Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **16** (2022), 379–399.
https://doi.org/10.2298/AADM200710034C

ASYMPTOTIC EXPANSIONS AND INEQUALITIES RELATING TO THE GAMMA FUNCTION

Chao-Ping Chen[*] and Cristinel Mortici

We present some asymptotic expansions and inequalities for $\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}}$ and $\Gamma(x+1)^{1/x}$.

1. INTRODUCTION

The function $\phi(n) = (n!)^{1/n}$ (for $n \in \mathbb{N} := \{1, 2, \ldots\}$) has many applications in pure and applied mathematics. For example, in 1963 Minc [**25**] (see also [**23**, Conjecture 4]) conjectured, then Brégman [**9**] and Schrijver [**30**] proved that the permanent of a $(0, 1)$-matrix with row sums $r_1, r_2, \ldots, r_n$ is less than or equal to $\phi(r_1)\,\phi(r_2)\cdots\phi(r_n)$. It is easy to see that

$$\frac{(n+1)^n}{n!} = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)^2\cdots\left(1+\frac{1}{n}\right)^n \le \left(1+\frac{1}{n}\right)^{n^2},$$

which yields

$$\frac{n+1}{\sqrt[n]{n!}} \le \left(1+\frac{1}{n}\right)^n < e, \qquad n \in \mathbb{N}. \tag{1}$$

[*]Corresponding author. Chao-Ping Chen
2020 Mathematics Subject Classification. Primary 33B15; Secondary 41A60, 26D15
Keywords and Phrases. Gamma function, Bernoulli numbers, Asymptotic expansion, Inequality.

By using (1), Hardy [**18**] presented a proof of Carleman's inequality

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}a_n,$$

where $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$.

When investigating a conjecture on a upper bound for permanents of $(0,1)$-matrices, in 1964 Minc and Sathre [**26**] discovered several noteworthy inequalities involving $(n!)^{1/n}$. One of them is the following:

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1, \qquad n \in \mathbb{N}. \tag{2}$$

It is known in the literature that for $r > 0$ and $n \in \mathbb{N}$,

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{3}$$

When investigating a problem on Lorentz sequence spaces, in 1988 Martins [**24**] published the right-hand inequality of (3). The left-hand inequality of (3) was proved in 1993 by Alzer [**4**]. In 1994, Alzer [**7**] showed that if $r < 0$, the Martins inequality is reversed. In 2005, Chen and Qi [**13**] proved that the Alzer inequality is valid for all real numbers r. Also in [**13**], these authors posed the following conjecture: For any given natural number n, the function

$$f(r) = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r}, & r \neq 0, \\ \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}, & r = 0 \end{cases}$$

is strictly decreasing on $(-\infty, \infty)$. As far as we know, this conjecture has not yet been proved.

The inequalities (2) and (3) have attracted much interest of from many mathematicians and have motivated a large number of research papers concerning new proofs, various generalizations and improvements; see, for example, [**1, 4, 5, 6, 7, 13, 14, 17, 21, 22, 27, 28, 29, 31, 32**] and the references cited therein. See also [**1**] for some historical notes.

Alzer [**5**] proved that for $n \in \mathbb{N}$,

$$\frac{n+1}{n+2} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \leq \frac{n+1}{n+2\sqrt{2}-1}. \tag{4}$$

The lower and upper bounds in (4) are the best possible. Guan [**17**] presented the following improvement:

$$\frac{n+1}{n+2} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < \frac{n+1}{n+2}(n+1)^{1/n(n+1)}, \qquad n \geq 2.$$

Chen [**10**] proved that for $n \in \mathbb{N}$,

$$\left(\sqrt{2\pi n}\right)^{\frac{1}{n(n+1)}}\left(1-\frac{1}{n+a}\right) < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \leq \left(\sqrt{2\pi n}\right)^{\frac{1}{n(n+1)}}\left(1-\frac{1}{n+b}\right)$$

with the best possible constants

$$a=\frac{1}{2} \quad \text{and} \quad b=\frac{1}{2^{3/4}\pi^{1/4}-1}=0.807\ldots.$$

Alzer [**6**] also proved the following continuous version for every real number $x \geq 2$:

$$\frac{x+2}{x+1} < \frac{\left(\Gamma\left(x+1\right)\right)^{1/x}}{\left(\Gamma\left(x+2\right)\right)^{1/(x+1)}},$$

where Γ denotes the gamma function. Mortici [**28**] proved that for $x \geq 2$,

$$L_1(x) < \frac{\Gamma\left(x+1\right)^{1/x}}{\Gamma\left(x+2\right)^{1/(x+1)}} < M_1(x), \tag{5}$$

where

$$L_1(x)=\frac{x+\frac{1}{2}\ln(2\pi)-\frac{1}{2}}{x+\frac{1}{2}\ln(2\pi)+\frac{1}{2}}x^{\frac{1}{2x(x+1)}}$$

and

$$M_1(x)=\frac{x+\frac{1}{2}\ln(2\pi)-\frac{1}{2}}{x+\frac{1}{2}\ln(2\pi)+\frac{1}{2}}x^{\frac{1}{2x(x+1)}}\exp\left\{\frac{\frac{1}{4}\ln^2(2\pi)-\frac{1}{2}\ln(2\pi)+\frac{2}{3}}{x^2(x+1)}\right\}.$$

We find by Maple software that Mortici's approximation

$$\frac{\Gamma\left(x+1\right)^{1/x}}{\Gamma\left(x+2\right)^{1/(x+1)}} \approx \frac{x+\frac{1}{2}\ln(2\pi)-\frac{1}{2}}{x+\frac{1}{2}\ln(2\pi)+\frac{1}{2}}x^{\frac{1}{2x(x+1)}}, \qquad x \to \infty$$

is better than Guan's approximation

$$\frac{\Gamma\left(x+1\right)^{1/x}}{\Gamma\left(x+2\right)^{1/(x+1)}} \approx \frac{x+1}{x+2}\left(x+1\right)^{1/x(x+1)}, \qquad x \to \infty, \tag{6}$$

since

$$\frac{\Gamma\left(x+1\right)^{1/x}}{\Gamma\left(x+2\right)^{1/(x+1)}} = \frac{x+\frac{1}{2}\ln(2\pi)-\frac{1}{2}}{x+\frac{1}{2}\ln(2\pi)+\frac{1}{2}}x^{\frac{1}{2x(x+1)}}+O\left(\frac{1}{x^3}\right)$$

and

$$\frac{\Gamma\left(x+1\right)^{1/x}}{\Gamma\left(x+2\right)^{1/(x+1)}} = \frac{x+1}{x+2}\left(x+1\right)^{1/x(x+1)}+O\left(\frac{\ln x}{x^2}\right).$$

