal constant $c_0 > 0$ such that*

$$u(x) \le 1 - c_0 \quad \text{on } \partial E_{2R} \cap \{x_n \ge 0\}.$$

*Proof.* We only need to set $u(x) = 1/2$ on $\partial(E_{4R}^+ \backslash \overline{E}_R^+) \cap \{x_n = 0\}$. Otherwise, one can consider a supersolution $v$ with $v(x) = \frac{1}{2}$ on $\partial(E_{4R}^+ \backslash \overline{E}_R^+) \cap \{x_n = 0\}$, and if it holds for $v$, by the comparison principle, so does for $u$. Let

$$\widehat{u}(x) = u(F_R x), \quad x \in E_4^+ \backslash \overline{E}_1^+.$$

By the definitions of $F_R$ and $E_R^+$ in Section 2, we have $F_R(E_4^+ \backslash \overline{E}_1^+) = E_{4R}^+ \backslash \overline{E}_R^+$. Then

$$\begin{aligned} \widetilde{L}\widehat{u} = 0 \quad &\text{in } E_4^+ \backslash \overline{E}_1^+, \\ \widehat{u} \le 1 \quad &\text{on } \partial(E_4^+ \backslash \overline{E}_1^+) \cap \{x_n > 0\}, \\ \widehat{u} = \frac{1}{2} \quad &\text{on } \partial(E_4^+ \backslash \overline{E}_1^+) \cap \{x_n = 0\}, \end{aligned} \tag{3.2}$$

where $\widetilde{L}$ is given by (2.6). Clearly, the coefficients of $\widetilde{L}$ also satisfy (1.2) and (1.3) in $E_4^+ \backslash \overline{E}_1^+$. Then by the third equality in (3.2) and Theorem 1.1, there exists a universal constant $0 < \tau \le 1$ such that

$$\widehat{u}(x) \le \frac{2}{3} \quad \text{on } \partial E_2 \cap \{0 \le x_n \le \tau\}. \tag{3.3}$$

By the comparison principle, we have $\widehat{u} \le 1$ in $E_4^+ \backslash \overline{E}_1^+$. Then $1 - \widehat{u}$ satisfies

$$\widetilde{L}(1 - \widehat{u}) = 0 \quad \text{in } E_4^+ \backslash \overline{E}_1^+.$$

By the interior Harnack inequality for $1 - \widehat{u}$, there exists a universal constant $C \ge 1$ such that

$$C \inf_{\partial E_2 \cap \{x_n \ge \tau\}} (1 - \widehat{u}) \ge \sup_{\partial E_2 \cap \{x_n \ge \tau\}} (1 - \widehat{u}) \ge \sup_{\partial E_2 \cap \{x_n = \tau\}} (1 - \widehat{u}) \ge \frac{1}{3}.$$

This implies

$$\widehat{u}(x) \le 1 - \frac{1}{3C} \quad \text{on } \partial E_2 \cap \{x_n \ge \tau\}. \tag{3.4}$$

This, the definition of $\widehat{u}$ ,and (3.3) implies the conclusion, via taking $c_0 = \frac{1}{3C}$. □

Applying Corollary 3.1, we have the following convergence result.

**Theorem 3.2.** *Let $u \in C(\overline{\mathbb{R}}^n_+ \backslash E_1^+)$ be a solution of $Lu = 0$ in $\mathbb{R}^n_+ \backslash \overline{E}_1^+$, where $L$ is given by* (1.1) *with the coefficients satisfying* (1.2) *and* (1.3) *in $\mathbb{R}^n_+ \backslash \overline{E}_1^+$. If*

- $|u| \le 1$ *on* $(\partial E_1 \cap \{x_n > 0\}) \cup \{x_n = 0, |x| \ge 1\}$,
- $u(x', 0) \to \beta$ *as* $|x'| \to \infty$
- $|Du(x)| \to 0$ *as* $|x| \to \infty$.

*Then $u(x) \to \beta$ as $|x| \to \infty$.*

*Proof.* The proof of this theorem is divided into two steps as follows.

