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Upper Bounds for the Complex Growth Rate of a Disturbance in Ferrothermohaline Convection

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Abstract: It is proved analytically that the complex growth rate $\sigma = \sigma_r + i\sigma_i$ (σ_r and σ_i are the real and imaginary parts of σ, respectively) of an arbitrary oscillatory motion of neutral or growing amplitude in ferrothermohaline convection in a ferrofluid layer for the case of free boundaries is located inside a semicircle in the right half of the $\sigma_r\sigma_i$-plane, whose center is at the origin and

$$\text{radius} = \sqrt{\frac{R_s\left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right]}{P_r'}},$$

where R_s is the concentration Rayleigh number, P_r' is the solutal Prandtl number, M_1' is the ratio of magnetic flux due to concentration fluctuation to the gravitational force, and M_5 is the ratio of concentration effect on magnetic field to pyromagnetic coefficient. Further, bounds for the case of rigid boundaries are also derived separately.

Keywords: Linear stability; Ferrofluid; Oscillatory motions; Ferrothermohaline convection.

1 Introduction

Ferrofluids, also known as magnetic fluids, are colloidal suspensions of nano-sized ferromagnetic particles stably dispersed in a carrier liquid. For most applications, it is absolutely essential that the ferrofluids must be very stable with regard to temperature and in the presence of magnetic field. The agglomeration of particles is avoided by some surfactant coating. Ferrofluids have wide range of practical applications, which include treatment of ulcers and brain tumors, destroying cancer cells, sealing of computer hard disc drives, cooling down of loudspeakers, noiseless jet printing system, etc. (Rosensweig [18], Odenbach [7, 8]).

The study of thermal convection in ferrofluids has gained much importance in recent decades. Finlayson [2] studied the convective instability of ferromagnetic fluids and explained the concept of thermomechanical interactions in ferrofluids. Lalas and Carmi [5] investigated the thermoconvective stability of ferrofluids without considering buoyancy effects. Rosensweig et al. [17] investigated experimentally the penetration of ferrofluids in a Hele-Shaw cell. For further details on the subject of ferroconvection, one may refer to Sekar et al. [20,21], Sekar and Vaidyanathan [19], Gupta and Gupta [3], Shliomis [26], Vaidyanathan et al. [29], Rahman and Suslov [16], Nataraj and Bhavya [6], Prakash [9,10,12], and Prakash et al. [15].

These researchers have performed their analysis by considering ferroconvection as a single diffusive system with heat as an only diffusive component. Since ferrofluids are mostly suspensions of magnetic salts in an organic carrier, it is equally important to study the convective instability in double diffusive systems, which is also known as ferrothermohaline convection configurations. Several researchers have contributed to the development of this problem. Vaidyanathan et al. [30,31] analyzed the ferrothermohaline instability problem in porous and nonporous medium, respectively, for stationary as well as oscillatory modes by using linear stability theory. Sekar and Raju [24] studied the effect of sparse distribution pores in thermohaline convection in a micropolar ferromagnetic fluid. Sunil et al. [27] investigated thermosolutal convection in a ferrofluid layer heated and soluted from below in the presence of uniform vertical magnetic field and obtained exact solutions for the case of two free boundaries. Sekar et al. [22] have analyzed ferrothermohaline convection in a rotating medium heated from below and salted from above and have shown that stationary mode of convection is more favorable in comparison to oscillatory mode of

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convection. The effect of rotation on ferromagnetic fluid heated and soluted from below saturating a porous medium was investigated by Sunil et al. [28]. Sekar et al. [23] performed a linear analytical study of Soret-driven ferrothermohaline convection in an anisotropic porous medium. Sekar and Murugan [25] studied the stability analysis of ferrothermohaline convection in a Darcy porous medium with Soret and magnetic field–dependent viscosity effects.

Since for a double diffusive ferroconvection problem, the exact solutions in closed form are not possible for the cases where at least one of the boundaries is rigid, in order to facilitate the experimentalists and numerical analysts with better estimates of the complex growth rate of an arbitrary oscillatory motion of neutral or growing amplitude, the problem of obtaining its upper bounds has its own importance. Initially, Banerjee et al. [1] and Gupta et al. [4] had derived the bounds for the complex growth rate of arbitrary oscillatory perturbations in some thermohaline convection problems. Later, this problem was extended to triply diffusive convection by Prakash et al. [13]. Recently, Prakash [9, 10] has also derived the upper bounds for the complex growth rates in some ferromagnetic convection problems in porous/nonporous medium. Prakash and Gupta [11] have extended his work to ferromagnetic convection with rotation and magnetic field–dependent viscosity. Recently, Prakash et al. [14] also derived the upper bounds for complex growth rates in ferromagnetic convection in a rotating porous medium.

In the present communication, as a further step, we have derived the upper bounds for the complex growth rate of a disturbance in ferrothermohaline convection in a ferrofluid layer heated and soluted from below in the presence of a uniform vertical magnetic field by using linear stability theory.

2 Mathematical Formulation of the Problem

A ferromagnetic Boussinesq fluid layer of infinite horizontal extension and finite vertical depth, heated and salted from below, has been considered. The lower (z=0) and upper (z=d) boundaries are, respectively, maintained at temperatures T_0 and T_1 (<T_0) and concentrations C_0 and C_1 (<C_0). A uniform magnetic field H acts along the vertical direction, which is taken as the z-axis (see Figure 1).