By mainly using the following asymptotic formula:

$$\ln \Gamma(x+1) = x \ln x - x + \ln \sqrt{2\pi x} + O(x^{-1}), \qquad x \to \infty, \tag{7}$$

we find that

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \approx \left(\sqrt{2\pi x}\right)^{\frac{1}{x(x+1)}}. \tag{8}$$

In this paper, we develop the approximation formula (8) to produce various asymptotic expansions. Based on the obtained expansions, we prove new upper and lower bounds for $\Gamma(x+1)^{1/x}/\Gamma(x+2)^{1/(x+1)}$. We also consider asymptotic expansions and inequalities for $\Gamma(x+1)^{1/x}$.

2. LEMMAS

The following lemmas are required in our present investigation.

Lemma 1 (See [**15**, Corollary 1])**.** *It follows that*

$$\exp\left(\sum_{j=1}^{\infty} \frac{p_j}{x^j}\right) \sim 1 + \sum_{j=1}^{\infty} \frac{q_j}{x^j}, \quad x \to \infty,$$

$$\ln\left(1 + \sum_{j=1}^{\infty} \frac{q_j}{x^j}\right) \sim \sum_{j=1}^{\infty} \frac{p_j}{x^j}, \quad x \to \infty,$$

the coefficients p_j and q_j have the following relations:

$$q_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}}{k_1! k_2! \cdots k_j!},$$

summed over all nonnegative integers k_j satisfying the equation

$$k_1 + 2k_2 + \cdots + jk_j = j,$$

and

$$p_j = \sum_{\substack{k_1+2k_2+\cdots+jk_j=j \\ k_1+k_2+\cdots+k_j=k \\ 1\le k\le j}} (-1)^{k-1}(k-1)! \frac{q_1^{k_1} q_2^{k_2} \cdots q_j^{k_j}}{k_1! k_2! \cdots k_j!},$$

where the summation is over all nonnegative integral solutions $(k_1, k_2, \ldots, k_j)$ of the equations

$$k_1 + 2k_2 + \cdots + jk_j = j, \quad k_1 + k_2 + \cdots + k_j = k, \quad k = 1, 2, \ldots, j.$$

Lemma 2 (See [**11**]). *Let*

$$A(x) \sim \sum_{j=1}^{\infty} a_j x^{-j}, \qquad x \to \infty$$

be a given asymptotical expansion. Then the composition $\exp(A(x))$ *has asymptotic expansion of the following form*

$$\exp(A(x)) \sim \sum_{j=0}^{\infty} b_j x^{-j}, \qquad x \to \infty,$$

where

$$b_0 = 1, \quad b_j = \frac{1}{j} \sum_{k=1}^{j} k a_k b_{j-k}, \qquad j \in \mathbb{N}.$$

3. ASYMPTOTIC EXPANSIONS

Theorem 3 develops the approximation formula (8) to produce a complete asymptotic expansion.

Theorem 3. *As* $x \to \infty$, *we have*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x(x+1)}} \exp\left(\sum_{j=1}^{\infty} \frac{\alpha_j}{x^j}\right) \tag{9}$$

with the coefficients α_j *given by*

$$\alpha_1 = -1, \quad \alpha_2 = 0, \quad \alpha_j = (-1)^{j-1} \sum_{k=0}^{j-3} \frac{1}{k+2} \left(\frac{(-1)^k B_{k+2}}{k+1} + 1 \right), \quad j \geq 3, \tag{10}$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ *are the Bernoulli numbers defined by the following generating function:*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi. \tag{11}$$

Proof. It follows from Stirling's series for the gamma function (see [**2**, p. 257, Eq. (6.1.40)]) that

$$\ln \Gamma(x+1) \sim \ln \sqrt{2\pi x} + x \ln x - x + \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)x^j}, \qquad x \to \infty, \tag{12}$$

where B_n are the Bernoulli numbers defined by (11). Using (12), we find that for $x \to \infty$,

$$\begin{aligned}
&\ln \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \\
&= \frac{1}{x^2}\left(1+\frac{1}{x}\right)^{-1}\left[\ln\Gamma(x+1) - x\ln x - x\ln\left(1+\frac{1}{x}\right)\right] \\
&\sim \sum_{j=2}^{\infty}\frac{(-1)^j}{x^j}\left[\ln\sqrt{2\pi x} - x + \sum_{j=1}^{\infty}\frac{B_{j+1}}{j(j+1)x^j} - \sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)x^j}\right] \\
&\sim \sum_{j=2}^{\infty}\frac{(-1)^j}{x^j}\left[\ln\sqrt{2\pi x} - x - 1 + \sum_{j=1}^{\infty}\frac{1}{j+1}\left(\frac{B_{j+1}}{j} - (-1)^j\right)\frac{1}{x^j}\right],
\end{aligned}$$

$$\begin{aligned}
&\ln \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \\
&\sim \sum_{j=2}^{\infty}\frac{(-1)^j\ln\sqrt{2\pi x}}{x^j} - \frac{1}{x} + \sum_{k=2}^{\infty}\frac{(-1)^k}{x^k}\sum_{j=1}^{\infty}\frac{1}{j+1}\left(\frac{B_{j+1}}{j} - (-1)^j\right)\frac{1}{x^j} \\
&\sim -\frac{1}{x} + \sum_{j=2}^{\infty}\frac{(-1)^j\ln\sqrt{2\pi x}}{x^j} + \sum_{j=3}^{\infty}\sum_{k=0}^{j-3}\frac{(-1)^{j-k-1}}{k+2}\left(\frac{B_{k+2}}{k+1} + (-1)^k\right)\frac{1}{x^j}.
\end{aligned}$$

We then obtain

$$\begin{aligned}
&\ln \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \\
&\sim -\frac{1}{x} + \sum_{j=2}^{\infty}(-1)^j\left[\ln\sqrt{2\pi x} - \sum_{k=0}^{j-3}\frac{1}{k+2}\left(\frac{(-1)^k B_{k+2}}{k+1} + 1\right)\right]\frac{1}{x^j},
\end{aligned} \tag{13}$$

where an empty sum is (elsewhere throughout this paper) understood to be nil.