**Step 1.** $|u| \le 1$ in $\overline{\mathbb{R}}^n_+ \backslash \overline{E}_1^+$. For any $\varepsilon > 0$, since $|Du| \to 0$ as $|x| \to \infty$, there exists $R_\varepsilon \ge 1$ such that

$$|Du| \le \varepsilon \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash Q^+_{R_\varepsilon}, \tag{3.5}$$

where $Q^+_{R_\varepsilon} := \{(x', x_n) : |x'| < R_\varepsilon, 0 < x_n < R_\varepsilon\}$ is a cylinder. By $|u| \le 1$ on $\{x_n = 0,\ |x| \ge 1\}$, (3.5) and Newton-Leibniz formula, we have

$$|u(x)| \le 1 + 2\varepsilon x_n \quad \text{on } \partial Q^+_{R_\varepsilon} \cap \{x_n > 0\}.$$

Since $|u| \le 1$ on $(\partial E_1 \cap \{x_n > 0\}) \cup \{x_n = 0, |x| \ge 1\}$, we obtain

$$|u(x)| \le 1 + 2\varepsilon x_n \quad \text{on } \partial(Q^+_{R_\varepsilon} \backslash \overline{E}_1^+).$$

Obviously, $1 + 2\varepsilon x_n$ solves (1.1) in $Q^+_{R_\varepsilon} \backslash \overline{E}_1^+$. Then by the comparison principle,

$$|u(x)| \le 1 + 2\varepsilon x_n \quad \text{in } Q^+_{R_\varepsilon} \backslash \overline{E}_1^+.$$

Letting $\varepsilon \to 0$, it completes the proof of step 1.

**Step 2.** $u(x) \to \beta$ as $|x| \to \infty$. We only need to set $\beta = 0$. Otherwise, we consider $\frac{u(x)-\beta}{1+|\beta|}$.

Now we argue by contradiction. If this step is not true, by Step 1, $u$ has finite superior limit $\overline{u} > 0$ or inferior limit $\underline{u} < 0$ at infinity. It suffices to assume that $\overline{u} > 0$.

By the definition of $\overline{u}$ and $u(x', 0) \to \beta$ as $|x'| \to \infty$, there exists large $R_1 \ge 1$ such that for all $R \ge R_1$,

$$u(x) \le \Big(1 + \frac{c_0}{2}\Big)\overline{u} \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash \overline{E}_R^+$$

and

$$u(x', 0) \le \frac{1}{2}\Big(1 + \frac{c_0}{2}\Big)\overline{u} \quad \text{if } |x'| \ge R,$$

where $c_0$ is given by Corollary 3.1. Then applying Corollary 3.1 to $\frac{u(x)}{(1+\frac{c_0}{2})\overline{u}}$ in $E_{4R}^+ \backslash \overline{E}_R^+$, we obtain for all $R \ge R_1$,

$$u(x) \le (1 - c_0)\big(1 + \frac{c_0}{2}\big)\overline{u} \le \big(1 - \frac{c_0}{2}\big)\overline{u} \quad \text{on } \partial E_{2R} \cap \{x_n \ge 0\}.$$

This implies

$$u(x) \le \big(1 - \frac{c_0}{2}\big)\overline{u} \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash \overline{E}_{2R_1}^+,$$

which reaches a contradiction. □

Theorem 3.2 implies the following corollary.

**Corollary 3.3.** *Let $u$ be as in Theorem 1.2. Then $u(x) \to 0$ as $|x| \to \infty$.*

The proofs is obvious and thus we omit it here.

3.2. **Asymptotic behavior at infinity.** In this subsection we obtain the asymptotic behavior at infinity of solutions in Theorem 1.2, through constructing a barrier function.

To get the barrier function, we first let

$$w(x', x_n) = \frac{x_n}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma}}, \tag{3.6}$$

where $\beta = \frac{1}{(1+\alpha)^2}$, $\gamma = \frac{n-1}{2} + \frac{1}{2(1+\alpha)}$. Simple calculations deduce that

$$\begin{aligned}
D_i w &= -\frac{2\gamma x_i x_n}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}}, \quad i < n;\\
D_n w &= \frac{1}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma}} - \frac{\gamma\beta(2+2\alpha)x_n^{2+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}};\\
D_{ij} w &= -\frac{2\gamma x_n \delta_{ij}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}} + \frac{4\gamma(\gamma+1)x_i x_j x_n}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+2}}, \quad i, j < n;\\
D_{in} w &= -\frac{2\gamma x_i}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}} + \frac{2\gamma\beta(2+2\alpha)x_i x_n^{2+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+2}}, \quad i < n;\\
D_{nn} w &= -\frac{\gamma\beta(2+2\alpha)x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}} - \frac{\gamma\beta(2+2\alpha)^2 x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}}\\
&\quad + \frac{\gamma(\gamma+1)\beta^2(2+2\alpha)^2 x_n^{3+4\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+2}}.
\end{aligned} \tag{3.7}$$

