

OSCILLATION FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH A SUB-LINEAR NEUTRAL TERM

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ABSTRACT. We study the oscillatory behavior of solution to the second order nonlinear differential equations with a sub-linear neutral term

$$(a(t)[(x(t) + p(t)x^\alpha(\tau(t)))']^\gamma)' + q(t)x^\beta(\sigma(t)) = 0, \quad t \geq t_0 > 0.$$

A new criterion is established that improves related results reported in the literature. Moreover, some examples are provided to illustrate the main results.

1. INTRODUCTION

This article concerns the second order nonlinear differential equation with a sub-linear neutral term

$$(a(t)[(x(t) + p(t)x^\alpha(\tau(t)))']^\gamma)' + q(t)x^\beta(\sigma(t)) = 0, \quad t \geq t_0 > 0, \quad (1.1)$$

under the following assumptions:

- (A1) α, β and γ are the ratios of odd positive integers with $\alpha \in (0, 1]$;
- (A2) $a \in C^1([t_0, \infty), (0, \infty))$, $a'(t) > 0$, $p, q \in C([t_0, \infty), (0, \infty))$, $\lim_{t \rightarrow \infty} p(t) = 0$ and q is not eventually zero on $[t^*, \infty)$ for $t^* \geq t_0$;
- (A3) $\tau \in C([t_0, \infty), R)$, $\sigma \in C^1([t_0, \infty), R)$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, and $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \sigma(t) = \infty$.

A function $x(t) \in C^1([T_x, \infty), R)$, $T_x \geq t_0$ is called a solution of (1.1) if $a(t)[(x(t) + p(t)x^\alpha(\tau(t)))']^\gamma \in C^1([T_x, \infty), R)$ and satisfies (1.1) on an interval $[T_x, \infty)$. We only consider the nontrivial solutions of (1.1), which means $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (1.1) is said to be oscillatory if it has an arbitrarily large zeros on $[T_x, \infty)$; otherwise, it is called non-oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Oscillation phenomena arise in various models from real world applications; see, e.g., [7, 13, 16, 20, 22]. In particular, we refer the reader to the papers [16, 22] for models from mathematical biology and physics where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. The increasing interest in oscillatory criteria for the second order nonlinear differential equations is motivated by their applications in the natural sciences and engineering, see, for

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example, [2, 3, 4, 8, 9, 10, 15, 17, 18, 21, 23, 26] and the references cited therein. For (1.1), one important special case is when $\gamma = 1$,

$$(a(t)(x(t) + p(t)x^\alpha(\tau(t)))')' + q(t)x^\beta(\sigma(t)) = 0. \quad (1.2)$$

Note that if $p(t) \equiv 0$, this equation is the well-known Emden-Fowler equation which has been widely applied in mathematics and theoretical physics (see [19, 25]). When $\alpha = 1$, Li et al. [17, 18, 21] obtained oscillation criteria under canonical and non-canonical conditions. In 2014, Agarwal et al. [1] discussed the oscillatory behavior of (1.2) with $\beta = 1$. Later on, Grace and Graef [12] and Tamilvanan et al. [24] considered the sub-linear ($0 < \beta < 1$) and super-linear ($\beta > 1$) cases under the non-canonical condition

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty \quad (1.3)$$

and obtained the oscillation criteria of all solutions of (1.2). Tamilvanan et al. [24] studied the sub-linear and super-linear cases under the canonical condition

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty \quad (1.4)$$

and obtained the oscillation of all solutions of (1.2).

Another important special case of (1.1) is (when $\beta = \gamma$)

$$(a(t)[(x(t) + p(t)x^\alpha(\tau(t)))^\gamma]')' + q(t)x^\gamma(\sigma(t)) = 0. \quad (1.5)$$

If $p(t) \equiv 0$, this equation is called the half-linear differential equation which is first studied by Hungarian mathematicians Bihari and Elbert in the 1970s, and has attracted considerable attentions in recent years, see, for example, [2, 4, 9, 10, 20, 11, 23, 25]. However, all of those results on the oscillation of (1.5) are for $\alpha = 1$ and few for $0 < \alpha < 1$.

Now in this article we shall use the Riccati transformation technique to study the oscillation behaviors of (1.1) for different positive constants α, β and γ . In Section 2, we establish a new oscillation criterion for (1.1). In Section 3, we present some examples to illustrate our results.

2. MAIN RESULTS

Without loss of generality, we only deal with the positive solution of (1.1) in the proofs of our results. In what follows, all functional inequalities are assumed to be satisfied for all t sufficiently large. We define the functions

$$z(t) := x(t) + p(t)x^\alpha(\tau(t)), \quad \pi(t) := \int_t^{\infty} \frac{1}{a^{1/\gamma}(s)} ds,$$

$$\varphi(t) := \frac{p(t)}{\pi^{1-\alpha}(\tau(t))}, \quad \psi(t) := \frac{p(t)\pi^\alpha(\tau(t))}{\pi^{2-\alpha}(t)}$$

and assume that

- (A4) $\pi(t_0) < \infty$;
- (A5) $\max\{\varphi(\sigma(t)), \psi(\sigma(t))\} < 1$.