Noting that

$$\frac{1}{x(x+1)} = \sum_{j=2}^{\infty}\frac{(-1)^j}{x^j}, \qquad x \geq 1, \tag{14}$$

from (13) we deduce (9). The proof is complete. □

Here, from (9), we give the following explicit asymptotic expansion:

$$\begin{aligned}
&\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \\
&\sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x(x+1)}}\exp\left(-\frac{1}{x} + \frac{7}{12x^3} - \frac{11}{12x^4} + \frac{419}{360x^5} - \frac{491}{360x^6} + \cdots\right).
\end{aligned} \tag{15}$$

By Lemma 2, we deduce from (15) that

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x(x+1)}} \left(1 - \frac{1}{x} + \frac{1}{2x^2} + \frac{5}{12x^3} - \frac{35}{24x^4} + \cdots\right). \tag{16}$$

We call the representation $\left(\sqrt{2\pi x}\right)^{\frac{1}{x(x+1)}}$ the kernel of asymptotic expansion of $\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}}$. The emergence of the kernel $\left(\sqrt{2\pi x}\right)^{\frac{1}{x(x+1)}}$ has a superiority, the coefficients of asymptotic expansions of $\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}}$ are rational numbers, see (15) and (16).

Stirling's series for the gamma function is given (see [**2**, p. 257, Eq. (6.1.40)]) by

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right), \qquad x \to \infty. \tag{17}$$

The following asymptotic formula is due to Laplace:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots\right) \tag{18}$$

as $x \to \infty$ (see [**2**, p. 257, Eq. (6.1.37)]).

The expression (18) is sometimes incorrectly called Stirling's series (see [**16**, pp. 2–3]). The formula (9) is an interesting analogue of Stirling's series (17), while the formula (16) is an interesting analogue of the Laplace formula (18). Stirling's formula

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$$

is in fact the first approximation to the asymptotic formula (18), while the formula (8) is the first approximation to the asymptotic formula (16).

Recall that a function f is said to be completely monotonic on $(0, \infty)$ if it has derivatives of all orders on $(0, \infty)$ and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for} \quad x \in (0, \infty) \quad \text{and} \quad n \in \mathbb{N}_0.$$

Alzer [**8**] first proved in 1997 that for every $m \in \mathbb{N}_0$, the function

$$R_m(x) = (-1)^m \left[\ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right] \tag{19}$$

is completely monotonic on $(0, \infty)$. In 2006, Koumandos [**19**] gave a simpler proof of the complete monotonicity property of $R_m(x)$. In 2009, Koumandos and Pedersen [**20**, Theorem 2.1] strengthened this result.

Some computer experiments led us to conjecture that for every $m \in \mathbb{N} \setminus \{1\}$, the function

$$F_m(x) = (-1)^m \left[\ln \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} - \frac{\ln\sqrt{2\pi x}}{x(x+1)} - \sum_{j=1}^{m} \frac{\alpha_j}{x^j} \right]$$

is completely monotonic on $(0, \infty)$, where α_j are given in (10). This conjecture is similar to the complete monotonicity property of $R_m(x)$ in (19). However, we have not been able to verify it.

Theorem 4 develops Guan's approximation formula (6) to produce a complete asymptotic expansion.

Theorem 4. *As $x \to \infty$, we have*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{x+1}{x+2} (x+1)^{\frac{1}{x(x+1)}} \exp\left(-\frac{\ln x}{2x(x+1)} + \sum_{j=2}^{\infty} \frac{b_j}{x^j} \right), \tag{20}$$

where

(21)

$$b_2 = \ln\sqrt{2\pi} - \frac{3}{2}, \quad b_j = (-1)^j \left(\ln\sqrt{2\pi} - \frac{2^j - 1}{j} - \sum_{k=0}^{j-3} \frac{(-1)^k B_{k+2} - 1}{(k+1)(k+2)} \right), \quad j \geq 3.$$

Proof. We find that for $x \to \infty$,

$$\begin{aligned}
&\ln\left(\frac{x+1}{x+2} (x+1)^{\frac{1}{x(x+1)}} \right) \\
&= \frac{1}{x^2}\left(1 + \frac{1}{x}\right)^{-1} \ln x + \frac{1}{x^2}\left(1 + \frac{1}{x}\right)^{-1} \ln\left(1 + \frac{1}{x}\right) \\
&\quad + \ln\left(1 + \frac{1}{x}\right) - \ln\left(1 + \frac{2}{x}\right) \\
&= \sum_{j=2}^{\infty} \frac{(-1)^j}{x^j} \ln x + \sum_{k=2}^{\infty} \frac{(-1)^k}{x^k} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j x^j} + \sum_{j=1}^{\infty} \frac{(-1)^j (2^j - 1)}{j} \frac{1}{x^j} \\
&= \sum_{j=2}^{\infty} \frac{(-1)^j}{x^j} \ln x + \sum_{j=3}^{\infty} \sum_{k=0}^{j-3} \frac{(-1)^{j-1}}{k+1} \frac{1}{x^j} + \sum_{j=1}^{\infty} \frac{(-1)^j (2^j - 1)}{j} \frac{1}{x^j} \\
&= -\frac{1}{x} + \sum_{j=2}^{\infty} (-1)^j \left(\ln x - \sum_{k=0}^{j-3} \frac{1}{k+1} + \frac{2^j - 1}{j} \right) \frac{1}{x^j}.
\end{aligned} \tag{22}$$