Then

$$\begin{aligned}
\mathfrak{L} w &= -\frac{2\gamma(n-1)x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}} + \frac{4\gamma(\gamma+1)|x'|^2 x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+2}} - \frac{\gamma\beta(2+2\alpha)x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}}\\
&\quad - \frac{\gamma\beta(2+2\alpha)^2 x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}} + \frac{\gamma(\gamma+1)\beta^2(2+2\alpha)^2 x_n^{3+4\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+2}}\\
&= \frac{\{-2\gamma(n-1) - \gamma\beta(2+2\alpha)(3+2\alpha)\}x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}}\\
&\quad + \frac{4\gamma(\gamma+1)\{|x'|^2 + \beta^2(1+\alpha)^2 x_n^{2+2\alpha}\}x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+2}}\\
&= \frac{\{-2\gamma(n-1) - \gamma\beta(2+2\alpha)(3+2\alpha)\}x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}} + \frac{4\gamma(\gamma+1)x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}}\\
&= \frac{2\gamma\{-(n-1) - (1+\alpha)^{-1}(3+2\alpha) + 2(\gamma+1)\}x_n^{1+2\alpha}}{\left(|x'|^2 + \beta x_n^{2+2\alpha}\right)^{\gamma+1}}
\end{aligned}$$

$$= \frac{2\gamma\{-n+1-(1+\alpha)^{-1}+2\gamma\}x_n^{1+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}} = 0$$

where $\gamma = \frac{n-1}{2} + \frac{1}{2(1+\alpha)}$, and $\mathfrak{L}$ is given by (1.4). Using $w$, we can construct a supersolution of (1.1) as follows.

**Lemma 3.4.** *Let $L$ be given by* (1.1) *with coefficients satisfying* (1.2), (1.3) *and* (1.7). *Then for each $\rho \in \big(0, \min\{\frac{s}{n-1}, 1\}\big)$, there exists $R_0 \geq 1$ depending only on $\rho$, $s$, $\alpha$ and $n$ such that*

$$L(w - w^{1+\rho}) \leq 0 \quad \text{in } \mathbb{R}^n_+ \backslash \overline{E}^+_{R_0}. \tag{3.8}$$

*Proof.* For $i, j < n$, we have

$$\begin{aligned}
|D_{ij}(w^{1+\rho})| &= \big|(1+\rho)w^\rho D_{ij}w + \rho(1+\rho)w^{\rho-1}D_iwD_jw\big| \\
&\leq (1+\rho)w^\rho\Big\{\frac{2\gamma x_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}} + \frac{4\gamma(\gamma+1)|x'|^2x_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+2}}\Big\} \\
&\quad + \rho(1+\rho)w^{\rho-1}\frac{4\gamma|x'|^2x_n^2}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2(\gamma+1)}} \\
&\leq \frac{C(\rho,\alpha,n)w^\rho x_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}} + \frac{C(\rho,\alpha,n)w^{\rho-1}x_n^2}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}} \\
&\leq \frac{C(\rho,\alpha,n)w^{\rho-1}x_n^2}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}},
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
|D_{in}(w^{1+\rho})| &= \big|\rho(1+\rho)w^{\rho-1}D_iwD_nw + (1+\rho)w^\rho D_{in}w\big| \\
&\leq \frac{2\gamma\rho(1+\rho)w^{\rho-1}|x'|x_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}}\Big\{\frac{1}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma}} + \frac{\gamma\beta(2+2\alpha)x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}}\Big\} \\
&\quad + (1+\rho)w^\rho\Big\{\frac{2\gamma|x'|}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}} + \frac{2\gamma\beta(2+2\alpha)|x'|x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+2}}\Big\} \\
&\leq \frac{C(\rho,\alpha,n)w^{\rho-1}|x'|x_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}} + \frac{C(\rho,\alpha,n)w^\rho|x'|}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}} \\
&\leq \frac{C(\rho,\alpha,n)w^{\rho-1}|x'|x_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}},
\end{aligned} \tag{3.10}$$

where $C(\rho,\alpha,n)$ is positive, depending only on $\rho$, $\alpha$ and $n$, and may change from line to line. Thus,