The mathematical equations governing the flow of the ferromagnetic fluid for the above model were given by Sunil et al. [27].

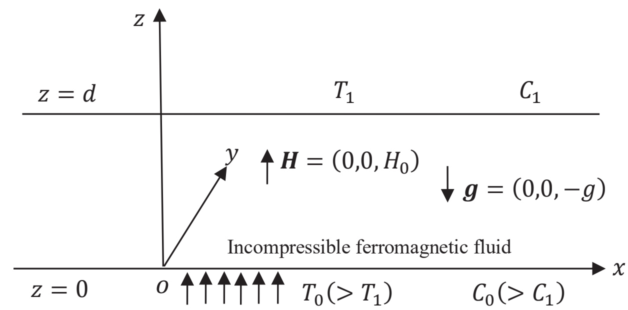

Figure 1: Geometrical configuration of the problem.

$$\nabla . \boldsymbol{q} = 0, \tag{1}$$

$$\rho_0 \frac{D\boldsymbol{q}}{Dt} = -\nabla p + \rho \boldsymbol{g} + \nabla . (\boldsymbol{HB}) + \mu \nabla^2 \boldsymbol{q}, \tag{2}$$

$$\left[\rho_0 \, C_{V,H} - \mu_0 \, \boldsymbol{H}.\left(\frac{\partial \boldsymbol{M}}{\partial T}\right)_{V,\boldsymbol{H}}\right] \frac{DT}{Dt} + \\ + \mu_0 T \left(\frac{\partial \boldsymbol{M}}{\partial T}\right)_{V,\boldsymbol{H}} . \frac{D\boldsymbol{H}}{Dt} = K_1 \nabla^2 T + \Phi_T, \tag{3}$$

$$\left[\rho_0 \, C_{V,H} - \mu_0 \, \boldsymbol{H}.\left(\frac{\partial \boldsymbol{M}}{\partial C}\right)_{V,\boldsymbol{H}}\right] \frac{DC}{Dt} + \\ + \mu_0 C \left(\frac{\partial \boldsymbol{M}}{\partial C}\right)_{V,\boldsymbol{H}} . \frac{D\boldsymbol{H}}{Dt} = K'_1 \nabla^2 C + \Phi_C, \tag{4}$$

where $\boldsymbol{q}$, t, p, $\boldsymbol{H}$, $\boldsymbol{B}$, μ, $\boldsymbol{g}$= (0,0-g) denote the velocity, time, pressure, magnetic field, magnetic induction, coefficient of viscosity, and acceleration due to gravity, respectively. $C_{V,H}$ is the heat capacity at constant volume and magnetic field, μ_0 is the magnetic permeability, T is the temperature, C is the solute concentration, $\boldsymbol{M}$ is magnetization, K_1 is thermal conductivity, K'_1 is the solute conductivity, and Φ_T and Φ_C are the viscous dissipation containing second-order terms in velocity. Φ_T and Φ_C, being small of second order, may be neglected.

The equation of state is given by

$$\rho = \rho_0 [1 - \alpha (T - T_0) + \alpha' (C - C_0)], \tag{5}$$

where ρ is the fluid density, ρ_0 is the reference density, α is the coefficient of volume expansion, and α' is an analogous solvent coefficient of expansion.

In Eq. (2), the viscosity is assumed to be isotropic and independent of the magnetic field.

Maxwell's equations, for a nonconducting fluid, with no displacement currents, are given by

$$\nabla . \boldsymbol{B} = 0, \tag{6a}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{0}. \tag{6b}$$

Further, the relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ is expressed as

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}). \tag{7}$$

It is assumed that magnetization is aligned with the magnetic field intensity and depends on the magnitude of magnetic field, temperature, and salinity, so that

$$\boldsymbol{M} = \frac{\boldsymbol{H}}{H} M(H, T, C), \tag{8}$$

and the linearized magnetic equation of state is given by

$$M = M_0 + \chi(H - H_0) - K_2(T - T_0) + K_3\,(C - C_0). \tag{9}$$

In the above equation, $M_0 = M(H_0, T_0, C_0)$ is magnetization when the magnetic field is H_0, temperature is T_0, and the concentration is C_0. $\chi = (\partial M/\partial C)_{H_0,T_0}$ is magnetic susceptibility, $K_2 = (\partial M/\partial C)_{H_0,T_0}$ is the pyromagnetic coefficient, $K_3 = (\partial M/\partial C)_{H_0,C_0}$ is the salinity magnetic coefficient, $\boldsymbol{H}$ is the magnitude of $\boldsymbol{H}$, and $\boldsymbol{M}$ is the magnitude of $\boldsymbol{M}$.

The basic state is assumed to be static and is given by

$$\boldsymbol{q} = \boldsymbol{q}_b = \boldsymbol{0}, p = p_b(z), \rho = \rho_b(z), T =$$
$$= T_b(z) = -\beta z + T_0, C = C_b(z) = -\beta' z + C_0,$$
$$\beta = \frac{T_0 - T_1}{d}, \beta' = \frac{C_0 - C_1}{d}, \boldsymbol{H}_b = \tag{10}$$
$$= \left[H_0 - \frac{K_2 \beta z}{1+\chi} + \frac{K_3 \beta' z}{1+\chi}\right]\hat{\boldsymbol{k}}, \boldsymbol{M}_b = \left[M_0 + \frac{K_2\,\beta z}{1+\chi} - \frac{K_3 \beta' z}{1+\chi}\right]\hat{\boldsymbol{k}},$$
$$H_0 + M_0 = H_0^{\text{ext}},$$

where $\hat{k}$ is the unit vector in the z direction.

