Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 31, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LOWER AND UPPER SOLUTIONS FOR DELAY EVOLUTION EQUATIONS WITH NONLOCAL AND IMPULSIVE CONDITIONS

XUPING ZHANG

Abstract. In this article, we apply the method of lower and upper solutions for studying delay evolution equations with nonlocal and impulsive conditions in infinite dimensional Banach spaces. Under wide monotone conditions and noncompactness measure condition of nonlinear term, we obtain the existence of extremal solutions and a unique solution between lower and upper solutions. A concrete application to partial differential equations is considered.

1. Introduction and main results

Many complex process in nature and technology are described by functional differential equations which are dominant nowadays because the functional components in equations allow one to consider after-effect or prehistory influence. Delay evolution equation is one of the important type of functional differential equations, in which the response of system depends not only on the current state of system, but also on the past history of system. For more details on this topic, see [1, 13, 18, 21, 22, 23, 25, 26, 27, 28] and the references therein.

The study of abstract nonlocal Cauchy problem was initiated by Byszewski in [5]. It is demonstrated that the nonlocal problems have better effects in applications than the traditional Cauchy problems, differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see [6, 8, 9, 15, 17, 24, 29, 30] and the references therein for more comments and citations. Particularly, there has been a significant development in the theory of impulsive evolution equations with nonlocal conditions in Banach spaces. In 2009, Liang, Liu and Xiao [24] combined impulsive conditions and nonlocal conditions, and investigated the nonlocal impulsive evolution equation in Banach spaces. Later, Balachandran, Kiruthika and Trujillo [4], Chang, Kavitha and Mallike Arjunan [7], Chen and Li [9], Debbouche and Baleanu [11], Ji, Li and Wang [16], Fan and Li [17] studied the impulsive evolution equation with nonlocal conditions.

2020 *Mathematics Subject Classification.* 35R12, 35K90, 47D06.

Key words and phrases. Evolution equations; delay; nonlocal and impulsive conditions; lower and upper solutions; measure of noncompactness.

Submitted June 2, 2021. Published April 18, 2022.

We mention that in 2012, Chuong and Ke [10] studied the delay evolution inclusions involving nonlocal and impulsive conditions

$$\begin{aligned} u'(t) + Au(t) &\in F(t, u(t), u_t), \quad t \in [0, a],\ t \neq t_k, \\ u(t_k^+) &= u(t_k^-) + I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(s) + g(u)(s) &= \varphi(s), \quad s \in [-r, 0], \end{aligned} \tag{1.1}$$

where X is a Banach space, $F : [0, a] \times X \times C([-r, 0], X) \to P(X)$ is a multi-valued map, $P(X)$ stands for the collection of all nonempty subsets of X, A is a closed linear operator on X. By using the fixed point theory for multi-valued maps and the theory of differential inclusions, the authors obtain the existence of mild solutions for nonlocal problem (1.1). Furthermore, by applying corresponding measure of noncompactness estimates, they also proved the continuity of the solution mapping, which demonstrates that the solution set depends continuously on initial data. But so far we have not seen relevant papers that study delay evolution equations involving nonlocal and impulsive conditions by applying the iterative method, perturbation technique and the method of lower and upper solutions.

Let X be a Banach space with norm $\|\cdot\|$, and a and h positive constants. We denote by $PC([-h, a], X)$ the space of piecewise continuous functions $u : [-h, a] \to X$ such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists for $k = 1, 2, \dots, m$. Evidently, $PC([-h, a], X)$ is a Banach space with norm $\|u\|_{PC} = \sup_{t\in[-h,a]} \|u(t)\|$. Let $u_t(\tau) = u(t + \tau)$ for $\tau \in [-h, 0]$. In this space $\mathcal{B}$ is considered as a Banach space of piecewise continuous functions $v : [-h, 0] \to X$ with the norm $\|v\|_{\mathcal{B}} = \sup_{-h\le s\le 0} \|v(s)\|$.

In this article, we use the method of lower and upper solutions to discuss the existence of solutions to the delay evolution equations with nonlocal and impulsive conditions in an order space X,

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t), u_t), \quad t \in [0, a],\ t \neq t_k, \\ u(t_k^+) - u(t_k^-) &= I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(s) &= g(u)(s) + \phi(s), \quad s \in [-h, 0], \end{aligned} \tag{1.2}$$

where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a positive strongly continuous semigroup (positive C_0-semigroup, in short) $T(t)$ $(t \ge 0)$ on X; $f : [0, a]\times X\times\mathcal{B} \to X$ is a Carathéodory continuous; $0 < t_1 < t_2 < \dots < t_m < a$ are pre-fixed numbers, $I_k \in C(X, X)$ is an impulsive function, $k = 1, 2, \dots, m$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and the left limits of $u(t)$ at $t = t_k$, respectively; $\phi \in C([-h, 0], X)$ is a priori given history, while the function $g : PC([-h, a], X) \to C([-h, 0], X)$ implicitly defines a complementary history, chosen by the system itself; u_t denotes the function in $\mathcal{B}$ defined as $u_t(\tau) = u(t + \tau)$ for $\tau \in [-h, 0]$ and $u_t(\cdot)$ represent the time history of the state from the time $t - h$ up to the present time t.

We denote $J_0 = [-h, 0]$, $J_1 = [0, t_1]$, $J_k = (t_{k-1}, t_k]$, $k = 2, 3, \dots, m + 1$, $t_{m+1} = a$, $I' = [-h, a]\backslash\{t_1, t_2, \dots, t_m\}$ and $I'' = [-h, a]\backslash\{0, t_1, t_2, \dots, t_m\}$, and use X_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. An abstract function $u \in PC([-h, a], X)\cap C^1(I'', X)\cap C(I', X_1)$ is called a solution of the delay evolution equations with nonlocal and impulsive conditions (1.2) if $u(t)$ satisfies all the equalities in (1.2).

Let X be an ordered Banach space with partial order $\leq$, whose positive cone $P = \{u \in X \mid u \geq \theta\}$ is normal with normal constant N. Evidently, $PC([-h,a],X)$ and $\mathcal{B}$ are also order Banach spaces with partial order " $\leq$ " reduced by the positive function cones $K_{PC} = \{u \in PC([-h,a],X) : u(t) \geq \theta,\ t \in [-h,a]\}$ and $K_{\mathcal{B}} = \{u \in \mathcal{B} \mid u(s) \geq \theta,\ s \in [-h,0]\}$ (θ is the zero element of X) respectively. For $v,w \in PC([-h,a],X)$ with $v \leq w$, we use $[v,w]$ to denote the order interval $\{u \in PC([-h,a],X) \mid v \leq u \leq w\}$, and $[v(t),w(t)]$ to denote the order interval $\{u \in X : v(t) \leq u(t) \leq w(t),\ t \in [-h,a]\}$. If a function $u \in PC([-h,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ satisfies

$$\begin{gathered} u'(t) + Au(t) \leq f(t,u(t),u_t), \quad t \in [0,a],\ t \neq t_k, \\ u(t_k^+) - u(t_k^-) \leq I_k(u(t_k)), \quad k = 1,2,\dots,m, \\ u(s) \leq g(u)(s) + \phi(s), \quad s \in [-h,0], \end{gathered} \tag{1.3}$$

we call it a lower solution of problem (1.2); if all inequalities of (1.3) are reversed, we call it an upper solution of problem (1.2).