$$\begin{aligned}
\mathfrak{L}(w^{1+\rho}) &= x_n^{2\alpha}\sum_{i=1}^{n-1}(1+\rho)w^\rho D_{ii}w + \rho(1+\rho)w^{\rho-1}D_iwD_iw + (1+\rho)w^\rho D_{nn}w \\
&\quad + \rho(1+\rho)w^{\rho-1}(D_nw)^2 \\
&= +\rho(1+\rho)w^{\rho-1}(D_nw)^2 \\
&= \rho(1+\rho)w^{\rho-1}\Big\{x_n^{2\alpha}\sum_{i=1}^{n-1}\Big(-\frac{2\gamma x_ix_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}}\Big)^2
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{1}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma}}-\frac{\gamma\beta(2+2\alpha)x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1}}\Big)^2\Big\}\\
&=\rho(1+\rho)w^{\rho-1}\Big\{\frac{4\gamma^2|x'|^2x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2(\gamma+1)}}+\frac{1}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma}}\\
&\quad-\frac{\gamma\beta(2+2\alpha)x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}}+\frac{\gamma^2\beta^2(2+2\alpha)^2x_n^{4+4\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2(\gamma+1)}}\Big\}\\
&=\rho(1+\rho)w^{\rho-1}\Big\{\frac{4\gamma^2x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}}-\frac{2\gamma(1+\alpha)^{-1}x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}}\\
&\quad+\frac{1}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma}}.\Big\}\\
&=\frac{(n-1)\big(n-1+\frac{1}{1+\alpha}\big)\rho(1+\rho)w^{\rho-1}x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}}+\frac{\rho(1+\rho)w^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma}},
\end{aligned}
$$

where $\gamma=\frac{n-1}{2}+\frac{1}{2(1+\alpha)}$, and $\mathfrak{L}$ is given by (1.4). This, (3.9), and (3.10) imply that

$$
\begin{aligned}
&L\left(w^{1+\rho}\right)\\
&\geq\mathfrak{L}\left(w^{1+\rho}\right)-\sum_{i,j=1}^{n-1}|a_{ij}(x)-\delta_{ij}||D_{ij}(w^{1+\rho})|x_n^{2\alpha}-\sum_{i=1}^{n-1}|a_{in}(x)||D_{in}(w^{1+\rho})|\\
&\geq\frac{\rho(1+\rho)w^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma}}-\big(|x'|+x_n^{1+\alpha}\big)^{-s}\frac{C(\rho,\alpha,n)w^{\rho-1}x_n^{2+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}}\\
&\quad-\big(|x'|+x_n^{1+\alpha}\big)^{-s}\frac{C(\rho,\alpha,n)w^{\rho-1}|x'|x_n}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+1}}\\
&\geq\frac{\rho(1+\rho)w^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma}}-\frac{C(\rho,\alpha,n)w^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+\frac{s}{2}}}\\
&\quad-\frac{C(\rho,\alpha,n)w^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+\frac{s}{2}+\frac12-\frac{1}{2(1+\alpha)}}}\\
&\geq\frac{\frac12\rho(1+\rho)w^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma}}\quad\text{in }\mathbb{R}^n_+\setminus\overline{E}^+_{R_0}
\end{aligned}\tag{3.11}
$$

for some $R_0\geq 1$ depending only on $\rho$, $s$, $\alpha$, and $n$. Similarly,

$$
\begin{aligned}
Lw&\leq\mathfrak{L}w+\sum_{i,j=1}^{n-1}|a_{ij}(x)-\delta_{ij}||D_{ij}w|x_n^{2\alpha}+\sum_{i=1}^{n-1}|a_{in}(x)||D_{in}w|\\
&\leq\frac{C(\rho,\alpha,n)x_n^{1+2\alpha}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1+\frac{s}{2}}}+\frac{C(\rho,\alpha,n)|x'|}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1+\frac{s}{2}}}\\
&\leq\frac{C(\rho,\alpha,n)}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{\gamma+1+\frac{s}{2}-\frac{1+2\alpha}{2(1+\alpha)}}}.
\end{aligned}\tag{3.12}
$$