From (13) and (22), we obtain that

$$\ln \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} - \ln\left(\frac{x+1}{x+2}(x+1)^{\frac{1}{x(x+1)}}\right) \sim \sum_{j=2}^{\infty}(-1)^{j-1}\left(\ln\sqrt{\frac{x}{2\pi}} + \frac{2^j-1}{j} + \sum_{k=0}^{j-3}\frac{(-1)^k B_{k+2}-1}{(k+1)(k+2)}\right)\frac{1}{x^j}. \tag{23}$$

Noting that (14) holds, we deduce from (23) that

$$\ln \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} - \ln\left(\frac{x+1}{x+2}(x+1)^{\frac{1}{x(x+1)}}\right) \sim -\frac{\ln x}{2x(x+1)} + \sum_{j=2}^{\infty}\frac{b_j}{x^j}, \tag{24}$$

where b_j are given in (21). The formula (24) can be written as (20). The proof is complete. □

Remark 5. *The formula* (20) *can be rewritten as*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{x+1}{x+2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{\frac{1}{x(x+1)}} \exp\left(\sum_{j=2}^{\infty}\frac{b_j}{x^j}\right), \qquad x \to \infty,$$

where b_j are given in (21)*. Namely,*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{x+1}{x+2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{\frac{1}{x(x+1)}} \times \exp\left(-\frac{\frac{3}{2}-\ln\sqrt{2\pi}}{x^2} + \frac{\frac{23}{12}-\ln\sqrt{2\pi}}{x^3} - \frac{\frac{19}{6}-\ln\sqrt{2\pi}}{x^4} + \cdots\right). \tag{25}$$

It follows form (25) *that*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \approx \frac{x+1}{x+2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{\frac{1}{x(x+1)}}, \qquad x \to \infty,$$

which is better than Guan's approximation (6)*, since*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} = \frac{x+1}{x+2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{\frac{1}{x(x+1)}} + O\left(\frac{1}{x^2}\right).$$

Remark 6. *Noting that* (14) *holds, we deduce from* (23) *that*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{x+1}{x+2}\left(\sqrt{2\pi x}\left(1+\frac{1}{x}\right)\right)^{\frac{1}{x(x+1)}} \exp\left(\sum_{j=2}^{\infty}\frac{c_j}{x^j}\right),$$

where c_j are given by

$$c_2 = -\frac{3}{2}, \quad c_j = (-1)^{j-1}\left(\frac{2^j - 1}{j} + \sum_{k=0}^{j-3} \frac{(-1)^k B_{k+2} - 1}{(k+1)(k+2)}\right), \qquad j \geq 3.$$

Namely,

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{x+1}{x+2}\left(\sqrt{2\pi x}\left(1+\frac{1}{x}\right)\right)^{\frac{1}{x(x+1)}} \times \exp\left(-\frac{3}{2x^2} + \frac{23}{12x^3} - \frac{19}{6x^4} + \frac{1991}{360x^5} - \frac{3521}{360x^6} + \cdots\right). \tag{26}$$

By Lemma 2, we deduce from (26) *that*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{x+1}{x+2}\left(\sqrt{2\pi x}\left(1+\frac{1}{x}\right)\right)^{\frac{1}{x(x+1)}} \left(1 - \frac{3}{2x^2} + \frac{23}{12x^3} - \frac{49}{24x^4} + \cdots\right). \tag{27}$$

It follows form (27) *that*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \approx \frac{x+1}{x+2}\left(\sqrt{2\pi x}\left(1+\frac{1}{x}\right)\right)^{\frac{1}{x(x+1)}}, \qquad x \to \infty,$$

which is better than Guan's approximation (6)*, since*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} = \frac{x+1}{x+2}\left(\sqrt{2\pi x}\left(1+\frac{1}{x}\right)\right)^{\frac{1}{x(x+1)}} + O\left(\frac{1}{x^2}\right). \tag{28}$$

In view of (28), we now introduce the approximations family:

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \approx \frac{x+a}{x+b}\left(\sqrt{2\pi x}\left(1+\frac{c}{x}\right)\right)^{\frac{1}{x(x+1)}}, \tag{29}$$

where a, b, $c \in \mathbb{R}$ are parameters. Setting

$$(a,b,c) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right)$$

in (29), we find (by Maple software) the following higher approximation:

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} = \frac{2x-1}{2x+1}\left(\sqrt{2\pi x}\left(1+\frac{2}{3x}\right)\right)^{\frac{1}{x(x+1)}} + O\left(\frac{1}{x^4}\right). \tag{30}$$

Likewise, we find (by Maple software) that

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} = \frac{2x-1}{2x+1}\left(\sqrt{2\pi x}\exp\left(\frac{2}{3x}\right)\right)^{\frac{1}{x(x+1)}} + O\left(\frac{1}{x^4}\right). \tag{31}$$

Theorem 7 develops the approximation formula (31) to produce a complete asymptotic expansion.

Theorem 7. *As $x \to \infty$, we have*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{2x-1}{2x+1}\left[\sqrt{2\pi x}\exp\left(\sum_{j=1}^{\infty}\frac{p_j}{x^j}\right)\right]^{\frac{1}{x(x+1)}} \tag{32}$$

with the coefficients p_j given by

$$p_j = \beta_j + \beta_{j+1}, \qquad j \in \mathbb{N} \tag{33}$$

and

$$\beta_j = \alpha_{j+1} + \frac{1-(-1)^{j-1}}{(j+1)\cdot 2^{j+1}}, \qquad j \in \mathbb{N},$$

where α_j given in (10)*. Namely,*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{2x-1}{2x+1}\left[\sqrt{2\pi x}\exp\left(\frac{2}{3x} - \frac{1}{4x^2} + \frac{187}{720x^3} - \frac{3}{16x^4} + \cdots\right)\right]^{\frac{1}{x(x+1)}}. \tag{34}$$

Proof. We first express (32) as follows:

$$\ln\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \sum_{j=1}^{\infty}\frac{(-1)^j-1}{j\cdot 2^j}\frac{1}{x^j} + \frac{\ln(\sqrt{2\pi x})}{x(x+1)} + \frac{1}{x(x+1)}\sum_{j=1}^{\infty}\frac{p_j}{x^j}. \tag{35}$$