The method of lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces. Early on, Du and Lakshmikantham [14] built the method of lower and upper solutions for addressing the initial ordinary differential equations in Banach space. Latter, Guo and Liu [19] built an upper and lower solution method for the initial value problem (IVP) impulsive differential equations in an ordered Banach space X,

$$\begin{gathered} u'(t) = f(t,u(t),Gu(t)), \quad t \in [0,a],\ t \neq t_k, \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad k = 1,2,\dots,m, \\ u(0) = x_0, \end{gathered} \tag{1.4}$$

where $f \in C([0,a] \times X \times X, X)$, $a > 0$, $0 < t_1 < t_2 < \dots < t_m < a$, $I_k \in C(X,X)$, $k = 1,2,\dots,m$,

$$Gu(t) = \int_0^t K(t,s)u(s)ds, \quad K \in C(\Delta,\mathbb{R}^+), \quad \Delta = \{(t,s) \mid 0 \leq s \leq t \leq a\}.$$

They proved that if IVP (1.4) has a lower solution $v^{(0)}$ and an upper solution $w^{(0)}$ with $v^{(0)} \leq w^{(0)}$, and the nonlinear term f and impulsive function I_k satisfy the monotonicity condition

$$\begin{gathered} f(t,\bar{x},\bar{y}) - f(t,x,y) \geq -C(\bar{x}-x) - C^*(\bar{y}-y), \quad I_k(\bar{x}) \geq I_k(x), \\ v^{(0)}(t) \leq x \leq \bar{x} \leq w^{(0)}(t), \quad Gv^{(0)}(t) \leq y \leq \bar{y} \leq Gw^{(0)}(t), \quad \forall t \in [0,a], \end{gathered} \tag{1.5}$$

with positive constant C and C^*, and noncompactness measure conditions

$$\alpha(f(t,U,V)) \leq L_1\alpha(U) + L_2\alpha(V), \tag{1.6}$$

$$\alpha(I_k(D)) \leq M_k\alpha(D), \tag{1.7}$$

where $U,V,D \subset X$ are arbitrarily sets, $\alpha(\cdot)$ denotes the Kurataowski measure of noncompactness in X, L_1, L_2 and M_k are positive constants and satisfy

$$2a(M + L_1 + aK_0L_2) + \sum_{k=1}^{m} M_k < 1, \tag{1.8}$$

where $K_0 = \max_{(t,s)\in\Delta} K(t,s)$, then IVP (1.4) has a minimal and maximal solutions, which can be obtained by a monotone iterative procedure staring from $v^{(0)}$

and $w^{(0)}$ respectively. Recently, Chen and Li [9] extended the results of [19] to the nonlocal impulsive problem evolution equations without delay in X,

$$\begin{gathered} u'(t)+Au(t)=f(t,u(t),Gu(t)), \quad t\in J,\ t\neq t_k,\\ u(t_k^+)-u(t_k^-)=I_k(u(t_k)), \quad k=1,2,\dots,m, \\ u(0)=g(u)+x_0, \end{gathered} \tag{1.9}$$

where g constitutes a nonlocal condition, $x_0\in X$.

The purpose of this article is to improve and extend the above-mentioned results to the delay evolution equation involving nonlocal and impulsive conditions (1.2). We will delete the noncompactness measure condition (1.7) for impulsive function I_k and the strong restriction condition (1.8) for the constants. Our main results are as follows.

Theorem 1.1. *Let X be an ordered Banach space and its positive cone P be normal. Assume that problem* (1.2) *has a lower solution $v^{(0)}\in PC([-h,a],X)\cap C^1(I'',X)\cap C(I',X_1)$ and an upper solution $w^{(0)}\in PC([-h,a],X)\cap C^1(I'',X)\cap C(I',X_1)$ with $v^{(0)}\leq w^{(0)}$. Suppose also that the following conditions are satisfied:*

- (H1) *There exists a constant $C>0$ such that*
$$f(t,\bar{x},\bar{y})-f(t,x,y)\geq -C(\bar{x}-x),$$
for $\forall\, t\in[0,a]$, $x,\bar{x}\in X$ and y, $\bar{y}\in\mathcal{B}$ with $v^{(0)}(t)\leq x\leq\bar{x}\leq w^{(0)}(t)$ and $(v^{(0)})_t\leq y\leq\bar{y}\leq(w^{(0)})_t$;
- (H2) *I_k is increasing on order interval $[v^{(0)}(t),w^{(0)}(t)]$ for $t\in[0,a]$, $k=1,2,\dots,m$;*
- (H3) *The nonlocal function $g(u)$ is continuous and compact and is increasing on order interval $[v^{(0)},w^{(0)}]$;*
- (H4) *There exists a constant $L_f>0$, such that for every $t\in[0,a]$,*
$$\begin{aligned} &\alpha\Big(\big\{f(t,u^{(n)}(t),(u^{(n)})_t)\big\}\Big)\\ &\leq L_f\Big[\alpha\Big(\big\{u^{(n)}(t)\big\}\Big)+\sup_{-h\leq\tau\leq 0}\alpha\Big(\big\{u^{(n)}(t+\tau)\big\}\Big)\Big], \end{aligned}$$
where $\{u^{(n)}\}\subset[v^{(0)},w^{(0)}]$ is countable and increasing or decreasing monotonic set and $\{(u^{(n)})_t\}\subset\mathcal{B}$.

Then problem (1.2) *has minimal and maximal mild solutions between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$ respectively.*

Theorem 1.1 improves the main results in [9] and [19]. In Theorem 1.1, if X is a weakly sequentially complete Banach space, then the condition (H4) holds automatically. Hence, we can easily obtain the following result from Theorem 1.1.