Only the spatially varying parts of H_0 and M_0 contribute to the analysis, so that the direction of the external magnetic field is unimportant and the convection is the same whether the external magnetic field is parallel or antiparallel to the gravitational force (Finlayson [2]).

Now, the stability of the system is analyzed by perturbing the basic state. The perturbed state is given by

$$\boldsymbol{q} = \boldsymbol{q}_b + \boldsymbol{q}', \rho = \rho_b(z) + \rho', p = p_b(z) +$$
$$+ p', T = T_b(z) + \theta', C = C_b(z) + \phi', \tag{11}$$
$$\boldsymbol{H} = \boldsymbol{H}_b(z) + \boldsymbol{H}', \boldsymbol{M} = \boldsymbol{M}_b(z) + \boldsymbol{M}',$$

where $q' = (u', v', w')$, ρ', p', θ', ϕ',$\boldsymbol{H}'$, and $\boldsymbol{M}'$ are infinitesimal perturbations in velocity, density, pressure, temperature, concentration, magnetic field intensity, and magnetization. Using Eq. (11) into Eqs (1)–(9) and using the basic state solutions, we obtain the following linearized perturbation equations:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{12}$$

$$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \mu_0(M_0 + H_0)\frac{\partial H_1'}{\partial z} + \mu \nabla^2 u', \tag{13}$$

$$\rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial y} + \mu_0(M_0 + H_0)\frac{\partial H_2'}{\partial z} + \mu \nabla^2 v', \tag{14}$$

$$\rho_0 \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} + \mu_0(M_0 + H_0)\frac{\partial H_3'}{\partial z} + \mu \nabla^2 w' -$$
$$- \frac{\mu_0 K_2\,\beta}{(1+\chi)}(H_3'(1+\chi) - k_2 \theta') + \tag{15}$$
$$+ \frac{\mu_0 k_3 \beta'}{(1+\chi)}(H_3'(1+\chi) + k_3 \phi') - \frac{\mu_0 k_2 k_3}{(1+\chi)}(\beta' \theta' + \beta \phi') +$$
$$\rho_0 g(\alpha \theta' - \alpha' \phi'),$$

$$\rho C_1 \frac{\partial \theta'}{\partial t} - \mu_0\, T_{0.} K_2 \frac{\partial}{\partial t}\left(\frac{\partial \Phi_1'}{\partial z}\right) =$$
$$= K_1 \nabla^2 \theta' + \left(\rho C_1 \beta - \frac{\mu_0 T_0 K_2{}^2 \beta}{1+\chi}\right) w', \tag{16}$$

where $\rho C_{1.} = \rho_0\, C_{V,H} + \mu_0\, K_2\, H_0$, (17)

$$\rho C_2 \frac{\partial \phi'}{\partial t} - \mu_0\, C_{0.} K_3 \frac{\partial}{\partial t}\left(\frac{\partial \Phi_2'}{\partial z}\right) = K_1' \nabla^2 \phi' +$$
$$+ \left(\rho C_2\, \beta' - \frac{\mu_0 C_0 K_3{}^2 \beta'}{1+\chi}\right) w', \tag{18}$$

where $\rho C_{2.} = \rho_0\, C_{V,H} + \mu_0\, K_3\, H_0$, and (19)

$$H_3' + M_3' = (1+\chi)\, H_3' - K_2\, \theta', H_3' +$$
$$+ M_3' =.(1+\chi)\, H_3' + K_3\, \phi', \tag{20}$$
$$H_i' + M_i' = \left(1 + \frac{M_0}{H_0}\right) H_i'\ (i = 1, 2),$$

where we have assumed $K_2\beta' d \ll (1+\chi)H_0$, $K_3\beta' d \ll (1+\chi)H_0$. Eq. (6b) means that we can write $\boldsymbol{H}=\nabla(\Phi_1'-\Phi_2')$, where Φ_1' is the perturbation magnetic scalar potential and Φ_2' is the perturbation magnetic scalar potential analogous to solute.

Now, following Finlayson [2] and Sunil et al. [27] and using the normal mode technique by assuming to all quantities describing the perturbation a dependence on x, y, and t of the form

$$(w', \theta', \phi', \Phi_1', \Phi_2')(x, y, z, t) = \\ = [w''(z), \theta''(z), \phi''(z), .\Phi_1''(z), \\ \Phi_2''(z).]exp\big[i\big(k_x x + k_y y\big).+.nt\big], \tag{21}$$

where k_x and k_y are the wave numb[illegible] x and y directions, respectively, [illegible] $k = \sqrt{k_x^2 + k_y^2}$ is the resultant wave number, n is a complex constant in general, and nondimentionalizing the variables by setting