Write (9) as

$$\ln\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{\ln(\sqrt{2\pi x})}{x(x+1)} + \sum_{j=1}^{\infty}\frac{\alpha_j}{x^j}. \tag{36}$$

It follows from from (35) and (36) that

$$\sum_{j=1}^{\infty}\frac{(-1)^j-1}{j\cdot 2^j}\frac{1}{x^j} + \frac{1}{x(x+1)}\sum_{j=1}^{\infty}\frac{p_j}{x^j} \sim \sum_{j=1}^{\infty}\frac{\alpha_j}{x^j},$$

$$\sum_{j=1}^{\infty}\frac{p_j}{x^j} \sim x(x+1)\sum_{j=1}^{\infty}\left(\alpha_j + \frac{1-(-1)^j}{j\cdot 2^j}\right)\frac{1}{x^j},$$

$$\sum_{j=1}^{\infty}\frac{p_j}{x^j} \sim \sum_{j=1}^{\infty}\left(\alpha_{j+2} + \frac{1-(-1)^j}{(j+2)\cdot 2^{j+2}} + \alpha_{j+1} + \frac{1-(-1)^{j-1}}{(j+1)\cdot 2^{j+1}}\right)\frac{1}{x^j}. \tag{37}$$

Equating coefficients of equal powers of x in (37) we obtain (33). The proof is complete. □

Theorem 8 develops the approximation formula (30) to produce a class of complete asymptotic expansions.

Theorem 8. *Let $r \neq 0$ be a given real number. The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{2x-1}{2x+1}\left[\sqrt{2\pi x}\left(1+\sum_{j=1}^{\infty}\frac{q_j(r)}{x^j}\right)^{1/r}\right]^{\frac{1}{x(x+1)}}, \qquad x \to \infty \tag{38}$$

with the coefficients $q_j \equiv q_j(r)$ given by

$$q_j = \sum_{k_1+2k_2+\cdots+jk_j=j} r^{k_1+k_2+\cdots+k_j}\frac{p_1^{k_1}p_2^{k_2}\cdots p_j^{k_j}}{k_1!k_2!\cdots k_j!}, \qquad j \in \mathbb{N},$$

where p_j given in (33).

Proof. We first express (38) as follows:

$$\ln\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \sum_{j=1}^{\infty}\frac{(-1)^j-1}{j\cdot 2^j}\frac{1}{x^j}+\frac{\ln(\sqrt{2\pi x})}{x(x+1)}+\frac{1}{rx(x+1)}\ln\left(1+\sum_{j=1}^{\infty}\frac{q_j(r)}{x^j}\right). \tag{39}$$

It follows from from (35) and (39) that

$$\ln\left(1+\sum_{j=1}^{\infty}\frac{q_j}{x^j}\right) \sim \sum_{j=1}^{\infty}\frac{rp_j}{x^j}. \tag{40}$$

By Lemma 1, we have

$$q_j = \sum_{k_1+2k_2+\cdots+jk_j=j} r^{k_1+k_2+\cdots+k_j}\frac{p_1^{k_1}p_2^{k_2}\cdots p_j^{k_j}}{k_1!k_2!\cdots k_j!},$$

where p_j given in (33). The proof is complete. □

Remark 9. *In fact, the coefficients $q_j \equiv q_j(r)$ in* (38) *can be given by the recurrence relation.*

Write (40) *as*

$$\exp\left(\sum_{j=1}^{\infty}\frac{rp_j}{x^j}\right) \sim 1+\sum_{j=1}^{\infty}\frac{q_j}{x^j}.$$

By Lemma 2, we have

$$q_0 = 1, \quad q_j = \frac{r}{j}\sum_{k=1}^{j} kp_k q_{j-k}, \qquad j \in \mathbb{N},$$

where p_j given in (33).

The representation using recursive algorithm is better for numerical evaluations. Setting $r = 1$ and $r = \frac{1}{2}$ in (38)*, respectively, we give two explicit asymptotic expansions:*

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{2x-1}{2x+1}\left[\sqrt{2\pi x}\left(1+\frac{2}{3x}-\frac{1}{36x^2}+\frac{923}{6480x^3}-\frac{1183}{38880x^4}+\cdots\right)\right]^{\frac{1}{x(x+1)}}$$

and

$$\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} \sim \frac{2x-1}{2x+1}\left[\sqrt{2\pi\left(x+\frac{4}{3}+\frac{7}{18x}+\frac{803}{3240x^2}+\frac{631}{4860x^3}+\cdots\right)}\right]^{\frac{1}{x(x+1)}}.$$

Allasia et al. [**3**, Theorem 3] proved that for $x \ge 1$,

$$\frac{x+1}{[\Gamma(x+1)]^{1/x}} \le \left(1+\frac{1}{x}\right)^x. \tag{41}$$

This is a continuous version of the first inequality in (1). Chen and Qi [**12**] showed that the inequality (41) is reversed for $0 < x \le 1$ and equality occurs for $x = 1$. Also in [**12**], the authors proved that for $x > 0$,

$$\frac{x}{[\Gamma(x+1)]^{1/x}} < \left(1+\frac{1}{x}\right)^x. \tag{42}$$

By mainly using the asymptotic formula (7), we find that

$$\left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} \approx \left(\sqrt{2\pi x}\right)^{\frac{1}{x}}. \tag{43}$$

Theorem 10 develops the approximation formula (43) to produce a complete asymptotic expansion.

Theorem 10. *As $x \to \infty$, we have*

$$\left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \exp\left(\sum_{j=1}^{\infty}\frac{\omega_j}{x^j}\right) \tag{44}$$

with the coefficients ω_j given by

$$\omega_1 = -\frac{1}{2}, \quad \omega_j = \frac{(-1)^j}{j+1} + \frac{B_j}{j(j-1)}, \quad j \ge 2, \tag{45}$$

where B_n are the Bernoulli numbers.