Since $\rho \in \big(0, \min\{\frac{s}{n-1}, 1\}\big)$, we obtain

$$\begin{aligned}\frac{w^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma}} &= \frac{x_n^{\rho-1}}{\big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{2\gamma+\gamma(\rho-1)}} \\ &\geq \big(|x'|^2+\beta x_n^{2+2\alpha}\big)^{-2\gamma-\gamma(\rho-1)-\frac{1-\rho}{2(1+\alpha)}},\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\Big(-2\gamma-\gamma(\rho-1)-\frac{1-\rho}{2(1+\alpha)}\Big)+\Big(\gamma+1+\frac{s}{2}-\frac{1+2\alpha}{2(1+\alpha)}\Big) \\ &= -\frac{n-1}{2}\rho+\frac{s}{2} > 0.\end{aligned} \tag{3.14}$$

By (3.11), (3.12), (3.13), and (3.14), we have

$$L(w-w^{1+\rho}) \leq 0 \quad \text{in } \mathbb{R}^n_+ \backslash \overline{E}^+_{R_0}$$

for larger $R_0 \geq 1$ depending only on $\rho$, $s$, $\alpha$ and $n$. $\square$

*Proof of Theorem 1.2.* By Lemma 3.4, for each fixed $\rho \in \big(0, \min\big\{\frac{s}{n-1}, 1\big\}\big)$, there exists $R > 1$ depending only on $s$, $\alpha$ and $n$ such that

$$L\left(w-w^{1+\rho}\right) \leq 0 \quad \text{in } \mathbb{R}^n_+ \backslash \overline{E}^+_R.$$

By $u(x)=0$ on $\{x_n=0\}$, $|Du(x)| \leq 1$ in $\mathbb{R}^n_+ \backslash \overline{E}^+_1$ and Newton-Leibniz formula,

$$|u(x)| \leq 2x_n \quad \text{on } \partial E_R \cap \{x_n \geq 0\}.$$

On $\partial E_R \cap \{x_n \geq 0\}$, it is clear that

$$w-w^{1+\rho} = w(1-w^\rho) \geq c(R,\alpha,n)x_n.$$

The above two inequalities imply that for some $C>0$ depending only on $s$, $\delta$, $\alpha$ and $n$,

$$|u(x)| \leq C(w-w^{1+\rho}), \quad \text{on } \partial E_R \cap \{x_n \geq 0\}. \tag{3.15}$$

For any $\varepsilon > 0$, by Corollary 3.3, there exists $R_\varepsilon > R$ such that

$$|u(x)| \leq \varepsilon, \quad x \in \partial E_{R_\varepsilon} \cap \{x_n \geq 0\}. \tag{3.16}$$

It follows from (3.15), (3.16) and $u(x)=0$ on $(E_{R_\varepsilon} \backslash \overline{E}_R) \cap \{x_n = 0\}$ that

$$|u(x)| \leq C(w-w^{1+\rho})+\varepsilon \quad \text{on } \partial(E^+_{R_\varepsilon} \backslash \overline{E}^+_R).$$

By the comparison principle,

$$|u(x)| \leq C(w-w^{1+\rho})+\varepsilon \quad \text{in } E^+_{R_\varepsilon} \backslash \overline{E}^+_R.$$

Then (1.8) is immediate by letting $\varepsilon \to 0$. $\square$

## 4. Appendix

*Proof of Lemma 2.1.* We denote

$$A'(x) = \begin{pmatrix} a_{11}(x) & \dots & a_{1,n-1}(x) \\ \vdots & \ddots & \vdots \\ a_{n-1,1}(x) & \dots & a_{n-1,n-1}(x) \end{pmatrix},$$

$$\widetilde{A}(x) = \begin{pmatrix} & & & a_{1,n}(x)x_n^\alpha \\ & A'(x)x_n^{2\alpha} & & \vdots \\ & & & a_{n-1,n}(x)x_n^\alpha \\ a_{n,1}(x)x_n^\alpha & \dots & a_{n,n-1}(x)x_n^\alpha & 1 \end{pmatrix},$$

where $a_{ij}(x)$ and $a_{in}(x)$ are given by (1.1). It suffices to show that eigenvalues of $\widetilde{A}(x)$ are positive in $\overline{B}_1^+$ and have uniformly bound (depending on the fixed number $\varepsilon_0$) in $\overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$.