$$z_* = \frac{z}{d}, w_* = \frac{d}{\nu}w'', a = kd, D_* = d\frac{d}{dz}, \phi_* = \\ = \frac{K_1' a {R_s}^{1/2}}{(\rho C_2)\beta' \nu d}\phi'', \theta_* = \frac{K_1 a R^{1/2}}{(\rho C_1)\beta \nu\, d}\theta'', {\Phi_1}_* = \\ = \frac{(1+\chi)K_1 a R^{1/2}}{K_2(\rho C_1)\beta\nu\, d^2}\Phi_1'', {\Phi_2}_* = \frac{(1+\chi)K_1' a {R_s}^{1/2}}{K_{3(\rho C_2)}\beta'\nu\, d^2}\Phi_2'', \nu = \frac{\mu}{\rho_0}, P_r' = \\ P_r' = \frac{\nu\rho C_2}{K_1'}, \; P_r = \frac{\nu \rho C_1}{K_1}, \; R = \frac{g\alpha\beta d^4 \rho C_1}{K_1 \nu}, \; R_s = \\ = \frac{g\alpha'\beta' d^4 \rho C_2}{K_1'\nu}, M_1 = \frac{\mu_0 {K_2}^2 \beta}{(1+\chi)\,\alpha\rho_0 g}, M_1' = \frac{\mu_0 {K_3}^2 \beta'.}{(1+\chi)\alpha'\rho_0 g}, \\ M_2 = \frac{\mu_0 T_0 {K_2}^2}{(1+\chi)\rho C_1}, M_2' = \frac{\mu_0 C_0 {K_3}^2}{(1+\chi)\rho C_2}', M_3 = \frac{1+\frac{M_0}{H_0}}{(1+\chi)}, \\ M_4 = \frac{\mu_0 K_2 K_3\, \beta'}{(1+\chi)\,\alpha\rho_0 g}, M_4' = \frac{\mu_0 K_2 K_3\, \beta}{(1+\chi)\,\alpha'\rho_0 g},$$

$$M_5 = \frac{M_4}{M_1} = \frac{M_1'}{M_4'} = \frac{K_3\beta'}{K_2\beta}, \sigma = \frac{nd^2}{\nu}, \tag{22}$$

we obtain the following nondimensional equations (dropping the asterisks for convenience):

$$(D^2 - a^2)(D^2 - a^2 - \sigma)w = aR^{1/2}[\,(1 + M_1 - M_4)\theta - \\ -(M_1 - M_4)D\Phi_1\,] - a{R_s}^{1/2}[(1 - M_1' + M_4')\phi - \\ -(M_4' - M_1')D\Phi_2], \tag{23}$$

$$(D^2 - a^2 - \sigma P_r)\theta = -(1 - M_2)aR^{1/2}w - P_r\, M_2 \sigma D\Phi_1, \tag{24}$$

$$(D^2 - a^2 - \sigma P_r')\phi = -(1 - M_2')a{R_s}^{1/2}w - P_r' M_2' \sigma D\Phi_2, \tag{25}$$

$$(D^2 - a^2 M_3)\Phi_1 = D\theta, \text{ and} \tag{26}$$

$$(D^2 - a^2 M_3)\Phi_2 = D\phi. \tag{27}$$

In the above equations, z is a real independent variable such that $0 \le z \le 1$, D is differentiation with respect to z, a^2 is square of the wave number, $P_r > 0$ is Prandtl number, $P_r' > 0$ is Prandtl number analogous to the solute, σ is the complex growth rate, $R > 0$ is thermal Rayleigh number, $R_s > 0$ is the concentration Rayleigh number, $M_1 > 0$ is the ratio of magnetic force due to temperature fluctuation to the gravitational force, $M_2 > 0$ is the ratio of thermal flux due to magnetization to magnetic flux, $M_1' > 0$ is the ratio of magnetic flux due to concentration fluctuation to the gravitational force, $M_2' > 0$ is the ratio of mass flux due to magnetization to magnetic flux, $M_4 > 0$ and $M_4' > 0$ are nondimensional parameters, $M_5 > 0$ is the ratio of concentration effect on magnetic field to pyromagnetic coefficient, $M_3 > 0$ is the measure of nonlinearity of magnetization, $\sigma = \sigma_r + i\sigma_i$ is a complex constant in general, such that σ_r and σ_i are real constants, and as a consequence, the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$, $\Phi(z) = \Phi_r(z) + i\Phi_i(z)$, and $\Phi_1(z) = \Phi_{1r}(z) + i\,\Phi_{1i}(z)$ are the complex valued functions of the real variable z, such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z)$, $\phi_i(z)$, $\Phi_{1r}(z)$, $\Phi_{1i}(z)$, $\Phi_{2r}(z)$, and $\Phi_{2i}(z)$ are the real valued functions of the real variable z.

Since M_2 and M_2' are of very small order (Finlayson [2]), they are neglected in the subsequent analysis, and therefore, Eqs (24) and (25) takes the forms

$$(D^2 - a^2 - \sigma P_r)\theta = -aR^{1/2}w \text{ and} \tag{28}$$

$$(D^2 - a^2 - \sigma P_r')\phi = -a{R_s}^{1/2}w, \tag{29}$$

respectively.

The boundary conditions are given by

$$w = 0 = \theta = \phi = D^2 w = D\Phi_1 = D\Phi_2 \\ \text{at } z = 0 \text{ and } z = 1 \tag{30}$$

(both the boundaries are free)

$$\text{or } w = 0 = \theta = \phi = Dw = \Phi_1 = \Phi_2 \quad \text{at } z = 0 \text{ and } z = 1 \tag{31}$$

(both the boundaries are rigid).