Before giving the proof of our main theorem, we need the following lemma derived by Zhang and Wang [27].

Lemma 2.1. *Let ξ be a ratio of odd positive integers, $A, B > 0$ and $w \geq 0$. Then*

$$Aw - Bw^{1+\frac{1}{\xi}} \leq \frac{\xi^\xi}{(\xi + 1)^{\xi+1}} A^{\xi+1} B^{-\xi}.$$

Theorem 2.2. *Let (A1)–(A5) hold. If there exist a positive nondecreasing function $\rho \in C^1([t_0, \infty), (0, \infty))$ and two positive constants K, M such that*

$$\lim_{t \rightarrow \infty} \int_T^t \left[\rho(s)q(s)(1 - \varphi(\sigma(s)))^\beta - \frac{a(\theta(s))(\rho'(s))^{\xi+1}}{(\xi + 1)^{\xi+1}(K\rho(s)\sigma'(s))^\xi} \right] ds = \infty, \tag{2.1}$$

$$\lim_{t \rightarrow \infty} \int_T^t \left[\pi^\eta(s)q(s)(1 - \psi(\sigma(s)))^\beta - \left(\frac{\eta}{\eta + 1}\right)^{\eta+1} \frac{(\eta/M)^\eta}{\pi(s)a^{1/\gamma}(s)} \right] ds = \infty \tag{2.2}$$

hold for sufficiently large $T \geq t_0$, where $\xi = \min\{\beta, \gamma\}$, $\eta = \max\{\beta, \gamma\}$ and

$$\theta(t) = \begin{cases} t, & \gamma > \beta, \\ \sigma(t), & \gamma \leq \beta, \end{cases}$$

then every solution of (1.1) is oscillatory.

Epecially, when $\beta = \gamma$, equation (1.1) is oscillatory if assumptions (2.1) and (2.2) hold for $K = 1$ and $M = \eta = \beta$.

Proof. Suppose to the contrary that (1.1) has an eventually positive solution $x(t)$, i.e., there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. It follows that $z(t) > 0$ for $t \geq t_1$. From (1.1), we see that

$$(a(t)(z'(t))^\gamma)' = -q(t)x^\beta(\sigma(t)) < 0, \tag{2.3}$$

which implies that $a(t)(z'(t))^\gamma$ is decreasing and thus $z'(t)$ does not change sign eventually. Therefore, there exists a $t_2 \geq t_1$ such that either $z'(t) > 0$ or $z'(t) < 0$ for all $t \geq t_2$.

Case I. First we assume that $z'(t) > 0$ for all $t \geq t_2$. Recall that $z(t) = x(t) + p(t)x^\alpha(\tau(t))$, hence we have $z(t) \geq x(t)$ and

$$x(t) \geq z(t) - p(t)z^\alpha(\tau(t)), \quad t \geq t_2. \tag{2.4}$$

Since $z(t)$ is a positive increasing function, $\pi(t)$ is a positive decreasing function and $\pi(t) \rightarrow 0$ as $t \rightarrow \infty$, then there exists a $t_3 \geq t_2$ such that

$$z(t) \geq \pi(t), \quad t \geq t_3. \tag{2.5}$$

Hence we obtain

$$x(t) \geq \left(1 - \frac{p(t)}{\pi^{1-\alpha}(\tau(t))}\right)z(t). \tag{2.6}$$

This, (1.1) and the definition of $\varphi(t)$, imply that

$$(a(t)(z'(t))^\gamma)' + q(t)[1 - \varphi(\sigma(t))]^\beta z^\beta(\sigma(t)) \leq 0, \quad t \geq t_3. \tag{2.7}$$

Now define a function $w(t)$ by

$$w(t) := \rho(t) \frac{a(t)(z'(t))^\gamma}{z^\beta(\sigma(t))}, \quad t \geq t_3. \tag{2.8}$$

It follows that $w(t) > 0$ for $t \geq t_3$, and

$$w'(t) = \rho'(t) \frac{a(t)(z'(t))^\gamma}{z^\beta(\sigma(t))} + \rho(t) \frac{(a(t)(z'(t))^\gamma)'}{z^\beta(\sigma(t))} - \frac{\beta\rho(t)a(t)(z'(t))^\gamma \sigma'(t)z'(\sigma(t))}{z^{\beta+1}(\sigma(t))}.$$

By (2.7) and (2.8), we obtain

$$w'(t) \leq -\rho(t)q(t)(1 - \varphi(\sigma(t)))^\beta + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta\rho(t)a(t)(z'(t))^\gamma\sigma'(t)z'(\sigma(t))}{z^{\beta+1}(\sigma(t))}. \quad (2.9)$$

We discuss this inequality in three cases. If $\gamma < \beta$, notice that $a^{1/\gamma}(t)z'(t) \leq a^{1/\gamma}(\sigma(t))z'(\sigma(t))$, then

$$w'(t) \leq -\rho(t)q(t)(1 - \varphi(\sigma(t)))^\beta + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta\sigma'(t)}{(\rho(t)a(\sigma(t)))^{1/\gamma}}[z(\sigma(t))]^{\frac{\beta-\gamma}{\gamma}}w^{\frac{\gamma+1}{\gamma}}(t).$$