Theorem 1.2. *Let X be an ordered and weakly sequentially complete Banach space and its positive cone P be normal. Assume that the problem* (1.2) *has a lower solution $v^{(0)}\in PC([-h,a],X)\cap C^1(I'',X)\cap C(I',X_1)$ and an upper solution $w^{(0)}\in PC([-h,a],X)\cap C^1(I'',X)\cap C(I',X_1)$ with $v^{(0)}\leq w^{(0)}$, and conditions* (H1)–(H3) *are satisfied, then problem* (1.2) *has minimal and maximal mild solution between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$ respectively.*

If we replace assumption (H4) by the assumption

(H5) There exist positive constants $\overline{L}_f$, $\overline{M}_f$ and $\overline{L}_g < \frac{1}{NM}$ such that for any $u, v \in [v^{(0)}, w^{(0)}]$ and $s \in [-h, 0]$,
$$\begin{aligned} &f(t, u(t), u_t) - f(t, v(t), v_t) \\ &\le \overline{M}_f(u(t) - v(t)) + \overline{L}_f(u(t+\tau) - v(t+\tau)), \quad \forall t \in [0, a], \\ &\quad g(u)(s) - g(v)(s) \le \overline{L}_g(u(s) - v(s)), \quad \forall\, s \in [-h, 0], \end{aligned}$$

then we have the following uniqueness result.

Theorem 1.3. *Let X be an order Banach space and its positive cone P be normal. If problem* (1.2) *has a lower solution $v^{(0)} \in PC([-h, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ and an upper solution $w^{(0)} \in PC([-h, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ with $v^{(0)} \le w^{(0)}$, such that conditions* (H1)–(H3), (H5) *hold, then problem* (1.2) *has a unique solution between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ or $w^{(0)}$.*

The proofs of Theorem 1.1 and 1.3 will be shown in the next section.

2. Proof of main results

Throughout this paper, let $A : D(A) \subset X \to X$ be a closed linear operator and let $-A$ generate a positive C_0-semigroup $T(t)$ $(t \ge 0)$ on ordered Banach space X. Then there exist constants $M_1 \ge 1$ and $\delta \in \mathbb{R}$ such that
$$\|T(t)\| \le M_1 e^{\delta t}, \quad t \ge 0. \tag{2.1}$$
Denote $\mathcal{L}(X)$ by the Banach space of all bounded linear operators from X to X equipped with its natural topology. From (2.1) we know that
$$M := \sup_{t \in [0,a]} \|T(t)\|_{\mathcal{L}(X)} \ge 1 \tag{2.2}$$
is a finite number. One can easily to see that for any constant $C \ge 0$, $-(A + CI)$ also generates a positive C_0-semigroup $S(t) = e^{-Ct}T(t)$ $(t \ge 0)$ in X, and
$$\sup_{t \in [0,a]} \|S(t)\|_{\mathcal{L}(X)} = \sup_{t \in [0,a]} \|e^{-Ct}T(t)\|_{\mathcal{L}(X)} = M \ge 1. \tag{2.3}$$

Definition 2.1. A function $u \in PC([-h, a], X)$ is said to be a mild solution of (1.2) if it satisfies the equation
$$u(t) = \begin{cases} g(u)(t) + \phi(t), & t \in [-h, 0], \\ T(t)[g(u)(0) + \phi(0)] + \int_0^t T(t-s) f(s, u(s), u_s) ds & \\ + \sum_{0 < t_k < t} T(t - t_k) I_k(u(t_k)), & t \in [0, a]. \end{cases} \tag{2.4}$$

We use $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness on the bounded set of X. For the details about the definition and properties of the measure of noncompactness, we refer to the monographs by Ayerbe, Domínguez and Lópezwo [2], Banaŝ and Goebel [3], Deimling [12]. The following lemma will be used in proof of Theorem 1.1.

Lemma 2.2 ([20])**.** *Let $D = \{u_n\}_{n=1}^{\infty} \subset PC([0, a], X)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $[0, a]$, and*
$$\alpha\Big(\Big\{ \int_0^a u_n(t) dt \mid n \in \mathbb{N} \Big\}\Big) \le 2 \int_0^a \alpha(D(t)) dt.$$

Proof of Theorem 1.1. It is easy to see that problem (1.2) is equivalent to the following delay evolution equations involving nonlocal and impulsive conditions

$$\begin{gathered} u'(t)+Au(t)+Cu(t)=f(t,u(t),u_t)+Cu(t), \quad t\in[0,a], \ t\neq t_k, \\ u(t_k^+)-u(t_k^-)=I_k(u(t_k)), \quad k=1,2,\dots,m, \\ u(s)=g(u)(s)+\phi(s), \quad s\in[-h,0], \end{gathered} \tag{2.5}$$

for any constant $C>0$. Therefore, we consider $\mathcal{F}:[v^{(0)},w^{(0)}]\to PC([-h,a],X)$ defined by

$$(\mathcal{F}u)(t)=\begin{cases} g(u)(t)+\phi(t), \quad \text{if } t\in[-h,0], \\ S(t)[g(u)(0)+\phi(0)]+\int_0^t S(t-s)[f(s,u(s),u_s)+Cu(s)]ds \\ +\sum_{0<t_k<t} S(t-t_k)I_k(u(t_k)), \quad \text{if } t\in[0,a], \end{cases} \tag{2.6}$$

where $S(t)=e^{-Ct}T(t)$ $(t\geq 0)$ is the positive C_0-semigroup generated by $-(A+CI)$. By Definition 2.1, the mild solution of problem (1.2) is equivalent to the fixed point of $\mathcal{F}$ defined by (2.6).

First, we prove that the operator $\mathcal{F}:[v^{(0)},w^{(0)}]\to PC([-h,a],X)$ defined by (2.6) is continuous. For this purpose, let $\{u^{(n)}\}_{n=1}^\infty\subset[v^{(0)},w^{(0)}]$ be a sequence such that $\lim_{n\to\infty}u^{(n)}=u$ in $[v^{(0)},w^{(0)}]$. Then for any $t\in[0,a]$, $\lim_{n\to\infty}(u^{(n)})_t=u_t$. If $t\in[-h,0]$, by (2.6) and the continuity of the nonlocal function g, we have that

$$\|(\mathcal{F}u^{(n)})(t)-(\mathcal{F}u)(t)\|=\|g(u^{(n)})(t)-g(u)(t)\|\to 0 \quad textas\ n\to\infty, \tag{2.7}$$

and if $t\in[0,a]$, by the Carathéodory continuity of the nonlinear function f, and the continuity of the impulsive function I_k for $k=1,2,\dots,m$, we obtain

$$\lim_{n\to\infty}\|f(s,u^{(n)}(s),(u^{(n)})_s)+Cu^{(n)}(s)-f(s,u(s),u_s)-Cu(s)\|=0 \tag{2.8}$$

a.e. $s\in[0,t]$; and

$$\lim_{n\to\infty}\|I_k(u^{(n)}(t_k))-I_k(u(t_k))\|=0 \quad \text{for } k=1,2,\dots,m. \tag{2.9}$$

Applying assumption (H1), we know that for any $u\in[v^{(0)},w^{(0)}]$ and $s\in[0,t]$, $t\in[0,a]$,

$$\begin{aligned} f(s,v^{(0)}(s),(v^{(0)})_s)+Cv^{(0)}(s) &\leq f(s,u(s),u_s)+Cu(s) \\ &\leq f(s,w^{(0)}(s),(w^{(0)})_s)+Cw^{(0)}(s). \end{aligned}$$