It may further be noted that Eqs (23) and (26)-(31) describe an eigenvalue problem for σ and govern thermosolutal ferromagnetic convection in ferrofluid layer heated and salted from below.

3 Mathematical Analysis

We now derive the upper bounds for the complex growth rate of the arbitrary oscillatory motions of neutral or growing amplitude for the cases of free and rigid boundaries separately, respectively, in the form of following theorems:

Theorem 1: If $R>0$, $R_s>0$, $M_1'>0$, $1-(1/M_5)<0$, $P_r'>0$, $\sigma_r \geq 0$, and $\sigma_i \neq 0$, then a necessary condition for the existence of a nontrivial solution (w, θ, ϕ, Φ_1, Φ_2, σ) of Eqs (23) and (26)–(29) together with the boundary conditions in Eq. (30) is that

$$|\sigma| < \sqrt{\frac{R_s\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]}{P_r'}}.$$

Proof: Multiplying Eq. (23) by w^* (the superscript * here denotes the complex conjugation) throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-\sigma)w\,dz = = aR^{1/2}(1+M_1-M_4)\int_0^1 w^*\,\theta\,dz - - aR^{1/2}(M_1-M_4)\int_0^1 w^*\,D\Phi_1 dz - aR_s^{1/2}(1-M_1'+M_4')\int_0^1 w^*\,\phi\,dz + aR_s^{1/2}(M_4'-M_1')\int_0^1 w^*\,D\Phi_2 dz. \tag{32}$$

Using Eqs (26)–(29) and the boundary conditions in Eq. (30), we can write

$$aR^{1/2}(1+M_1-M_4)\int_0^1 w^*\,\theta\,dz = = -(1+M_1(1-M_5))\int_0^1 \theta(D^2-a^2-P_r\sigma^*)\theta^* dz, \tag{33}$$

$$-aR^{1/2}(M_1-M_4)\int_0^1 w^*\,D\Phi_1 dz = M_1(1- - M_5)\int_0^1 D\Phi_1(D^2-a^2-P_r\sigma^*)\theta^* dz = -M_1(1-M_5)\int_0^1 D^2\Phi_1\,D\theta^* dz + M_1(1- - M_5)(a^2+P_r\sigma^*)\int_0^1 \Phi_1\,D\theta^* dz = -M_1(1-M_5)\int_0^1 D^2\Phi_1(D^2-a^2M_3)\Phi_1{}^* dz + + M_1(1-M_5)(a^2+P_r\sigma^*)\int_0^1 \Phi_1(D^2- - a^2M_3)\Phi_1{}^* dz \text{ (utilizing Eq. (26))}, \tag{34}$$

$$-aR_s{}^{1/2}(1-M_1'+M_4')\int_0^1 w^*\,\phi dz = = \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1 \phi\,(D^2-a^2-P_r'\sigma^*)\phi^* dz, \tag{35}$$

$$aR_s{}^{1/2}(M_4'-M_1')\int_0^1 w^*\,D\Phi_2 dz = = -M_4'(1-M_5)\int_0^1 D\Phi_2(D^2-a^2-P_r'\sigma^*)\phi^* dz = M_4'(1-M_5)\int_0^1 D^2\Phi_2\,D\phi^* dz - M_4'(1- - M_5)(a^2+P_r'\sigma^*)\int_0^1 \Phi_2\,D\phi^* dz = M_4'(1-M_5)\int_0^1 D^2\Phi_2(D^2-a^2M_3)\Phi_2{}^* dz - - M_4'(1-M_5)(a^2+P_r'\sigma^*)\int_0^1 \Phi_2(D^2- - a^2M_3)\Phi_2{}^* dz \text{ (utilizing Eq. (27))}. \tag{36}$$

Combining Eqs (32)–(36), we get

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-\sigma)w\,dz = -(1+M_1(1-M_5))\int_0^1 \theta(D^2-a^2- - P_r\sigma^*)\theta^* dz - M_1(1-M_5)\int_0^1 D^2\Phi_1(D^2- - a^2M_3)\Phi_1{}^* dz + M_1(1-M_5)(a^2+P_r\sigma^*)\int_0^1 \Phi_1(D^2- a^2M_3)\Phi_1{}^* dz + \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1 \phi\,(D^2- - a^2-P_r'\sigma^*)\phi^* dz + M_4'(1- - M_5)\int_0^1 D^2\Phi_2(D^2-a^2M_3)\Phi_2{}^* dz - M_4'(1-M_5)(a^2+P_r'\sigma^*)\int_0^1 \Phi_2(D^2-a^2M_3)\Phi_2{}^* dz. \tag{37}$$