Since $z(\sigma(t))$ is increasing, then there exist constants $k_1 > 0$ and $t_4 \geq t_3$ such that $[z(\sigma(t))]^{\frac{\beta-\gamma}{\gamma}} \geq k_1$ for $t \geq t_4$. It follows that

$$w'(t) \leq -\rho(t)q(t)(1 - \varphi(\sigma(t)))^\beta + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\gamma k_1 \sigma'(t)}{(\rho(t)a(\sigma(t)))^{1/\gamma}}w^{\frac{\gamma+1}{\gamma}}(t). \quad (2.10)$$

An easy computation shows that $k_1 = 1$ as $\gamma = \beta$. Now if $\gamma > \beta$, we claim that $[z'(t)]^{\frac{\beta-\gamma}{\beta}}$ is increasing. Observing that $(a(t)(z'(t))^\gamma)' \leq 0$ and $a'(t) > 0$, then $z''(t) \leq 0$, which implies that $z'(t)$ is decreasing and hence $[z'(t)]^{\frac{\beta-\gamma}{\beta}}$ is increasing. Therefore, there exist constants $k_2 > 0$, $t_5 \geq t_4$ such that $[z'(t)]^{\frac{\beta-\gamma}{\beta}} \geq k_2$ for $t \geq t_5$. It follows from (2.9) that

$$\begin{aligned} w'(t) &\leq -\rho(t)q(t)(1 - \varphi(\sigma(t)))^\beta + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta\sigma'(t)}{(\rho(t)a(t))^{\frac{1}{\beta}}}[z'(t)]^{\frac{\beta-\gamma}{\beta}}w^{\frac{\beta+1}{\beta}}(t) \\ &\leq -\rho(t)q(t)(1 - \varphi(\sigma(t)))^\beta + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta k_2 \sigma'(t)}{(\rho(t)a(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t), \quad t \geq t_5. \end{aligned}$$

Combining this inequality and (2.10), we have

$$w'(t) \leq -\rho(t)q(t)(1 - \varphi(\sigma(t)))^\beta + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\xi K \sigma'(t)}{(\rho(t)a(\theta(t)))^{\frac{1}{\xi}}}w^{\frac{\xi+1}{\xi}}(t) \quad (2.11)$$

for $t \geq t_3$, where $\xi = \min\{\gamma, \beta\}$, $K = \min\{k_1, k_2\}$, and

$$\theta(t) = \begin{cases} t, & \gamma > \beta, \\ \sigma(t), & \gamma \leq \beta. \end{cases}$$

Obviously, $K = 1$ if $\beta = \gamma$ and K is a positive constant if $\beta \neq \gamma$.

For the inequality (2.11), applying Lemma 2.1 with $A = \frac{\rho'(t)}{\rho(t)}$ and

$$B = \frac{\xi K \sigma'(t)}{(\rho(t)a(\theta(t)))^{\frac{1}{\xi}}},$$

we obtain

$$w'(t) \leq -\rho(t)q(t)(1 - \varphi(\sigma(t)))^\beta + \frac{(\rho'(t))^{\xi+1}a(\theta(t))}{(\xi + 1)^{\xi+1}(K\rho(t)\sigma'(t))^\xi}. \quad (2.12)$$

Integrating this inequality from $T > t_5$ to $t \geq T$, we have

$$\begin{aligned} &\int_T^t [\rho(s)q(s)(1 - \varphi(\sigma(s)))^\beta - \frac{(\rho'(s))^{\xi+1}a(\theta(s))}{(\xi + 1)^{\xi+1}(K\rho(s)\sigma'(s))^\xi}] ds \\ &\leq w(T) - w(t) \leq w(T), \end{aligned}$$

which contradicts (2.1).

Case II. Assume that $z'(t) < 0$ for $t > t_2$. In view of (1.1) we have that $(a(t)(-z'(t))^\gamma)' \geq 0$, $t \geq t_2$. Then $a^{1/\gamma}(t)(-z'(t))$ is an increasing function and thus

$$z'(s) \leq \left(\frac{a(t)}{a(s)}\right)^{1/\gamma} z'(t), \quad s \geq t \geq t_2. \quad (2.13)$$

Integrating this inequality from t to u , we obtain

$$z(u) - z(t) \leq a^{1/\gamma}(t) z'(t) \int_t^u a^{-1/\gamma}(s) ds, \quad t \geq t_2.$$

Letting $u \rightarrow \infty$ we then obtain

$$z(t) \geq \pi(t) a^{1/\gamma}(t) (-z'(t)), \quad t \geq t_2. \quad (2.14)$$

We define the function

$$V(t) := \frac{a(t)(-z'(t))^\gamma}{z^\beta(t)}, \quad t \geq t_2. \quad (2.15)$$