The above inequality combined with the normality of the positive cone P, we know that there exists a constant $C_1>0$, such that

$$\|f(s,u(s),u_s)+Cu(s)\|\leq C_1, \quad s\in[0,t],\ t\in[0,a]. \tag{2.10}$$

By (2.3), (2.6), (2.8)-(2.10) and the Lebesgue dominated convergence theorem, we know that for any $t\in[0,a]$,

$$\begin{aligned} &\|(\mathcal{F}u^{(n)})(t)-(\mathcal{F}u)(t)\| \\ &\leq M\|g(u^{(n)})(0)-g(u)(0)\| \\ &\quad +M\int_0^t\|f(s,u^{(n)}(s),(u^{(n)})_s)+Cu^{(n)}(s)-f(s,u(s),u_s)-Cu(s)\|ds \\ &\quad +M\sum_{0<t_k<t}\|I_k(u^{(n)}(t_k))-I_k(u(t_k))\|\to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{2.11}$$

Hence, from (2.6), (2.8) and (2.11) we obtain

$$\|\mathcal{Q}u^{(n)} - \mathcal{Q}u\|_{PC} \to 0 \quad \text{as } n \to \infty,$$

which means that $\mathcal{F} : [v^{(0)}, w^{(0)}] \to PC([-h, a], X)$ is a continuous operator.

Secondly, we prove that $\mathcal{F}$ maps $[v^{(0)}, w^{(0)}]$ to $[v^{(0)}, w^{(0)}]$ is a monotonic increasing operator. By assumptions (H1)–(H3), $\mathcal{F}$ is increasing in $[v^{(0)}, w^{(0)}]$, and maps any bounded set in $[v^{(0)}, w^{(0)}]$ into a bounded set. Next, we show that $v^{(0)} \leq \mathcal{F}v^{(0)}$ and $\mathcal{F}w^{(0)} \leq w^{(0)}$. Letting

$$h(t) = (v^{(0)})'(t) + Av^{(0)}(t) + Cv^{(0)}(t), \quad t \in [0, a],\ t \neq t_k,\ k = 1, 2, \dots, m.$$

By Definition 2.1, we obtain that $h \in PC([0, a], X)$ and $h(t) \leq f(t, v^{(0)}(t), (v^{(0)})_t) + Cv^{(0)}(t)$ for $t \in [0, a]$. Therefore, by Definitions 2.1, the definition of lower solution and the positivity of the C_0-semigroup $S(t)$ $(t \geq 0)$, we obtain that for $t \in [0, a]$,

$$\begin{aligned}
v^{(0)}(t) &= S(t)v^{(0)}(0) + \int_0^t S(t-s)h(s)ds + \sum_{0<t_k<t} S(t-t_k)[v^{(0)}(t_k^+) - v^{(0)}(t_k^-)] \\
&\leq S(t)[g(v^{(0)})(0) + \varphi(0)] + \int_0^t S(t-s)[f(s, v^{(0)}(s), (v^{(0)})_s) + Mv^{(0)}(s)]ds \\
&\quad + \sum_{0<t_k<t} S(t-t_k)I_k(v^{(0)}(t_k)) \\
&= (\mathcal{F}v^{(0)})(t), \quad t \in [0, a],
\end{aligned}$$

and we know that for $t \in [-h, 0]$,

$$v^{(0)}(t) \leq g(v^{(0)})(t) + \phi(t) = (\mathcal{F}v^{(0)})(t).$$

The two inequalities above imply that $v^{(0)} \leq \mathcal{F}v^{(0)}$. Similarly, it can be shown that $\mathcal{F}w^{(0)} \leq w^{(0)}$. Therefore, $\mathcal{F} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonic increasing operator.

Now, we define two sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ in the ordered interval $[v^{(0)}, w^{(0)}]$ by the following iterative scheme:

$$v^{(n)} = \mathcal{F}v^{(n-1)}, \quad w^{(n)} = \mathcal{F}w^{(n-1)}, \quad n = 1, 2, \dots. \tag{2.12}$$

From the monotonicity of $\mathcal{F}$, it follows that

$$v^{(0)} \leq v^{(1)} \leq v^{(2)} \leq \dots \leq v^{(n)} \leq \dots \leq w^{(n)} \leq \dots \leq w^{(2)} \leq w^{(1)} \leq w^{(0)}. \tag{2.13}$$

In what follows, we prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are convergent on $[-h, a]$. For convenience, let $U = \{v^{(n)} \mid n \in \mathbb{N}\}$ and $U^* = \{v^{(n-1)} \mid n \in \mathbb{N}\}$. Then $U = \mathcal{F}(U^*)$. From $U = U^* \cup \{v^{(0)}\}$ it follows that $\alpha(U(t)) = \alpha(U^*(t))$ for $t \in [-h, a]$. Let

$$\varphi(t) := \alpha(U(t)) = \alpha(U^*(t)), \quad t \in [-h, a].$$

Going from J_0 to J_{m+1} interval by interval we show that $\varphi(t) \equiv 0$ in $[-h, a]$.

For $t \in J_0$, by (2.6) and assumption (H3), we have that

$$\varphi(t) = \alpha(U(t)) = \alpha(\mathcal{F}(U^*)(t)) = \alpha\big\{g(v^{(n-1)})(t) + \phi(t)\big\} = 0.$$

For $t \in J_1$, by (2.6), Lemma 2.2, the assumptions (H3) and (H4), we have

$$\begin{aligned}
\varphi(t) &= \alpha(U(t)) = \alpha(\mathcal{F}(U^*)(t)) \\
&= \alpha\Big(\Big\{S(t)[g(v^{(n-1)})(0) + \phi(0)] \\
&\quad + \int_0^t S(t-s)[f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + Cv^{(n-1)}(s)]\Big\}\Big) \\
&\le 2M \int_0^t \alpha\Big(\Big\{f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + Cv^{(n-1)}(s)\Big\}\Big) ds \\
&\le 2M \int_0^t \Big[L_f \alpha(U^*(s)) + L_f \sup_{-h \le \tau \le 0} \alpha(U^*(s+\tau)) + C\alpha(U^*(s))\Big] ds \\
&= 2M \int_0^t \Big[L_f \varphi(s) + L_f \sup_{-h \le \tau \le 0} \varphi(s+\tau) + C\varphi(s)\Big] ds.
\end{aligned} \tag{2.14}$$

Let $\rho_1(t) = \sup\{\varphi(s) : -h \le s \le t\}$, $t \in J_1$. Then we have for any $t \in J_1$,

$$\varphi(t) \le \rho_1(t), \quad \sup_{-h \le \tau \le 0} \varphi(t+\tau) \le \rho_1(t). \tag{2.15}$$

Combining with (2.14) and (2.15), we obtain that for any $t \in J_1$,

$$\rho_1(t) \le 2M(2L_f + C) \int_0^t \rho_1(s) ds.$$

Hence, by the Gronwall's inequality on gets that $\rho_1(t) \equiv 0$ on J_1. This means that $\varphi(t) \equiv 0$ on J_1. In particular, $\alpha(U(t_1)) = \alpha(U^*(t_1)) = \varphi(t_1) = 0$, this implies that $U(t_1)$ and $U^*(t_1)$ are precompact on X. Thus $I_1(U^*(t_1))$ is precompact on X and $\alpha(I_1(U^*(t_1))) = 0$.