Integrating the various terms of Eq. (37) by parts, for a suitable number of times and making use of the boundary conditions in Eq. (30) and the equality

$$\int_0^1 \psi^* D^{2n}\psi dz = (-1)^n \int_0^1 |D^n\psi|^2 dz, \tag{38}$$

where $=w$ (n=1,2) or $\psi=\theta,\phi,\Phi_1,\Phi_2$ (n=1), we obtain

$$\int_0^1(|D^2w|^2 + 2a^2|Dw|^2 + a^4 |w|^2)dz + \sigma \int_0^1(|Dw|^2 + a^2|w|^2)dz = [1 + M_1(1 - M_5)] \int_0^1(|D\theta|^2 + a^2|\theta|^2 + P_r\sigma^*|\theta|^2)dz - M_1(1 - M_5) \int_0^1(|D^2\Phi_1|^2 + a^2M_3|D\Phi_1|^2)dz - M_1(1 - M_5)(a^2 + P_r\sigma^*) \int_0^1(|D\Phi_1|^2 + a^2M_3|\Phi_1|^2)dz - \left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right] \int_0^1(|D\phi|^2 + a^2|\phi|^2 + P_r'\sigma^*|\phi|^2)dz + M_4'(1 - M_5)\int_0^1(|D^2\Phi_2|^2 + a^2M_3|D\Phi_2|^2)dz + M_4'(1 - M_5)(a^2 + P_r'\sigma^*)\int_0^1(|D\Phi_2|^2 + a^2M_3|\Phi_2|^2)dz. \tag{39}$$

Equating the imaginary parts of both sides of Eq. (39) and cancelling σ_i ($\neq 0$) throughout from the resulting equation, we get

$$\int_0^1(|Dw|^2 + a^2|w|^2)dz = -P_r [1 + M_1(1 - M_5)] \int_0^1|\theta|^2 dz + M_1(1 - M_5)P_r \int_0^1(|D\Phi_1|^2 + a^2M_3|\Phi_1|^2)dz + \left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right] P_r' \int_0^1|\phi|^2 dz - M_4'(1 - M_5)P_r' \int_0^1(|D\Phi_2|^2 + a^2M_3|\Phi_2|^2)dz. \tag{40}$$

Now, multiplying Eq. (26) by Φ_1^* and integrating over the vertical range of z, we get

$$\int_0^1(|D\Phi_1|^2 + a^2M_3|\Phi_1|^2) dz = -\int_0^1 \Phi_1^* D\theta dz = \int_0^1 \theta D\Phi_1^* dz$$
$$\leq \left|\int_0^1 \theta D\Phi_1^* dz\right|$$
$$\leq \int_0^1 |\theta||D\Phi_1^*|dz$$
$$\leq \int_0^1 |\theta||D\Phi_1|dz$$
$$\leq \left(\int_0^1|\theta|^2dz\right)^{1/2}\left(\int_0^1|D\Phi_1|^2dz\right)^{1/2} \text{(using Schwartz inequality)}, \tag{41}$$

which implies that

$$\int_0^1|D\Phi_1|^2 dz \leq \left(\int_0^1|\theta|^2dz\right)^{1/2}\left(\int_0^1|D\Phi_1|^2dz\right)^{1/2}$$

and thus,

$$\left(\int_0^1|D\Phi_1|^2dz\right)^{1/2} \leq \left(\int_0^1|\theta|^2dz\right)^{1/2}. \tag{42}$$

Upon using a similar procedure, Eq. (27) yields

$$\left(\int_0^1|D\Phi_2|^2dz\right)^{1/2} \leq \left(\int_0^1|\phi|^2dz\right)^{1/2}. \tag{43}$$

Combining the inequalities in Eqs (41) and (42), we get

$$\int_0^1(|D\Phi_1|^2 + a^2M_3|\Phi_1|^2) dz \leq \int_0^1|\theta|^2 dz. \tag{44}$$

Now, multiplying Eq. (29) by its complex conjugate and integrating over the vertical range of z for an appropriate number of times and using the boundary conditions in Eq. (30), we obtain

$$\int_0^1(|D^2\phi|^2 + 2a^2|D\phi|^2 + a^4|\phi|^2)dz + 2\sigma_r P_r' \int_0^1(|D\phi|^2 + a^2|\phi|^2)dz + P_r'^2|\sigma|^2 \int_0^1|\phi|^2dz = R_s a^2 \int_0^1|w|^2dz. \tag{45}$$

Since $\sigma_r \geq 0$, it follows from Eq. (45) that

$$\int_0^1|\phi|^2dz < \frac{R_s a^2}{P_r'^2|\sigma|^2} \int_0^1|w|^2dz. \tag{46}$$

Using the inequalities in Eqs (44) and (46) in Eq. (40), we get

$$\int_0^1|Dw|^2dz + a^2\left[1 - \frac{R_s\left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right]}{|\sigma|^2P_r'}\right]\int_0^1|w|^2dz + P_r \int_0^1|\theta|^2 dz + M_4'(1 - M_5)P_r' \int_0^1(|D\Phi_2|^2 + a^2M_3|\Phi_2|^2)dz < 0, \tag{47}$$

which clearly implies that

$$|\sigma| < \sqrt{\frac{R_s\left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right]}{P_r'}}.$$

This completes the proof of the result.