Proof. Write (44) as

$$\ln\left(\left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x\left(\sqrt{2\pi x}\right)^{\frac{1}{x}}}\right) \sim \sum_{j=1}^{\infty} \frac{\omega_j}{x^j}. \tag{46}$$

The Maclaurin series of $\ln(1+t)$ with $t=\frac{1}{x}$ gives

$$\ln\left(1+\frac{1}{x}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{jx^j}. \tag{47}$$

Using (12) and (47), we deduce from (46) that for $x \to \infty$,

$$-\frac{1}{2x} + \sum_{j=2}^{\infty}\left(\frac{(-1)^j}{j+1} + \frac{B_j}{j(j-1)}\right) \sim \sum_{j=1}^{\infty} \frac{\omega_j}{x^j}. \tag{48}$$

Equating coefficients of equal powers of x in (48) we obtain (45). The proof is complete. □

Here, from (44), we give the following explicit asymptotic expansion:

$$\left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \exp\left(-\frac{1}{2x} + \frac{5}{12x^2} - \frac{1}{4x^3} + \cdots\right). \tag{49}$$

By Lemma 2, we deduce from (49) that

$$\left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \left(1 - \frac{1}{2x} + \frac{13}{24x^2} - \frac{23}{48x^3} + \cdots\right). \tag{50}$$

Some computer experiments led us to conjecture that for every $m \in \mathbb{N}$, the function

$$G_m(x) = (-1)^{m-1}\left[\ln\left(\left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x\left(\sqrt{2\pi x}\right)^{\frac{1}{x}}}\right) - \sum_{j=1}^{m} \frac{\omega_j}{x^j}\right]$$

is completely monotonic on $(0,\infty)$, where ω_j are given in (45).

It follows form (50) that

$$\left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} = \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} + O\left(\frac{1}{x}\right).$$

We find (by Maple software) the following higher approximation:

$$\left(\frac{2x+1}{2x-1}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} = \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} + O\left(\frac{1}{x^2}\right). \tag{51}$$

Theorem 11 develops the approximation formula (51) to produce a complete asymptotic expansion.

Theorem 11. *As $x \to \infty$, we have*

$$\left(\frac{2x+1}{2x-1}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \exp\left(\sum_{j=1}^{\infty} \frac{\lambda_j}{x^j}\right) \tag{52}$$

with the coefficients λ_j given by

$$\lambda_1 = 0, \quad \lambda_j = \frac{1+(-1)^j}{(j+1)2^{j+1}} + \frac{B_j}{j(j-1)}, \quad j \geq 2, \tag{53}$$

where B_n are the Bernoulli numbers.

Proof. Using (47), we have

$$x \ln \frac{2x+1}{2x-1} = x\left[\ln\left(1+\frac{1}{2x}\right) - \ln\left(1-\frac{1}{2x}\right)\right] = 1 + \sum_{j=2}^{\infty} \frac{1+(-1)^j}{(j+1)2^{j+1}} \frac{1}{x^j}.$$

Taking the logarithm of (52) yields

$$\sum_{j=2}^{\infty} \frac{1+(-1)^j}{(j+1)2^{j+1}} \frac{1}{x^j} + \frac{1}{x}\left(\Gamma(x+1) - x\ln x + x - \ln(\sqrt{2\pi x})\right) \sim \sum_{j=1}^{\infty} \frac{\lambda_j}{x^j}. \tag{54}$$

Using (12), we obtain from (54) that

$$\sum_{j=2}^{\infty} \left(\frac{1+(-1)^j}{(j+1)2^{j+1}} + \frac{B_j}{j(j-1)}\right) \frac{1}{x^j} \sim \sum_{j=1}^{\infty} \frac{\lambda_j}{x^j}. \tag{55}$$

Equating coefficients of equal powers of x in (55) we obtain (53). The proof is complete. □

Remark 12. *The asymptotic expansion* (52) *can be written as*

$$\left(\frac{2x+1}{2x-1}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \exp\left(\sum_{j=1}^{\infty} \frac{\lambda_{2j}}{x^{2j}}\right), \tag{56}$$

where

$$\lambda_{2j} = \frac{1}{(2j+1)2^{2j}} + \frac{B_{2j}}{2j(2j-1)}, \quad j \in \mathbb{N}.$$

Here, from (56), *we give the following explicit asymptotic expansion:*

$$\left(\frac{2x+1}{2x-1}\right)^x \frac{\Gamma(x+1)^{1/x}}{x} \sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \exp\left(\frac{1}{6x^2} + \frac{7}{720x^4} + \frac{61}{20160x^6} - \frac{13}{80640x^8} + \cdots\right),$$

which deduces that

$$\left(\frac{2x+1}{2x-1}\right)^x \frac{\Gamma(x+1)^{1/x}}{x}$$
$$\sim \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \left(1+\frac{1}{6x^2}+\frac{17}{720x^4}+\frac{983}{181440x^6}+\frac{12139}{21772800x^8}+\cdots\right).$$

4. INEQUALITIES

The formula (34) motivated us to observe the following Theorem 13.

Theorem 13. *For $x \geq 1$,*

$$L_2(x) < \frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} < M_2(x), \tag{57}$$

where

$$L_2(x) = \frac{2x-1}{2x+1}\left[\sqrt{2\pi x}\exp\left(\frac{2}{3x}-\frac{1}{4x^2}\right)\right]^{\frac{1}{x(x+1)}}$$

and

$$M_2(x) = \frac{2x-1}{2x+1}\left[\sqrt{2\pi x}\exp\left(\frac{2}{3x}\right)\right]^{\frac{1}{x(x+1)}}.$$

Proof. It follows from the known result (see [**8**, Theorem 8]) that for $x > 0$,

(58)
$$x\ln x - x + \ln(\sqrt{2\pi x}) + \frac{1}{12x} - \frac{1}{360x^3} < \ln\Gamma(x+1) < x\ln x - x + \ln(\sqrt{2\pi x}) + \frac{1}{12x}.$$

It is well known that

$$\sum_{j=1}^{2m}\frac{(-1)^{j-1}}{j}t^j < \ln(1+t) < \sum_{j=1}^{2m-1}\frac{(-1)^{j-1}}{j}t^j \tag{59}$$

for $-1 < t \leq 1$ and $m \in \mathbb{N}$. The proof of Theorem 13 is based on the inequalities (58) and (59).