Now, for $t \in J_2$, by (2.6) and the above argument, we have

$$\begin{aligned}
\varphi(t) &= \alpha(U(t)) = \alpha(\mathcal{F}(U^*)(t)) \\
&= \alpha\Big(\{S(t)[g(v^{(n-1)})(0) + \phi(0)]\}\Big) \\
&\quad + \alpha\Big(\Big\{\int_0^t S(t-s)[f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + Cv^{(n-1)}(s)] ds\Big\}\Big) \\
&\quad + \alpha(\{I_1(v^{(n-1)}(t_1))\}) \\
&\le 2M \int_0^t \big[L_f \varphi(s) + L_f \sup_{-h \le \tau \le 0} \varphi(s+\tau) + C\varphi(s)\big] ds + \alpha(I_1(U^*(t_1))) \\
&= 2M \int_{t_1}^t \big[L_f \varphi(s) + L_f \sup_{-h \le \tau \le 0} \varphi(s+\tau) + C\varphi(s)\big] ds.
\end{aligned} \tag{2.16}$$

Let $\rho_2(t) = \sup\{\varphi(s) : -h \le s \le t\}$, $t \in J_2$. Then for any $t \in J_2$, we have

$$\varphi(t) \le \rho_2(t), \quad \sup_{-h \le \tau \le 0} \varphi(t+\tau) \le \rho_2(t). \tag{2.17}$$

Combining with (2.16) and (2.17), we obtain that for any $t \in J_2$,

$$\rho_2(t) \le 2M(2L_f + C) \int_{t_1}^t \rho_2(s) ds.$$

Again by the Gronwall's inequality one gets that $\rho_2(t) \equiv 0$ on J_2. Thus $\varphi(t) \equiv 0$ on J_2, from which we obtain that $\alpha(U^*(t_2)) = 0$ and $\alpha(I_2(U^*(t_2))) = 0$.

Continuing such a process interval by interval up to J_{m+1}, we can prove that $\varphi(t) \equiv 0$ on every J_k $(k = 0, 1, \dots, m+1)$. Thus, $\{v_n(t)\}$ is precompact on X and $\{v^{(n)}(t)\}$ has a convergent subsequence for every $t \in [-h, a]$. Combining this with the monotonicity (2.13), we easily prove that $\{v^{(n)}(t)\}$ itself is convergent for any $t \in [-h, a]$. Using a similar argument with $\{v^{(n)}(t)\}$, we can prove that $\{w^{(n)}(t)\}$ is also convergent for any $t \in [-h, a]$. Set

$$\lim_{n\to\infty} v^{(n)}(t) = \underline{u}(t), \quad \lim_{n\to\infty} w^{(n)}(t) = \overline{u}(t), \quad t \in [-h, a].$$

Evidently, $\{v^{(n)}\} \subset PC([-h, a], X)$, so $\underline{u}$ is bounded and integrable on $[-h, a]$. By (2.6), we have that

$$\begin{aligned} v^{(n)}(t) &= \mathcal{F}(v^{(n-1)})(t) \\ &= \begin{cases} g(v^{(n-1)})(t) + \phi(t), \quad \text{if } t \in [-h, 0], \\ S(t)[g(v^{(n-1)})(0) + \phi(0)] + \int_0^t S(t-s)[f(s, v^{(n-1)}(s), (v^{(n-1)})_s) \\ +Cv^{(n-1)}(s)]ds + \sum_{0<t_k<t} S(t-t_k) I_k(v^{(n-1)}(t_k)), \quad \text{if } t \in [0, a]. \end{cases} \end{aligned}$$

Letting $n \to \infty$ in the above equalities, by the Lebesgue dominated convergent theorem, we have

$$\begin{aligned} \underline{u}(t) &= (\mathcal{F}\underline{u})(t) \\ &= \begin{cases} g(\underline{u})(t) + \phi(t), \quad \text{if } t \in [-h, 0], \\ S(t)[g(\underline{u})(0) + \phi(0)] + \int_0^t S(t-s)[f(s, \underline{u}(s), (\underline{u})_s) + C\underline{u}(s)]ds \\ + \sum_{0<t_k<t} S(t-t_k) I_k(\underline{u}(t_k)), \quad \text{if } t \in [0, a]. \end{cases} \end{aligned}$$

This means that $\underline{u} \in PC([-h, a], X)$ and $\underline{u} = \mathcal{F}\underline{u}$. Similarly, we can prove that $\overline{u} \in PC([-h, a], X)$ and $\overline{u} = \mathcal{F}\overline{u}$. Letting $n \to \infty$ in (2.13), we see that $v^{(0)} \le \underline{u} \le \overline{u} \le w^{(0)}$. By the monotonicity of $\mathcal{F}$, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of $\mathcal{F}$ in $[v^{(0)}, w^{(0)}]$. Therefore, $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of the problem (1.2) in $[v^{(0)}, w^{(0)}]$, and they can be obtained by a monotone iterative procedure (2.12) starting from $v^{(0)}$ and $w^{(0)}$, respectively. This completes the proof. $\square$

Proof of Theorem 1.3. We firstly proof that (H1) and (H5) imply (H4). For this purpose, let $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$ be a increasing sequence. For $m, n \in \mathbb{N}$ with $m > n$, by (H1) and (H5), we obtain that for every $t \in [0, a]$ and $\tau \in [-h, 0]$,

$$\begin{aligned} \theta &\le f(t, u^{(m)}(t), (u^{(m)})_t) - f(t, u^{(n)}(t), (u^{(n)})_t) + C[u^{(m)}(t) - u^{(n)}(t)] \\ &\le (C + \overline{M}_f)[u^{(m)}(t) - u^{(n)}(t)] + \overline{L}_f[u^{(m)}(t+\tau) - u^{(n)}(t+\tau)]. \end{aligned}$$