The above theorem, from the physical point of view, states that the complex growth rate of an arbitrary oscillatory motion of neutral or growing amplitude in ferrothermohaline convection, for the case of free boundaries, must lie inside a semicircle in the right half of the $\sigma_r\sigma_i$-plane, whose center is at the origin and

$$\text{radius} = \sqrt{\frac{R_s\left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right]}{P_r'}}.$$

Theorem 2: If $R>0$, $R_s>0$, $M_1>0$, $M_1'>0$, $1\text{-}M_5>0$, $P_r>0$, $P_r'>0$,. $\sigma_r\geq 0$, and $\sigma_i\neq 0$, then a necessary condition for the existence of a nontrivial solution (w, θ, ϕ, Φ_1, Φ_2, σ) of Eqs (23) and (26)-(29) together with the boundary conditions in Eq. (31) is that

$$|\sigma|^2\sigma_i^2 < \left\{\frac{R\,M_1(1-M_5)}{P_r} + \frac{R_s}{P_r'}\left(1+M_1'\left|1-\frac{1}{M_5}\right| - M_1'\left(1-\frac{1}{M_5}\right)\right)\right\}^2.$$

Proof: Multiplying Eq. (23) by w^* throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-\sigma)w\,dz = = aR^{1/2}\,(1+M_1-M_4)\int_0^1 w^*\,\theta\,dz - - aR^{1/2}(M_1-M_4)\int_0^1 w^*\,D\Phi_1 dz - aR_s^{1/2}\,(1-M_1'+M_4')\int_0^1 w^*\,\phi\,dz + aR_s^{1/2}(M_4'-M_1')\int_0^1 w^*\,D\Phi_2 dz. \quad (48)$$

Using Eqs (28) and (29), we can write

$$aR^{1/2}\,(1+M_1-M_4)\int_0^1 w^*\,\theta\,dz = = -[1+M_1(1-M_5)]\int_0^1 \theta(D^2-a^2-P_r\sigma^*)\theta^* dz, \quad (49)$$

and

$$-aR_s^{1/2}\,(1-M_1'+M_4')\int_0^1 w^*\,\phi dz = = \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1 \phi\,(D^2-a^2-P_r'\sigma^*)\phi^* dz. \quad (50)$$

Combining Eqs (48)–(50), we obtain

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-\sigma)w\,dz = = -[1+M_1(1-M_5)]\int_0^1 \theta(D^2-a^2-P_r\sigma^*)\theta^* dz - - aR^{1/2}M_1(1-M_5)\int_0^1 w^*\,D\Phi_1 dz + \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right] \int_0^1 \phi\,(D^2-a^2-P_r'\sigma^*)\phi^* dz - aR_s^{1/2}M_1'\left(1-\frac{1}{M_5}\right)\int_0^1 w^*\,D\Phi_2 dz. \quad (51)$$

Integrating the various terms of Eq. (51) by parts, for an appropriate number of times and making use of the boundary conditions in Eq. (31) and equality in Eq. (38), we obtain

$$\int_0^1(|D^2w|^2+2a^2|Dw|^2+a^4|w|^2)dz + + \sigma\int_0^1(|Dw|^2+a^2|w|^2)dz = [1+M_1(1- - M_5)]\int_0^1(|D\theta|^2+a^2|\theta|^2+P_r\sigma^*|\theta|^2)dz - aR^{1/2}M_1(1-M_5)\int_0^1 w^*\,D\Phi_1 dz - \left[1-M_1'\left(1- -\frac{1}{M_5}\right)\right]\int_0^1(|D\phi|^2+a^2|\phi|^2+P_r'\sigma^*|\phi|^2)dz - - aR_s^{1/2}\,M_1'\left(1-\frac{1}{M_5}\right)\int_0^1 w^*\,D\Phi_2 dz. \quad (52)$$

Equating the imaginary parts on both sides of Eq. (52) and dividing the resulting equation by σ_i ($\neq 0$), we get

$$\int_0^1(|Dw|^2+a^2|w|^2)dz = - [1+M_1(1-M_5)]P_r\int_0^1|\theta|^2 dz - \frac{aR^{1/2}M_1(1-M_5)}{\sigma_i}\text{imaginary part of}\int_0^1 w^*D\Phi_1 dz + \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]P_r'\int_0^1|\phi|^2 dz - -\frac{aR_s^{1/2}M_1'\left(1-\frac{1}{M_5}\right)}{\sigma_i}\text{imaginary part of}\int_0^1 w^*D\Phi_2 dz. \quad (53)$$

Now multiplying Eq. (28) by its complex conjugate and integrating over the vertical range of z by parts, for a suitable number of times, by making use of the boundary conditions in Eq. (31) and then by equating the real parts on both sides, we obtain

$$\int_0^1(|D^2\theta|^2+2a^2|D\theta|^2+a^4|\theta|^2)dz + + 2\sigma_r P_r\int_0^1(|D\theta|^2+a^2|\theta|^2)dz + + |\sigma|^2 P_r^{\,2}\int_0^1|\theta|^2 dz = a^2R\int_0^1|w|^2 dz\,. \quad (54)$$

Since $\sigma_r\geq 0$, it follows from Eq. (54) that

$$\int_0^1|\theta|^2 dz \leq \frac{a^2R}{P_r^2|\sigma|^2}\int_0^1|w|^2 dz\,. \quad (55)$$

Combining the inequalities in Eqs (42) and (55), we obtain

$$\left(\int_0^1|D\Phi_1|^2 dz\right)^{1/2} \leq \frac{aR^{1/2}}{P_r|\sigma|}\left(\int_0^1|w|^2 dz\right)^{1/2}. \quad (56)$$

On similar lines, from the inequalities in Eqs (43) and (46), we obtain

$$\left(\int_0^1|D\Phi_2|^2 dz\right)^{1/2} \leq \frac{aR_s^{1/2}}{P_r'|\sigma|}\left(\int_0^1|w|^2 dz\right)^{1/2}. \quad (57)$$