It follows that $V(t) > 0$ for all $t \geq t_2$. By (2.14) we have

$$z^\gamma(t) \geq \pi^\gamma(t) (a(t)(-z'(t)))^\gamma. \quad (2.16)$$

For this inequality, we first observe that if $\gamma \geq \beta$, then $z^{\gamma-\beta}(t)$ is a decreasing function and thus there exist constants $l_1 > 0$ and $t_3 \geq t_2$ such that $z^{\gamma-\beta}(t) \leq l_1$ for $t \geq t_3$. By (2.16) we obtain

$$l_1 \geq z^{\gamma-\beta}(t) \geq \pi^\gamma(t) V(t), \quad \gamma \geq \beta, \quad t \geq t_3. \quad (2.17)$$

Now from (2.14) we see that

$$z^\beta(t) \geq \pi^\beta(t) (a^{1/\gamma}(t)(-z'(t)))^{\beta-\gamma+\gamma}. \quad (2.18)$$

If $\gamma < \beta$, then $(a^{1/\gamma}(t)(-z'(t)))^{\beta-\gamma}$ is an increasing function. Hence there exist constants $l_2 > 0$ and $t_4 \geq t_3$, such that

$$l_2 \geq \left(r^{1/\gamma}(t)(-z'(t))\right)^{\gamma-\beta} \geq \pi^\beta(t) V(t), \quad t \geq t_4. \quad (2.19)$$

Combining (2.17) and (2.19), we have

$$0 < \pi^\eta(t) V(t) \leq l, \quad t \geq t_4, \quad (2.20)$$

where $\eta = \max\{\gamma, \beta\}$ and $l = \max\{l_1, l_2\}$.

Observing (2.14) again we obtain

$$\left(\frac{z(t)}{\pi(t)}\right)' \geq 0, \quad t \geq t_4. \quad (2.21)$$

It follows that $\frac{z(t)}{\pi(t)}$ is a positive increasing function. From assumption (A3) we have $\tau(t) \leq t$ and thus

$$\frac{z(\tau(t))}{\pi(\tau(t))} \leq \frac{z(t)}{\pi(t)}.$$

By this inequality and (2.4) we deduce that

$$x(t) \geq z(t) - p(t) \frac{\pi^\alpha(\tau(t))}{\pi^\alpha(t)} z^\alpha(t). \quad (2.22)$$

Since $\frac{z(t)}{\pi(t)}$ is a positive increasing function and $\pi(t)$ is a positive decreasing function tends to 0 as $t \rightarrow \infty$, then there exists $t_5 \geq t_4$ such that

$$z(t) \geq \pi^2(t), \quad t \geq t_5. \quad (2.23)$$

In view of the above two inequalities, for $t \geq t_5$, we have

$$\begin{aligned} x(t) &\geq \left(1 - \frac{p(t)\pi^\alpha(\tau(t))}{\pi^\alpha(t)z^{1-\alpha}(t)}\right)z(t) \\ &\geq \left(1 - \frac{p(t)\pi^\alpha(\tau(t))}{\pi^{2-\alpha}(t)}\right)z(t) \\ &= (1 - \psi(t))z(t). \end{aligned}$$

By (1.1) and the fact $z'(t) < 0$, we have

$$(a(t)(-z'(t))^\gamma)' \geq q(t)(1 - \psi(\sigma(t)))^\beta z^\beta(t). \quad (2.24)$$

Taking derivative of the function $V(t)$ defined by (2.15) and then using (2.24) we obtain

$$V'(t) \geq q(t)(1 - \psi(\sigma(t)))^\beta + \frac{\beta a(t)(-z'(t))^{\gamma+1}}{z^{\beta+1}(t)}, \quad t \geq t_5. \quad (2.25)$$

For this inequality, we first treat the case $\gamma > \beta$. Since $[z(t)]^{\frac{\beta-\gamma}{\gamma}}$ is an increasing function, then there exist constants $m_1 > 0$ and $t_6 \geq t_5$, such that $[z(t)]^{\frac{\beta-\gamma}{\gamma}} \geq m_1$ for $t \geq t_6$. Hence,

$$\begin{aligned} V'(t) &\geq q(t)(1 - \psi(\sigma(t)))^\beta + \frac{\beta}{a^{1/\gamma}(t)} [z(t)]^{\frac{\beta-\gamma}{\gamma}} V^{\frac{\gamma+1}{\gamma}}(t) \\ &\geq q(t)(1 - \psi(\sigma(t)))^\beta + \frac{\beta m_1}{a^{1/\gamma}(t)} V^{\frac{\gamma+1}{\gamma}}(t), \quad t \geq t_6. \end{aligned} \quad (2.26)$$

If $\gamma < \beta$, we find that $[a^{1/\gamma}(t)(-z'(t))]^{\frac{\beta-\gamma}{\beta}}$ is an increasing function. Thus there exist constants $m_2 > 0$ and $t_7 \geq t_6$ such that $[a^{1/\gamma}(t)(-z'(t))]^{\frac{\beta-\gamma}{\beta}} \geq m_2$ for $t \geq t_7$. In view of (2.25), we then have

$$\begin{aligned} V'(t) &\geq q(t)(1 - \psi(\sigma(t)))^\beta + \frac{\beta}{a^{1/\gamma}(t)} [a^{1/\gamma}(t)(-z'(t))]^{\frac{\beta-\gamma}{\beta}} V^{\frac{\beta+1}{\beta}}(t) \\ &\geq q(t)(1 - \psi(\sigma(t)))^\beta + \frac{\beta m_2}{a^{1/\gamma}(t)} V^{\frac{\beta+1}{\beta}}(t), \quad t \geq t_7. \end{aligned} \quad (2.27)$$

Now if $\gamma = \beta$, it is easy to see that $m_1 = m_2 = 1$. Hence (2.26) and (2.27) still hold.