When $A'(x)$ has eigenvalues $\lambda_1(x), \dots, \lambda_{n-1}(x)$, by (1.2), we obtain $\lambda \leq \lambda_i(x) \leq \Lambda$, $i = 1, 2, \dots, n-1$. Then there exists a orthogonal matrix $P'_{(n-1)\times(n-1)}$ such that

$$(P')^T A' P' = \operatorname{diag}\{\lambda_1(x), \dots, \lambda_{n-1}(x)\}.$$

Observe that eigenvalues of $\widetilde{A}(x)$ are that of the matrix

$$B(x) := P^T A P = \begin{pmatrix} \lambda_1(x) x_n^{2\alpha} & & & \widetilde{a}_{1,n}(x) x_n^{\alpha} \\ & \ddots & & \vdots \\ & & \lambda_{n-1}(x) x_n^{2\alpha} & \widetilde{a}_{n-1,n}(x) x_n^{\alpha} \\ \widetilde{a}_{n,1}(x) x_n^{\alpha} & \dots & \widetilde{a}_{n,n-1}(x) x_n^{\alpha} & 1 \end{pmatrix}$$

with

$$\widetilde{a}_{i,n}(x) = \sum_{j=1}^{n-1} P'_{ij} a_{j,n}(x), \quad i = 1, \dots, n-1; \quad P = \begin{pmatrix} P' & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, we only need to show that all eigenvalues of $B(x)$ are positive in $\overline{B}_1^+$ and have uniformly bound in $\overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$. For any $i = 1, 2, \dots, n$, let $e_i \in \mathbb{R}^n$ be the unit vector with its $i^{th}$ component is 1. Then

$$e_i^T B(x) e_i = \lambda_i x_n^{2\alpha}, \quad e_i^T B(x) e_n = \widetilde{a}_{in} x_n^{\alpha}, \quad e_i^T B(x) e_j = 0,$$

for $i, j \leq n-1$, and $e_n^T B(x) e_n = 1$.

For each $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, there exists a unique sequence $\{b_i\}_{i=1}^n$ such that $\xi = \sum_{i=1}^n b_i e_i$ and $\sum_{i=1}^n b_i^2 = 1$. Then, by (1.2),

$$\xi^T B(x) \xi = \sum_{i,j=1}^n (b_i e_i)^T B_{ij}(x) (b_j e_j) \geq \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + \sum_{i=1}^{n-1} 2 b_i b_n \widetilde{a}_{i,n} x_n^{\alpha} + b_n^2.$$

Applying Cauchy's inequality to $2 b_i b_n \widetilde{a}_{i,n} x_n^{\alpha}$, we have that for each $\tau \in (0,1)$,

$$\begin{aligned} \Big|\sum_{i=1}^{n-1} 2 b_i b_n \widetilde{a}_{i,n} x_n^{\alpha}\Big| &\leq \tau \sum_{i=1}^{n-1} \Big\{\lambda^{\frac{1}{2}} b_i x_n^{\alpha}\Big\}^2 + \tau^{-1} \sum_{i=1}^{n-1} \Big\{\lambda^{-1/2} b_n \widetilde{a}_{i,n}\Big\}^2 \\ &= \tau \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + \tau^{-1} b_n^2 \lambda^{-1} \sum_{i=1}^{n-1} \widetilde{a}_{i,n}^2. \end{aligned}$$

Therefore, for each $\tau \in (1-\delta, 1)$,

$$\begin{aligned} \xi^T B(x) \xi &\geq \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + b_n^2 - \tau \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 - \tau^{-1} b_n^2 \lambda^{-1} \sum_{i=1}^{n-1} \widetilde{a}_{i,n}^2 \\ &\geq (1-\tau) \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + b_n^2 \left\{1 - \tau^{-1}(1-\delta)\right\} \quad \text{(by (1.3))}, \end{aligned}$$

which implies that $L$ is elliptic in $\overline{B}_1^+$. And if $\{x_n \geq \varepsilon_0\}$, then

$$\xi^T A(x) \xi \geq (1-\tau) \lambda \varepsilon_0^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + b_n^2 \{1 - \tau^{-1}(1-\delta)\}$$

$$\geq \min\left\{(1-\tau)\lambda\varepsilon_0^{2\alpha}, 1-\tau^{-1}(1-\delta)\right\}.$$

In particular, taking $\tau = 1 - \frac{1}{2}\delta$, we have that for each $x \in \overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$,

$$\xi^T A(x)\xi \geq \min\left\{\frac{1}{2}\delta\lambda\varepsilon_0^{2\alpha}, 1-\left(1-\frac{1}{2}\delta\right)^{-1}(1-\delta)\right\} > 0.$$

Therefore, eigenvalues of $B(x)$ have uniformly below bound in $\overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$.

Similarly, one can obtain the uniformly upper bound of eigenvalues of $B(x)$ in $\overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$. □

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