By this and the normality of cone P, we have that for any $t \in [0, a]$ and some $\tau \in [-h, 0]$,

$$\begin{aligned} &\|f(t, u^{(m)}(t), (u^{(m)})_t) - f(t, u^{(n)}(t), (u^{(n)})_t)\| \\ &\le N(C + \overline{M}_f)\|u^{(m)}(t) - u^{(n)}(t)\| \\ &\quad + N\overline{L}_f\|u^{(m)}(t+\tau) - u^{(n)}(t+\tau)\| + C\|u^{(m)}(t) - u^{(n)}(t)\| \\ &\le [N(C + \overline{M}_f) + C]\|u^{(m)}(t) - u^{(n)}(t)\| \\ &\quad + N\overline{L}_f \sup_{-h\le\tau\le0} \|u^{(m)}(t+\tau) - u^{(n)}(t+\tau)\|. \end{aligned}$$

From this inequality and the definition of the measure of noncompactness, it follows that for every $t \in [0,a]$,

$$\mu\Big(\big\{f(t,u^{(n)}(t),(u^{(n)})_t)\big\}\Big) \le L_f\Big[\mu\Big(\big\{u^{(n)}(t)\big\}\Big) + \sup_{-h\le\tau\le 0}\mu\Big(\big\{u^{(n)}(t+\tau)\big\}\Big)\Big],$$

where $L_f = \max\big\{N(C+\overline{M}_f)+C, N\overline{L}_f\big\}$. If $\{u^{(n)}\}\subset[v^{(0)},w^{(0)}]$ is decreasing, the above inequality is also valid. Hence (H4) holds.

Therefore, by Theorem 1.3, problem (1.2) has minimal mild solution $\underline{u}$ and maximal mild solution $\overline{u}$ in $[v^{(0)},w^{(0)}]$. Next, going from J_0 to J_{m+1} interval by interval we show that $\underline{u}(t)\equiv\overline{u}(t)$ on every J_k $(k=0,1,\dots,m+1)$.

For $t\in J_0$, by (2.6) and assumption (H5), we obtain

$$\theta \le \overline{u}(t)-\underline{u}(t) = \mathcal{F}\overline{u}(t)-\mathcal{F}\underline{u}(t) = g(\overline{u})(t)-g(\underline{u})(t) \le \overline{L}_g(\overline{u}(t)-\underline{u}(t)).$$

Combining this and the normality of cone P, we have

$$\|\overline{u}(t)-\underline{u}(t)\| \le N\overline{L}_g\|\overline{u}(t)-\underline{u}(t)\|, \quad t\in J_0.$$

Hence, for any $t\in J_0$,

$$(1-N\overline{L}_g)\|\overline{u}(t)-\underline{u}(t)\| \le 0.$$

From the above inequality and the assumption $1-N\overline{L}_g>0$, one gets that $\overline{u}(t)\equiv\underline{u}(t)$ on J_0.

For $t\in J_1$, by (2.6) and assumption (H5), we have

$$\begin{aligned}
\theta &\le \overline{u}(t)-\underline{u}(t) = \mathcal{F}\overline{u}(t)-\mathcal{F}\underline{u}(t)\\
&= S(t)[g(\overline{u})(0)-g(\underline{u})(0)]\\
&\quad + \int_0^t S(t-s)[f(s,\overline{u}(s),(\overline{u})_s)-f(s,\underline{u}(s),(\underline{u})_s)+C(\overline{u}(s)-\underline{u}(s))]ds\\
&\le \overline{L}_g S(t)(\overline{u}(0)-\underline{u}(0))\\
&\quad + \int_0^t S(t-s)[(C+\overline{M}_f)(\overline{u}(s)-\underline{u}(s))+\overline{L}_f(\overline{u}(s+\tau)-\underline{u}(s+\tau))]ds.
\end{aligned}$$

Since $\overline{u}(0)=\underline{u}(0)$, using the above inequality and normality of cone P, we can prove that

$$\|\overline{u}(t)-\underline{u}(t)\| \le NM\int_0^t[(C+\overline{M}_f)\|\overline{u}(s)-\underline{u}(s)\|+\overline{L}_f\|\overline{u}(s+\tau)-\underline{u}(s+\tau)\|]ds. \tag{2.18}$$

We define a non-negative function $\rho_1'(t)=\sup\{\|\overline{u}(s)-\underline{u}(s)\| : -h\le s\le t\}$ on J_1. Then, for any $t\in J_1$,

$$\|\overline{u}(t)-\underline{u}(t)\| \le \rho_1'(t), \quad \sup_{-h\le\tau\le 0}\|\overline{u}(t+\tau)-\underline{u}(t+\tau)\| \le \rho_1'(t). \tag{2.19}$$

From (2.18) and (2.19)) it follows that

$$\rho_1'(t) \le NM(C+\overline{M}_f+\overline{L}_f)\int_0^t \rho_1'(s)ds, \quad t\in J_1.$$

By this fact and Gronwall's inequality, we obtain that $\overline{u}(t)\equiv\underline{u}(t)$ on J_1.

For $t \in J_2$, since $I_1(\overline{u}(t_1)) = I_1(\underline{u}(t_1))$, using completely similar argument as above for $t \in J_1$, we can prove that

$$\begin{aligned}
&\|\overline{u}(t) - \underline{u}(t)\| \\
&\le NM \int_0^t [(C + \overline{M}_f)\|\overline{u}(s) - \underline{u}(s)\| + \overline{L}_f \|\overline{u}(s+\tau) - \underline{u}(s+\tau)\|]ds \\
&\le NM \int_{t_1}^t [(C + \overline{M}_f)\|\overline{u}(s) - \underline{u}(s)\| + \overline{L}_f \|\overline{u}(s+\tau) - \underline{u}(s+\tau)\|]ds.
\end{aligned} \tag{2.20}$$

Defining a non-negative function $\rho_2'(t) = \sup\{\|\overline{u}(s) - \underline{u}(s)\| : -h \le s \le t\}$ on J_2. We know that for $t \in J_2$,

$$\|\overline{u}(t) - \underline{u}(t)\| \le \rho_2'(t), \quad \sup_{-h \le \tau \le 0} \|\overline{u}(t+\tau) - \underline{u}(t+\tau)\| \le \rho_2'(t). \tag{2.21}$$

Combining (2.20) and (2.21), we have

$$\rho_2'(t) \le NM(C + \overline{M}_f + \overline{L}_f) \int_{t_1}^t \rho_2'(s)ds, \quad t \in J_2.$$

Again by the Gronwall's inequality, we obtain that $\rho_2'(t) \equiv 0$ on J_2. Hence, $\overline{u}(t) \equiv \underline{u}(t)$ on J_2.