Combining (2.26) and (2.27) we obtain

$$V'(t) \geq q(t)(1 - \psi(\sigma(t)))^\beta + \frac{M}{a^{1/\gamma}(t)} V^{\frac{\eta+1}{\eta}}(t), \quad t \geq t_7, \quad (2.28)$$

where $\eta = \max\{\gamma, \beta\}$ and $M = \begin{cases} \beta, & \gamma = \beta, \\ \text{const} > 0, & \gamma \neq \beta. \end{cases}$ Multiplying (2.28) by $\pi^\eta(t)$

and integrating the resulting inequality from $T \geq t_7$ to t , we have

$$\begin{aligned} &\int_T^t \pi^\eta(s)q(s)(1 - \psi(\sigma(s)))^\beta ds \\ &\leq \int_T^t \pi^{\eta-1}(s)a^{-1/\gamma}(s)[\eta V(s) - M\pi(s)V^{\frac{\eta+1}{\eta}}(s)]ds + \pi^\eta(t)V(t) - \pi^\eta(T)V(T). \end{aligned}$$

Again using Lemma 2.1 we obtain

$$\eta V(s) - M\pi(s)V^{\frac{\eta+1}{\eta}}(s) \leq \left(\frac{\eta}{\eta+1}\right)^{\eta+1} \left(\frac{\eta}{M}\right)^\eta \pi^{-\eta}(s).$$

In view of the above two inequalities and (2.3), we have

$$\int_T^t [\pi^\eta(s)q(s)(1 - \psi(\sigma(s)))^\beta - (\frac{\eta}{\eta + 1})^{\eta+1} \frac{(\eta/M)^\eta}{\pi(s)a^{1/\gamma}(s)}] ds \leq l. \quad (2.29)$$

Letting $t \rightarrow \infty$ in the above inequality we then get a contradiction to (2.2). The proof is complete. \square

Recently, Jadlovská and Džurina [14] used the characteristic equation method to establish a variant of Kneser oscillation theorem for the second order differential equation

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + q(t)|y(\tau(t))|^{\alpha-1}y(\tau(t)) = 0 \quad (2.30)$$

under non-canonical conditions. This result is extended by Chatzarakis et al. [6] for the equation

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + \sum_{i=1}^m q_i(t)|y(\tau_i(t))|^{\alpha-1}y(\tau_i(t)) = 0. \quad (2.31)$$

Another extension is due to Bohner et al. [5], who studied the half-linear neutral differential equation

$$(a(t)(z'(t))^\alpha)' + q(t)y^\alpha(\sigma(t)) = 0 \quad (2.32)$$

where $z(t) = y(t) + p(t)y(\tau(t))$, and obtained a generalization of the Kneser theorem.

Theorem 2.3 ([5, Theorem 2]). *Assume*

$$\lambda_* := \liminf_{t \rightarrow \infty} \frac{\pi(\sigma(t))}{\pi(t)} < \infty.$$

If

$$\liminf_{t \rightarrow \infty} a^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) > \frac{\alpha \max\{K^\alpha(1-K)\lambda_*^{-\alpha K} : 0 < K < 1\}}{(1-p_0)^\alpha},$$

then equation (2.32) is oscillatory.

Obviously, (2.32) is a special case of equation (1.1) (when $\alpha = 1$ and $\beta = \gamma$). Now by Theorem 2.2 we derive the following Kneser oscillation criterion for (1.1) under non-canonical conditions.

Corollary 2.4. *Theorem 2.2 still holds if the conditions (2.1) and (2.2) are replaced by*

$$\limsup_{t \rightarrow \infty} \int_T^t q(s)(1 - \varphi(\sigma(s)))^\beta ds = \infty, \quad (2.33)$$

$$\liminf_{t \rightarrow \infty} \pi^{\eta+1}(t)a^{1/\gamma}(t)q(t)(1 - \psi(\sigma(t)))^\beta > (\frac{\eta}{\eta + 1})^{\eta+1}(\frac{\eta}{M})^\eta, \quad (2.34)$$

respectively.

Proof. Equality (2.33) follows by substituting $\rho(t) \equiv 1$ into (2.1). Now suppose (2.34) holds and denote $\mu = (\frac{\eta}{\eta+1})^{\eta+1}(\frac{\eta}{M})^\eta$. Then, for any $\varepsilon > 0$, there exists a sufficiently large T , such that

$$\pi^\eta(t)q(t)(1 - \psi(\sigma(t)))^\beta > \frac{\mu - \varepsilon}{\pi(t)a^{1/\gamma}(t)}, \quad t \geq T.$$

Integrating the above inequality from T to t we obtain

$$\begin{aligned} & \int_T^t \left[\pi^\eta(s)q(s)(1 - \psi(\sigma(s)))^\beta - \frac{\mu}{\pi(s)a^{1/\gamma}(s)} \right] ds \\ & > \int_t^T \frac{\varepsilon}{\pi(s)a^{1/\gamma}(s)} ds \\ & = -\varepsilon \int_T^t \frac{d\pi(s)}{\pi(s)} = \varepsilon \left(\ln \frac{1}{\pi(T)} - \ln \frac{1}{\pi(t)} \right). \end{aligned} \quad (2.35)$$

Then (2.2) holds as $t \rightarrow \infty$. This completes the proof. \square

The following theorem is for (1.2), the special case of (1.1) with $\gamma = 1$, and the theorem is Kneser oscillation criteria for Emden-Fowler neutral (1.2) and sub-linear neutral (1.5).