In order to prove the inequality (57), it suffices to show that for $x \geq 1$,

$$\ln\frac{2x-1}{2x+1} + \frac{\ln(\sqrt{2\pi x})}{x(x+1)} + \frac{1}{x(x+1)}\left(\frac{2}{3x}-\frac{1}{4x^2}\right)$$
$$< \ln\frac{\Gamma(x+1)^{1/x}}{\Gamma(x+2)^{1/(x+1)}} = \frac{1}{x(x+1)}\left[\ln\Gamma(x+1) - x\ln x - x\ln\left(1+\frac{1}{x}\right)\right]$$
$$< \ln\frac{2x-1}{2x+1} + \frac{\ln(\sqrt{2\pi x})}{x(x+1)} + \frac{1}{x(x+1)}\frac{2}{3x}.$$

By (58), it suffices to show that for $x \geq 1$,

(60)
$$\ln\frac{2x-1}{2x+1}+\frac{1}{x(x+1)}\left(\frac{2}{3x}-\frac{1}{4x^2}\right)<\frac{1}{x(x+1)}\left[-x+\frac{1}{12x}-\frac{1}{360x^3}-x\ln\left(1+\frac{1}{x}\right)\right]$$

and

(61)
$$\frac{1}{x(x+1)}\left[-x+\frac{1}{12x}-x\ln\left(1+\frac{1}{x}\right)\right]<\ln\frac{2x-1}{2x+1}+\frac{1}{x(x+1)}\frac{2}{3x}$$

In order to prove (60) and (61), it suffices to show

$$f(x)>0 \quad \text{and} \quad g(x)>0 \quad \text{for} \quad x\geq 1,$$

where

$$f(x)=-\frac{360x^4+210x^2-90x+1}{360x^4(x+1)}-\frac{1}{x+1}\ln\left(1+\frac{1}{x}\right)-\ln\frac{2x-1}{2x+1}$$

and

$$g(x)=\frac{12x^2+7}{12x^2(x+1)}+\frac{1}{x+1}\ln\left(1+\frac{1}{x}\right)+\ln\frac{2x-1}{2x+1}$$

By (59), we find that for $x \geq 1$,

$$\begin{aligned}
f'(x)&=\frac{1}{(x+1)^2}\ln\left(1+\frac{1}{x}\right)-\frac{1440x^6-720x^5+120x^4+1690x^3+44x^2-265x+4}{360x^5(x+1)^2(4x^2-1)}\\
&<\frac{1}{(x+1)^2}\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}\right)\\
&\quad-\frac{1440x^6-720x^5+120x^4+1690x^3+44x^2-265x+4}{360x^5(x+1)^2(4x^2-1)}\\
&=-\frac{1510x^3+164x^2-265x+4}{360x^5(x+1)^2(4x^2-1)}<0
\end{aligned}$$

and

$$\begin{aligned}
-g'(x)&=\frac{1}{(x+1)^2}\ln\left(1+\frac{1}{x}\right)-\frac{48x^4-24x^3-44x^2+21x+14}{12x^3(x+1)^2(4x^2-1)}\\
&>\frac{1}{(x+1)^2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)-\frac{48x^4-24x^3-44x^2+21x+14}{12x^3(x+1)^2(4x^2-1)}\\
&=\frac{3+49(x-1)+32(x-1)^2}{12x^3(x+1)^2(4x^2-1)}>0.
\end{aligned}$$

Hence, the functions $f(x)$ and $g(x)$ are strictly decreasing for $x \geq 1$, and we have

$$f(x)>\lim_{t\to\infty}f(t)=0 \quad \text{and} \quad g(x)>\lim_{t\to\infty}g(t)=0 \quad \text{for} \quad x\geq 1.$$

The proof is complete. □

Remark 14. *It is observed from Table 1 that for $x \geq 2$,*

$$L_1(x) < L_2(x) \quad \textit{and} \quad M_2(x) < M_1(x).$$

This shows that the inequality (57) *is sharper than the inequality* (5).

Table 1. Comparison between inequalities (5) and (57).

x	$L_2(x) - L_1(x)$	$M_1(x) - M_2(x)$
2	3.362×10^{-2}	4.288×10^{-3}
10	4.629×10^{-4}	1.260×10^{-5}
100	5.766×10^{-7}	1.568×10^{-9}
1000	5.905×10^{-10}	1.608×10^{-13}
10000	5.920×10^{-13}	1.612×10^{-17}

The formula (49) motivated us to observe the following Theorem 15.

Theorem 15. *For $x \geq 1$,*

$$\exp\left(-\frac{1}{2x}\right) < \left(1+\frac{1}{x}\right)^x \frac{\Gamma(x+1)^{1/x}}{x\left(\sqrt{2\pi x}\right)^{\frac{1}{x}}} < \exp\left(-\frac{1}{2x}+\frac{5}{12x^2}\right). \tag{62}$$

Proof. In order to prove the inequality (62), it suffices to show that for $x \geq 1$,

$$-\frac{1}{2x} < x\ln\left(1+\frac{1}{x}\right) + \frac{1}{x}\left[\ln\Gamma(x+1) - x\ln x - \ln(\sqrt{2\pi x})\right] < -\frac{1}{2x}+\frac{5}{12x^2}.$$

By (58), it suffices to show that for $x \geq 1$,

$$F(x) := x\ln\left(1+\frac{1}{x}\right) + \frac{1}{x}\left(-x+\frac{1}{12x}-\frac{1}{360x^3}\right) + \frac{1}{2x} > 0$$

and

$$G(x) := x\ln\left(1+\frac{1}{x}\right) + \frac{1}{x}\left(-x+\frac{1}{12x}\right) + \frac{1}{2x} - \frac{5}{12x^2} < 0.$$

By (59), we find that for $x \geq 1$,

$$F(x) > x\left(\frac{1}{x}-\frac{1}{2x^2}\right) + \frac{1}{x}\left(-x+\frac{1}{12x}-\frac{1}{360x^3}\right) + \frac{1}{2x} = \frac{30x^2-1}{360x^4} > 0$$

and

$$G(x) < x\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}\right) + \frac{1}{x}\left(-x+\frac{1}{12x}\right) + \frac{1}{2x} - \frac{5}{12x^2} = 0.$$