Continuing such a process interval by interval up to J_{m+1}, we see that $\overline{u}(t) \equiv \underline{u}(t)$ over the whole of $[-h, a]$. Therefore, $\widetilde{u} := \overline{u} = \underline{u}$ is the unique mild solution of problem (1.2) in $[v^{(0)}, w^{(0)}]$, which can be obtained by the monotone iterative procedure (2.13) starting form $v^{(0)}$ or $w^{(0)}$. This completes the proof. $\square$

3. Example

In this section, we give an example to illustrate the applicability of our abstract results obtained in Section 2. We consider the delay parabolic partial differential equation involving nonlocal and impulsive conditions of the form

$$\begin{gathered}
\frac{\partial}{\partial t} w(x,t) - \iota \frac{\partial^2}{\partial x^2} w(x,t) = L\big(\frac{|w(x,t)|}{1 + |w(x,t)|}\big) + \int_{-h}^0 G(s) w(x, t+s) ds, \\
x \in [c,d],\ t \in [0,a],\ t \ne t_k, \\
w(x, t_k^+) = w(x, t_k^-) + \frac{\sqrt{|w(x,t_k)|}}{1 + |w(x,t_k)|}, \quad x \in [c,d],\ k = 1, 2, \dots, m, \\
]w(c,t) = w(d,t) = 0, \quad t \in [0,a], \\
w(x,s) = \int_0^a \rho(s,t) \lg(1 + |w(x,t)|) dt + \phi(x,s),\ x \in [c,d],\ s \in [-h, 0],
\end{gathered} \tag{3.1}$$

where $\iota > 0$ is the coefficient of heat conduction, $a, h, L > 0$ are all constants, $0 < t_1 < t_2 < \dots < t_m < a$, $G \in L([-h,0], \mathbb{R}^+)$, $\rho(s,t)$ is a continuous function from $[-h,0] \times [0,a]$ to $\mathbb{R}^+$, $\phi \in C([c,d] \times [-h,0], \mathbb{R}^+)$.

Let $X = L^2([c,d], \mathbb{R})$ with the norm $\|\cdot\|_2$ and let $P = \{v \in L^2([c,d], \mathbb{R}) : v(x) \ge 0 \text{ a.e. } x \in [c,d]\}$. Then X is a Banach space, P is a normal cone of X with normal constant $N = 1$. Define an operator $A : D(A) \subset X \to X$ by

$$Aw = -\iota \frac{\partial^2}{\partial x^2} w, \quad w \in D(A).$$

The domain $D(A)$ is defined by

$$D(A) = H^2(c,d) \cap H_0^1(c,d).$$

It is well know that A has discrete spectrum with eigenvalues $\lambda_n = \iota n^2\pi^2(d-c)^2$ $(n \in \mathbb{N})$ and the corresponding normalized eigenvectors are $e_n(x) = \sqrt{2/z}\sin n\pi x(d-c)$, $z = \sqrt{d-c+(\sin 2n\pi c - \sin 2n\pi d)/(2n\pi)}$, the set $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and

$$Aw = \sum_{n=1}^{\infty}(w, e_n)e_n, \quad w \in D(A).$$

Furthermore, $-A$ generates a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X, which is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-\frac{\iota n^2\pi^2 t}{(d-c)^2}}(w, e_n)e_n, \quad w \in X,\ t > 0.$$

and $\|T(t)\| \leq e^{-\frac{\iota \pi^2 t}{(d-c)^2}}$, for any $t \geq 0$. Let

$$\begin{gathered}
u(t) = w(\cdot, t), \quad t \in [-h, a], \\
f(t, u(t), u_t) = L\Big(\frac{|w(\cdot,t)|}{1+|w(\cdot,t)|}\Big) + \int_{-h}^{0} G(s)w(\cdot, t+s)ds, \quad t \in [0, a], \\
I_k(u(t_k)) = \frac{\sqrt{|w(x,t_k)|}}{1+|w(x,t_k)|}, \quad k = 1, 2, \dots, m, \\
g(u)(s) = \int_0^a \rho(s,t)\lg(1+|w(\cdot,t)|)dt, \quad \phi(s) = \phi(\cdot, s), \quad s \in [-h, 0].
\end{gathered}$$

Then the delay parabolic partial differential equation involving nonlocal and impulsive conditions (3.1) can be transformed into the abstract form of problem (1.2).

Theorem 3.1. *Assume that there exists a function $v = v(x,t) \in PC([c,d] \times [-h,a], \mathbb{R}) \cap C^1([0,1] \times I'', \mathbb{R})$ such that*

$$\begin{gathered}
\frac{\partial}{\partial t}v(x,t) - \iota\frac{\partial^2}{\partial x^2}v(x,t) \geq L\big(\frac{|v(x,t)|}{1+|v(x,t)|}\big) + \int_{-h}^{0} G(s)w(v, t+s)ds, \\
t \in [0, a],\ t \neq t_k, \\
v(x, t_k^+) \geq v(x, t_k^-) + \frac{\sqrt{|w(x,t_k)|}}{1+|w(x,t_k)|}, \quad k = 1, 2, \dots, m, \\
v(0,t) = v(1,t) = 0, \quad t \in [0, a], \\
v(x,s) \geq \int_0^a \rho(s,t)\lg(1+|v(x,t)|)dt + \phi(x,s), \quad s \in [-h, 0].
\end{gathered}$$

Then the delay parabolic partial differential equation involving nonlocal and impulsive conditions (3.1) *exist a minimal mild solution and a maximal mild solution between* 0 *and $v(x,t)$, which can be obtained by a monotone iterative procedure starting from* 0 *and $v(x,t)$, respectively.*

Proof. From the assumption and the definition of nonlinear term f, impulsive function I_k for $k = 1, 2, \dots, m$ and nonlocal function g, we can verify that that $v^{(0)} = 0$ and $w^{(0)} = v(x,t)$ are the lower and the upper solutions of the delay parabolic partial differential equation involving nonlocal and impulsive conditions (3.1) respectively. Furthermore, the conditions (H1), (H2) and (H3) are satisfied with $C = \frac{1}{2}$ and the nonlocal function $g : \mathcal{PC}([-h,a], X) \to \mathcal{B}$ is completely continuous.