Theorem 2.5. *Suppose $\gamma = 1$. Then Corollary 2.4 still holds if (2.2) is replaced by any one of the following conditions:*

- (i) $\beta > 1$, $\liminf_{t \rightarrow \infty} \pi^{\eta+1}(t)a(t)q(t)(1 - \psi(\sigma(t)))^\beta > \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \left(\frac{\beta}{M}\right)^\beta$;
- (ii) $\beta < 1$, $\liminf_{t \rightarrow \infty} \pi^2(t)a(t)q(t)(1 - \psi(\sigma(t)))^\beta > \frac{1}{4M}$;
- (iii) $\beta = 1$, $\liminf_{t \rightarrow \infty} \pi^2(t)a(t)q(t)(1 - \psi(\sigma(t))) > 1/4$.

The following theorem is for (1.5).

Theorem 2.6. *Suppose $\beta = \gamma$. Then Corollary 2.4 still holds if (2.2) is replaced by*

$$\liminf_{t \rightarrow \infty} \pi^{\gamma+1}(t)a^{1/\gamma}(t)q(t)(1 - \psi(\sigma(t)))^\gamma > \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}.$$

Note that Theorem 2.5 improves [1, Theorem 2.2] and [12, Theorems 1-4]. Theorem 2.6 improves [2, Theorem 2.2], [4, Theorems 2.2-2.4], [10, Theorem 3 and Corollary 1].

3. EXAMPLES

In this section, we provide some examples to illustrate our results.

Example 3.1. Consider the Emden-Fowler neutral differential equation

$$\left(t^{3/2} \left(x(t) + p_0 x\left(\frac{t}{2}\right) \right)' \right)' + q_0 t^\delta x^\beta(t) = 0, \quad t \geq 1, \quad (3.1)$$

where β is a ratio of odd positive integers, $p_0 \in [0, \frac{\sqrt{2}}{2})$, $q_0 > 0$ and $\delta \geq -1$.

We are going to use Theorem 2.5 to show that (3.1) is oscillatory. Note that $a(t) = t^{3/2}$, $\tau(t) = t/2$, $\sigma(t) = t$, $p(t) = p_0$, $q(t) = q_0 t^\delta$ and $\alpha = \gamma = 1$. Thus we have

$$\begin{aligned} \pi(t) &= \int_t^\infty \frac{1}{a^{1/\gamma}(s)} ds = \int_t^\infty s^{-\frac{3}{2}} ds = 2t^{-1/2}, \\ \varphi(t) &= \frac{p(t)}{\pi^{1-\alpha}(\tau(t))} = p_0, \quad \psi(t) = \frac{p(t)\pi^\alpha(\tau(t))}{\pi^{2-\alpha}(t)} = \sqrt{2}p_0. \end{aligned}$$

Therefore, (A1)–(A5) and (2.33) hold.

Now we see that if $\beta > 1$, then $\eta = \max\{\beta, 1\} = \beta$ and

$$\liminf_{t \rightarrow \infty} \pi^{\eta+1}(t)a(t)q(t)(1 - \psi(\sigma(t)))^\beta$$

$$= \liminf_{t \rightarrow \infty} 2^{\beta+1}(t)q_0(1 - \sqrt{2}p_0)^\beta t^{\delta - \frac{\beta}{2} + 1} = \infty$$

holds for $\delta > \frac{\beta}{2} - 1$. Hence Theorem 2.5-(i) is satisfied as $\beta > 1$ and $\delta > \frac{\beta}{2} - 1$.

If $\beta < 1$, we have $\eta = 1$ and thus

$$\liminf_{t \rightarrow \infty} \pi^2(t)a(t)q(t)(1 - \psi(\sigma(t)))^\beta = \liminf_{t \rightarrow \infty} 4q_0(1 - \sqrt{2}p_0)^\beta t^{\delta + \frac{1}{2}} = \infty$$

holds for $\delta > -1/2$. Then Theorem 2.5-(ii) is satisfied as $\beta < 1$ and $\delta > -1/2$.

If $\beta = 1$, we have $\eta = 1$ and thus

$$\liminf_{t \rightarrow \infty} \pi^2(t)a(t)q(t)(1 - \psi(\sigma(t)))^\beta = \liminf_{t \rightarrow \infty} 4q_0(1 - \sqrt{2}p_0)t^{\delta + \frac{1}{2}} > \frac{1}{4}$$

holds for $\delta = -1/2$. Now Theorem 2.5-(iii) is satisfied as $\beta = 1$ and $\delta = -1/2$.