The proof is complete. $\square$

Remark 16. *The inequality* (62) *can be written for* $x \geq 1$ *as*

$$\frac{x}{\Gamma(x+1)^{1/x}} \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \exp\left(-\frac{1}{2x}\right) < \left(1+\frac{1}{x}\right)^{x} \tag{63}$$
$$< \frac{x}{\Gamma(x+1)^{1/x}} \left(\sqrt{2\pi x}\right)^{\frac{1}{x}} \exp\left(-\frac{1}{2x}+\frac{5}{12x^2}\right).$$

For $x \geq 1$*, the lower in* (63) *is sharper than the lower in* (42)*. For* $x \geq 3$*, the lower in* (63) *is sharper than the lower in* (41)*.*

Acknowledgments. The first author was supported by Key Science Research Project in Universities of Henan (20B110007), the Fundamental Research Funds for the Universities of Henan Province (NSFRF210446). The second author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI project number PN-II-ID-PCE-2011-3-0087. Computations in this paper were made using Maple software.

REFERENCES

1. S. Abramovich, J. Barić, M. Matić, J. Pečarić: *On Van de Lune–Alzer's inequality*. J. Math. Inequal. **1** (2007), 563–587.
2. M. Abramowitz, I.A. Stegun (Editors): *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series **55**, Ninth printing, National Bureau of Standards, Washington, D.C., 1972.
3. G. Allasia, C. Giordano, J. Pečarić: *On the arithmetic and logarithmic means with applications to Stirling's formula*, Atti Sem. Mat. Fis. Univ. Modena **47** (1999), no. 2, 441–455.
4. H. Alzer: *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.
5. H. Alzer: *On some inequalities involving* $(n!)^{1/n}$, Rocky Mountain J. Math. **24** (1994), no. 3, 867–873.
6. H. Alzer: *On some inequalities involving* $(n!)^{1/n}$ *, II*, Period. Math. Hungar. **28** (1994), no. 3, 229–233.
7. H. Alzer: *Refinement of an inequality of G. Bennett*, Discrete Math. **135** (1994), 39–46.
8. H. Alzer: *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), 373–389.
9. L.M. Brégman: *Some properties of nonnegative matrices and their permanents*, Soviet Math. Dokl. **14** (1973), 945–949.
10. C.-P. Chen: *Sharp inequalities involving* $(n!)^{1/n}$, Mediterr. J. Math. (2018) 15:177 `https://doi.org/10.1007/s00009-018-1212-y`
11. C.-P. Chen, N. Elezović, L. Vukšić: *Asymptotic formulae associated with the Wallis power function and digamma function*, J. Classical Anal. **2** (2013), 151–166.

12. C.-P. Chen, F. Qi: *Inequalities relating to the gamma function*, Aust. J. Math. Anal. Appl. **1** (2004), no.1, Article 3. 7 pp.

13. C.-P. Chen, F. Qi: *Extension of an inequality of H. Alzer for negative powers*, Tamkang J. Math. **36** (2005), no.1, 69–72.

14. C.-P. Chen, F. Qi, P. Cerone, S.S. Dragomir: *Monotonicity of sequences involving convex and concave functions*, Math. Inequal. Appl. **6** (2003), no. 2, 229–239.

15. W.-S. Chou, L.C. Hsu, P.J.-S. Shiue: *Application of Faà di Bruno's formula in characterization of inverse relations*, J. Comput. Appl. Math. **190** (2006), 151–169.

16. E. T. Copson: *Asymptotic Expansions*, Cambridge University Press, 1965.

17. K. Guan: *Multiplicative convexity and its applications*, J. Math. Anal. Appl. **362** (2010), 156–166.

18. G. H. Hardy: *Notes on a theorem of Hilbert*, Math. Z., **6** (1920), 314–317.

19. S. Koumandos: *Remarks on some completely monotonic functions*, J. Math. Anal. Appl. **324** (2006), 1458–1461.

20. S. Koumandos, H.L. Pedersen: *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function*, J. Math. Anal. Appl. **355** (2009), 33–40.

21. J.-C. Kuang: *Some extensions and refinements of Minc-Sathre's inequality*, Math. Gaz. **83** (1999), 123–127.

22. J. Liao, K. Guan: *On Alzer's inequality and its generalized forms*, J. Math. Inequal. **4** (2010), no. 2, 161–170.

23. M. Marcus, H. Minc: *Permanents*, Amer. Math. Monthly **72** (1965), no. 6, 577–591.

24. J.S. Martins: *Arithmetic and geometric means, an application to Lorentz sequence spaces*, Math. Nachr. **139** (1988), 281–288.

25. H. Minc: *Upper bound for permanents of* $(0,1)$*-matrices*, Bull. Amer. Math. Soc. **69** (1963), 789–791.

26. H. Minc, L. Sathre: *Some inequalities involving* $(r!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1964/65), 41–46.

27. C. Mortici: *A lower bound for a quotient of roots of factorials*, Bul. Acad. Şt. Rep. Moldova. Mat. **61** (2009), no. 3, 64–69.

28. C. Mortici: *Estimates of the function and quotient by Minc–Sathre*, Appl. Math. Comp. **253** (2015), 52–60.

29. J. Sándor: *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), 1034–1035.

30. A. Schrijver: *A short proof of Minc's conjecture*, J. Combinatorial Theory Ser. A **25** (1978), no. 1, 80–83.

31. J. Rooin, H. Dehghan: *ASome monotonicity properties of convex functions with applications*, Mediterr. J. Math. **12** (2015), no. 3, 593–604.

32. J.S. Ume: *An elementary proof of H. Alzer's inequality*, Math. Japonica **44** (1996), 521–522.

Chap-Ping Chen
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454000, Henan Province, China
E-mail: *chenchaoping@sohu.com*

(Received 10. 07. 2020.)
(Revised 16. 10. 2022.)

Cristinel Mortici
Valahia University of Târgovişte,
Bd. Unirii 18, 130082 Târgovişte,
Romania
Academy of Romanian Scientists,
Splaiul Independenţei 54, 050094 Bucharest, Romania
E-mail: *cristinel.mortici@hotmail.com*