On the other hand, from the definition of nonlinear term f, we know that

$$\theta \leq f(t,v_2,\varphi_2)-f(t,v_1,\varphi_1) \leq L\Big(\frac{|v_2|}{1+|v_2|}-\frac{|v_1|}{1+|v_1|}\Big)+\int_{-h}^{0} G(s)(\varphi_2(s)-\varphi_1(s))ds,$$

for all $t \in [0,a]$, $v_1, v_2 \in X$ and $\varphi_1, \varphi_2 \in \mathcal{B}$ with $\theta \leq \varphi_1 \leq \varphi_2$. Combining this and the normality of cone P, we have

$$\begin{aligned}
&\|f(t,v_2,\varphi_2)-f(t,v_1,\varphi_1)\|_2\\
&\leq L(\|v_2-v_1\|_2)+\int_{-h}^{0} G(s)\|\varphi_2(s)-\varphi_1(s)\|_2 ds\\
&\leq L(\|v_2-v_1\|_2)+\int_{-h}^{0} G(s)ds \sup_{-h\leq s\leq 0}\|\varphi_2(s)-\varphi_1(s)\|_2.
\end{aligned}$$

Then for every $t \in [0,a]$,

$$\begin{aligned}
&\mu\Big(\Big\{f(t,u^{(n)}(t),(u^{(n)})_t)\Big\}\Big)\\
&\leq \max\Big\{L, \int_{-h}^{0} G(s)ds\Big\}\Big[\mu\big(\{u^{(n)}(t)\}\big)+\sup_{-h\leq s\leq 0}\mu\big(\{u^{(n)}(t+s)\}\big)\Big],
\end{aligned}$$

where $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$ is countable and increasing or decreasing monotonic set. This means that the condition (H4) holds with $L_f = \max\big\{L, \int_{-h}^{0} G(s)ds\big\}$. Therefore, our conclusion follows from Theorem 1.1. This completes the proof. $\square$

Acknowledgments. The authors would like to express sincere thanks to the anonymous referee for his/her carefully reading the manuscript and valuable comments and suggestions. This research was supported by the NSF of Gansu Province (No. 20JR5RA522), NNSF of China (No. 12061063), Science Research Project for Colleges and Universities of Gansu Province (No. 2019B-047), Project of NWNU-LKQN2019-13 and Project of NWNU-LKQN2019-3.

References

[1] S. Antontsev, J. Ferreira, E. Piskin, H. Yuksekkaya, M. Shahrouzi; *Blow up and asymptotic behavior of solutions for a $p(x)$-Laplacian equation with delay term and variable exponents*, Electron. J. Differential Equations, 2021 no. 84 (2021), 1–20.

[2] J. M. Ayerbe, T. Domínguez, G. López; *Measures of Noncompactness in Metric Fixed Point Theory*, Adv. Appl., vol. 99, Birkhäuser Verlag, Basilea, 1997.

[3] J. Banaś, K. Goebel; *Measures of Noncompactness in Banach Spaces*, In Lecture Notes in Pure and Applied Mathematics, Volume 60, Marcel Dekker, New York, 1980.

[4] K. Balachandran, S. Kiruthika, J. J. Trujillo; *On fractional impulsive equations of Sobolev type with nonlocal condition in Banach spaces*, Comput. Math. Appl., 62 (2011), 1157–1165.

[5] L. Byszewski; *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., 162 (1991), 494–505.

[6] L. Byszewski; *Application of preperties of the right hand sides of evolution equations to an investigation of nonlocal evolution problems*, Nonlinear Anal., 33 (1998), 413–426.

[7] Y. K. Chang, V. Kavitha, M. Mallike Arjunan; *Existence results for impulsive neutral differential and integrodifferential equations with nonlocal conditions via fractional operators*, Nonlinear Anal.: Hybrid Syst., 4 (2010), 32–43.

[8] P. Chen, Y. Li; *Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions*, Results Math., 63 (2013), 731–744.

[9] P. Chen, Y. Li; *Monotone iterative method for abstract impulsive integro-differential equations with nonlocal conditions in Banach spaces*, Appl. Math., 59 (2014), 99–120.

[10] N. M. Chuong, T. D. Ke; *Generalized Cauchy problems involving nonlocal and impulsive conditions*, J. Evol. Equ., 12 (2012), 367–392.

[11] A. Debbouche, D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems*, Comput. Math. Appl., 62 (2011), 1442–1450.

[12] K. Deimling; *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[13] Q. X. Dong, G. Li; *Measure of noncompactness and semilinear nonlocal functional differential equations in Banach spaces*, Acta Math. Sin. (Engl. Ser.), 31 (2015) 140–150.

[14] S.W. Du, V. Lakshmikantham, *Monotone iterative technique for differential equations in Banach spaces*, J. Math. Anal. Appl., 87 (1982), 454–459.

[15] K. Ezzinbi, X. Fu, K. Hilal; *Existence and regularity in the α-norm for some neutral partial differential equations with nonlocal conditions*, Nonlinear Anal., 67 (2007), 1613–1622.

[16] S. Ji, G. Li, M. Wang; *Controllability of impulsive differential systems with nonlocal conditions*, Appl. Math. Comput., 217 (2011), 6981–6989.

[17] Z. Fan, G. Li; *Existence results for semilinear differential equations with nonlocal and impulsive conditions*, J. Functional Anal., 258 (2010), 1709–1727.

[18] X. Fu; *Approximate controllability of semilinear non-autonomous evolution systems with state-dependent delay*, Evol. Equ. Control Theory, 6 (2017): 517–534.

[19] D. Guo, X. Liu; *Extremal solutions of nonlinear impulsive integro-differential equations in Banach spaces*, J. Math. Anal. Appl., 177 (1993), 538–552.

[20] H.P. Heinz; *On the behaviour of measure of noncompactness with respect to differentiation and integration of rector-valued functions*, Nonlinear Anal. 7, (1983), 1351–1371.

[21] Y. Li; *Existence and asymptotic stability of periodic solution for evolution equations with delays*, J. Functional Anal., 261 (2011), 1309–1324.

[22] K. Li, J. Jia; *Existence and uniqueness of mild solutions for abstract delay fractional differential equations*, Comput. Math. Appl., 62 (2011), 1398–1404.

[23] J. Liang, T. J. Xiao; *Solvability of the Cauchy problem for infinite delay equations*, Nonlinear Anal., 58 (2004), 271–297.

[24] J. Liang, J. Liu, T. J. Xiao; *Nonlocal impulsive problems for integrodifferential equations*, Math. Comput. Modelling, 49 (2009), 798–804.

[25] C. C. Travis, G. F. Webb; *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. 200 (1974) 395–418.

[26] I. I. Vrabie; *Existence in the large for nonlinear delay evolution inclusions with nonlocal initial conditions*, J. Functional Anal., 262 (2012), 1363-1391.

[27] I. I. Vrabie; *Delay evolution equations with mixed nonlocal plus local initial conditions*, Commun. Contemp. Math., 17(2015), 1350035, 22 pp.

[28] J. Wang, Y. Zhou; *Existence of mild solutions for fractional delay evolutions systems*, Appl. Math. Comput., 218 (2011), 357–367.

[29] R. N. Wang, T. J. Xiao, J. Liang; *A note on the fractional Cauchy problems with nonlocal initial conditions*, Appl. Math. Lett., 24 (2011), 1435–1442.

[30] X. Zhang, Y. Li, *Fractional retarded evolution equations with measure of noncompactness subjected to mixed nonlocal plus local initial conditions*, Int. J. Nonlinear Sci. Numer. Simul., 19 (2018), 69–81.

Xuping Zhang

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Email address: `lanyu9986@126.com`