We, therefore, can make clear from the above discussion that (3.1) is oscillatory if (i) $\beta > 1$ and $\delta > \frac{\beta}{2} - 1$, or (ii) $\beta < 1$ and $\delta > -1/2$, or (iii) $\beta = 1$ and $\delta = -1/2$.

Note that Li et al. [15] considered the equation

$$\left(t^{3/2}y'(t)\right)' + y(t) = 0, \tag{3.2}$$

which is a special case of (3.1) with $p_0 = 0, q_0 = 1, \delta = 0, \beta = 1$.

The following example illustrates Theorem 2.6.

Example 3.2. Consider the half-linear neutral differential equation

$$\left(t^{\gamma+1}[(x(t) + p_0x(\frac{t}{2}))']^\gamma\right)' + q_0x^\gamma(\lambda t) = 0, \quad t \geq 1, \tag{3.3}$$

where γ is a ratio of odd positive integers, $q_0 \in (0, \infty), p_0 \in [0, \sqrt[3]{0.5}), \lambda \in (0, 1]$.

Observing that $a(t) = t^{\gamma+1}, \tau(t) = t/2, \sigma(t) = \lambda t, p(t) = p_0,$ and $q(t) = q_0,$ we have

$$\begin{aligned} \pi(t) &= \int_t^\infty \frac{1}{a^{1/\gamma}(s)} ds = \frac{\gamma}{t^{1/\gamma}}, \\ \varphi(t) &= \frac{p(t)}{\pi^{1-\gamma}(\tau(t))} = p_0, \quad \psi(t) = \frac{p(t)\pi(\tau(t))}{\pi(t)} = \sqrt[3]{2}p_0. \end{aligned}$$

Hence, the conditions (A1)–(A5) and (2.33) hold.

An easy calculation shows that

$$\liminf_{t \rightarrow \infty} \pi^{\gamma+1}(t)a^{1/\gamma}q(t)(1 - \psi(\sigma(t)))^\gamma = \gamma^{\gamma+1}(t)q_0(1 - \sqrt[3]{2}p_0)^\gamma.$$

Then by Theorem 2.6, we conclude that (3.3) is oscillatory if

$$q_0(1 - \sqrt[3]{2}p_0)^\gamma > \left(\frac{1}{\gamma + 1}\right)^{\gamma+1}. \tag{3.4}$$

Remark 3.3. This example is also studied by Bohner et al. [4]. According to [4, Theorem 2.1], equation (3.3) is oscillatory if

$$q_0(1 - \sqrt[3]{2}p_0)^\gamma > \left(\frac{1}{\gamma}\right)^\gamma. \tag{3.5}$$

Obviously, this restriction is contained in (3.4).

In [2], the authors considered the special case of (3.3) when $\gamma = 1, \lambda = 1,$ i.e.,

$$\left(t^2\left(x(t) + p_0x\left(\frac{t}{2}\right)\right)'\right)' + q_0x(t) = 0. \tag{3.6}$$

By [2, Theorem 2.1], equation (3.6) is oscillatory if

$$q_0(1 - 2p_0) > \frac{1}{4}, \tag{3.7}$$

which is just a special case of (3.4) when $\gamma = 1$. Note that if $p_0 = 0$ and $\gamma = 1$, then (3.4) or (3.7) reduces to $q_0 > 1/4$, which is sharp for oscillation of the Euler differential equation (see [2, 4])

$$(t^2x(t))' + q_0x(t) = 0.$$

Another special case of (3.3) is studied by Džurina and Jadlovská [10], which reads

$$(t^{\gamma+1}(x'(t))^\gamma)' + q_0x^\gamma(\lambda t) = 0, \tag{3.8}$$

where $q_0 > 0, \gamma \geq 1, \lambda \in (0, 1]$. Due to [10, Theorem 3], one can conclude that (3.8) is oscillatory if $q_0 > 1$. However, by Theorem 2.6 we can obtain a sharp condition

$$q_0 > \left(\frac{1}{\gamma + 1}\right)^{\gamma+1}. \tag{3.9}$$

Example 3.4. Consider the sub-linear neutral Emden-Fowler equation

$$\left(t^2\left(x(t) + \frac{1}{t^2}x^\alpha\left(\frac{t}{2}\right)\right)'\right)' + \lambda t^b x^\beta\left(\frac{t}{2}\right) = 0, \quad t \geq 4, \tag{3.10}$$

where α, β are ratios of odd positive integers with $\alpha \in (0, 1]$.

For this example we set $a(t) = t^2, p(t) = 1/t^2, q(t) = \lambda t^b, b > -1, \lambda > 0, \tau(t) = \sigma(t) = t/2$. Hence we obtain

$$\pi(t) = \frac{1}{t}, \quad \varphi(\sigma(t)) = \frac{4^\alpha}{t^{1+\alpha}}, \quad \psi(\sigma(t)) = \frac{8^\alpha}{t^{2\alpha}}.$$

Therefore, (A1)–(A5) and (2.33) hold.