Now $-\frac{aR^{1/2}M_1(1-M_5)}{\sigma_i}$ imaginary part of $\int_0^1 w^*D\Phi_1 dz$

$$\leq aR^{1/2}M_1(1-M_5)\left|\frac{1}{\sigma_i}\int_0^1 w^* D\Phi_1 dz\right|$$

$$\leq. \frac{aR^{1/2}M_1(1-M_5)}{|\sigma_i|}\int_0^1 |w^* D\Phi_1|\,dz$$

$$\leq \frac{aR^{1/2}M_1(1-M_5)}{|\sigma_i|}\int_0^1 |w||D\Phi_1|\,dz$$

$$\leq \frac{aR^{1/2}M_1(1-M_5)}{|\sigma_i|}\left(\int_0^1 |w|^2 dz\right)^{1/2}\left(\int_0^1 |D\Phi_1|^2 dz\right)^{1/2}$$

(using Schwartz inequality)

$$\leq \frac{a^2 R M_1(1-M_5)}{P_r.|\sigma||\sigma_i|}\int_0^1 |w|^2 dz$$

(utilizing the inequality in Eq. (56)). (58)

Further, $-\dfrac{aR_s^{1/2}M_1'\left(1-\frac{1}{M_5}\right)}{\sigma_i}$ imaginary part of $\int_0^1 w^* D\Phi_2 dz$

$$\leq \frac{aR_s^{1/2}M_1'\left|1-\frac{1}{M_5}\right|}{|\sigma_i|}\left|\int_0^1 w^* D\Phi_2 dz\right|$$

$$\leq \frac{aR_s^{1/2}M_1'\left|1-\frac{1}{M_5}\right|}{|\sigma_i|}\int_0^1 |w||D\Phi_2|\,dz$$

$$\leq \frac{aR_s^{1/2}M_1'\left|1-\frac{1}{M_5}\right|}{|\sigma_i|}\left(\int_0^1 |w|^2 dz\right)^{1/2}\left(\int_0^1 |D\Phi_2|^2 dz\right)^{1/2}$$

(using Schwartz inequality)

$$\leq \frac{a^2 R_s M_1'\left|1-\frac{1}{M_5}\right|}{P_r'.|\sigma||\sigma_i|}\int_0^1 |w|^2 dz$$

(utilizing the inequality in Eq. (57)). (59)

Multiplying Eq. (29) by ϕ^* and integrating the resulting equation by parts, for an appropriate number of times over the vertical range of z, and then from the imaginary part of the final equation, we obtain

$$\int_0^1 |\phi|^2 dz = \frac{1}{\sigma_i}\text{ imaginary part of}\frac{aR_s^{1/2}}{P_r'}\int_0^1 \phi^* w dz.$$

$$\leq \frac{aR_s^{1/2}}{|\sigma_i|P_r'}\left|\int_0^1 \phi^* w dz\right|.$$

$$\leq \frac{aR_s^{1/2}}{|\sigma_i|P_r'}.\int_0^1 |\phi||w|\,dz.$$

$$\leq \frac{aR_s^{1/2}}{|\sigma_i|P_r'}\left(\int_0^1 |\phi|^2 dz\right)^{\frac{1}{2}}\left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}}.$$

(using Schwartz inequality)

$$\leq \frac{a^2 R_s}{|\sigma||\sigma_i|P_r'^2}\int_0^1 |w|^2 dz$$

(utilizing the inequality in Eq. (46)). (60)

Thus, utilizing the inequalities in Eqs (58)–(60) in Eq. (53), we finally obtain

$$\int_0^1 |Dw|^2 dz + a^2\left(1 - \frac{R\,M_1(1-M_5)}{P_r|\sigma||\sigma_i|} - \frac{R_s M_1'\left|1-\frac{1}{M_5}\right|}{P_r'\,|\sigma||\sigma_i|} - \frac{R_s\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]}{|\sigma||\sigma_i|P_r'}\right)\int_0^1 |w|^2 dz +$$

$$+\ [1+M_1(1-M_5)]P_r\int_0^1 |\theta|^2\,dz \leq 0,$$

which clearly implies that

$$|\sigma|^2\sigma_i^{2.} < \left\{\frac{R\,M_1(1-M_5)}{P_r} + \frac{R_s}{P_r'}\left(1 + M_1'\left|1-\frac{1}{M_5}\right| - M_1'\left(1-\frac{1}{M_5}\right)\right)\right\}^2. \quad (61)$$

The above theorem may be stated, from a physical point of view, as: the complex growth rate of an arbitrary oscillatory perturbation of growing amplitude in ferrothermohaline convection, for the case of rigid boundaries, must lie inside the region represented by the inequality in Eq. (61).

Note: It may be noted that the parametric value M_5, which represents the ratio of salinity effect on magnetic field to pyromagnetic coefficient, varies between 0.1 and 0.5 for most of the ferrofluids which are formed by changing ferric oxides and carrier organic fluids like kerosene, alcohol, hydrocarbon, etc. (Finlayson [2] and Gupta and Gupta [3]), so that the condition 1-M_5>0, and hence, 1-(1/M_5) <0 remain valid.

4 Conclusion

The linear stability theory has been used to derive the bounds for the complex growth rates in ferrothermohaline convection heated and salted from below in the presence of a uniform vertical magnetic field. Further, the results derived herein involve only dimensionless quantities and are wave number independent; thus, the present results are of uniform validity and applicability.

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