Now if $b > \eta - 1$, we have

$$\liminf_{t \rightarrow \infty} \pi^{\eta+1}(t)a(t)q(t)(1 - \psi(\sigma(t)))^\beta = \liminf_{t \rightarrow \infty} \lambda\left(1 - \frac{8^\alpha}{t^{2\alpha}}\right)^\beta t^{b-\eta+1} = \infty.$$

Then (3.10) is oscillatory because of Corollary 2.4.

If $b = \eta - 1$, we see that when $\eta = \beta = 1$,

$$\liminf_{t \rightarrow \infty} \pi^{\eta+1}(t)a(t)q(t)(1 - \psi(\sigma(t)))^\beta = \liminf_{t \rightarrow \infty} \pi^2(t)a(t)q(t)(1 - \psi(\sigma(t))) = \lambda.$$

By Theorem 2.5(iii), we conclude that (3.10) is oscillatory for $\lambda > 1/4$.

Remark 3.5. Grace and Graef [12] considered (3.10) when $\alpha = 1/3, \lambda = 1$, and used Theorems 1–4 to obtain oscillatory results for (1) $\beta = 3$ and $b = 9$, (2) $\beta = 3$ and $b > 4$, (3) $\beta = 1$ and $b > 0$, and (4) $\beta = 1/3$ and $b > 0$. Another result for (3.10) when $\beta = 1$ and $b = 0$ from Agarwal et al. [1], showed that this equation is oscillatory if $\lambda > 1/4$.

Example 3.6 ([8, Example 3.10]). Consider the equation

$$\left(t^{\gamma+1}\left[\left(x(t) + p(t)x^\alpha\left(\frac{t}{2}\right)\right)'\right]^\gamma\right)' + \lambda t^\delta x^\beta\left(\frac{t}{2}\right) = 0, \tag{3.11}$$

where $a(t) = t^{\gamma+1}, \pi(t) = \frac{\gamma}{t^{1/\gamma}}, q(t) = \lambda t^\delta, \delta \geq 0, \psi(t) = \frac{p(t)\pi^\alpha(\tau(t))}{\pi^{2-\alpha}(t)}$, and $p(t)$ is a positive function such that

$$\lim_{t \rightarrow \infty} \frac{p(t)\pi^\alpha(\tau(t))}{\pi^{2-\alpha}(t)} = 0.$$

When $\beta > \gamma$, equation (3.11) is a super-linear equation. From Theorem 2.2 we see that if

$$g(\beta, \gamma) := \liminf_{t \rightarrow \infty} \pi^{\beta+1}(t) a^{1/\gamma}(t) q(t) (1 - \psi(\sigma(t)))^\beta > \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \left(\frac{\beta}{M}\right)^\beta, \quad (3.12)$$

then (3.11) is oscillatory. A direct computation shows that

$$g(\beta, \gamma) = \liminf_{t \rightarrow \infty} \pi^{\beta+1}(t) a^{1/\gamma}(t) q(t) (1 - \psi(\sigma(t)))^\beta = \liminf_{t \rightarrow \infty} t^{1+\frac{1}{\gamma}-\frac{\beta}{\gamma}-\frac{1}{\gamma}+\delta}. \quad (3.13)$$

Setting $\delta + 1 > \frac{\beta}{\gamma}$ then gives $g = \infty$ and hence the inequality (3.12) holds. Note that from [8, Theorem 3.1], one can also deduce that (3.11) is oscillatory for $\delta \geq \frac{\beta}{\gamma}$.

Now if $\beta = \gamma$, equation (3.11) is a half-linear differential equation. Again using Theorem 2.2 we conclude that (3.11) is oscillatory if

$$g(\beta, \beta) = \liminf_{t \rightarrow \infty} \pi^{\beta+1}(t) a^{\frac{1}{\beta}}(t) q(t) (1 - \psi(\sigma(t)))^\beta > \left(\frac{\beta}{\beta+1}\right)^{\beta+1}. \quad (3.14)$$

Letting $\delta = 0$, we obtain

$$g(\beta, \beta) = \liminf_{t \rightarrow \infty} \lambda \left(\frac{\beta}{t^{\frac{1}{\beta}}}\right)^{\beta+1} t^{\frac{\beta+1}{\beta}}. \quad (3.15)$$

It follows that if $\lambda > \frac{1}{(\beta+1)^{\beta+1}}$, then (3.14) holds. We also mention here that if $\beta = \gamma$ and $\delta = 0$, from [8, Theorem 3.2], one can deduce that (3.11) is oscillatory for $\lambda > 1/\beta^\beta$. Therefore, Theorem 2.2 improves both [8, Theorems 3.1 and 3.2].

Remark 3.7. The theorems of [8] only hold for $0 < \alpha < 1$. However, Theorem 2.2 in this paper can also be applied to the case of $\alpha = 1$. In addition, the conditions of the theorem only require to find the limit.

4. CONCLUSION

In this article, we use Riccati transformation and some inequality techniques to establish some new Kneser oscillation criteria for second order nonlinear differential equations with sublinear term. Our method is different from those reported in [5, 6, 14] and the results can be used to deal with the half-linear, the sublinear, the linear and the Emden-Fowler neutral differential equations. Those examples given in the last section show that our results improve some well-known results published recently in the literature.

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