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ENUMERATION OF HAMILTONIAN CYCLES ON A THICK GRID CYLINDER — PART II: CONTRACTIBLE HAMILTONIAN CYCLES

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Here, in Part II, we proceeded further with the enumeration of Hamiltonian cycles (HC's) on the grid cylinder graphs of the form $P_{m+1} \times C_n$, where n is allowed to grow and m is fixed. We proposed two novel characterisations of the contractible HC's. Finally, we made a conjecture concerning the dependency of the asymptotically dominant type of HC's on the parity of m.

1. INTRODUCTION

Determining and enumerating Hamiltonian cycles in some specific grid graphs (such as thick grid cylinder graphs, which are studied here) is of quite some relevance to statistical physics [**6**] and polymer science [**2**]. An ample amount of references related to this topic may be found in Part I [**1**]. A few novel applications of this type of research can be found within the field of network systems, which revolves around computer network functionality. Hamiltonian cycles play a vital role there, because they cover all the nodes of the system. In [**10**] the issue of handling indeterminacy for interval data under neutrosophic environment is considered. Another field, which may benefit from our research, is that of cyber security. There, digital microfluidic biochips (DMFBs) are making the transition to the marketplace for commercial exploitation. For example, the microelectrode dot array (MEDA) is a next-generation DMFB platform that supports real-time sensing of droplets and has the added advantage of important security protection [**7**].

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When m is fixed, the graphs $P_{m+1} \times C_n$ are referred to as the ***thick grid cylinders*** (see Figure 1a). When $n \geq 2$, there are two kinds of Hamiltonian cycles on such graphs. The first kind, denoted by HC ^{nc}'s, are not contractible when perceived as closed Jordan curves (see Figure 1b) on the infinite cylindrical surface on which the graph $P_{m+1} \times C_n$ is settled. They were examined in Part I [**1**] of this series. The second kind of HC's, denoted by HC ^{c}'s, are the contractible ones. They are studied in Part II of the series (this exposition). In both parts, we study the topological properties of the HC's. Based on these properties, we construct digraphs from which the HC's can be counted. The motivation behind our investigations is made clear in Part I, together with the reasons why we have opted for the cell-coding approach.

Contractible HC's are more complicated than the non-contractible ones. These contractible HC's divide the underlying infinite cylindrical surface into two separate regions. The first is bounded and is called the ***interior***, whereas the second one is called the ***exterior*** of the HC in question (see Figure 1b-c). Moreover, we refer to these regions as the ***zero*** and ***non-zero region*** depending on whether a zero is assigned to the squares of the interior or the exterior region.

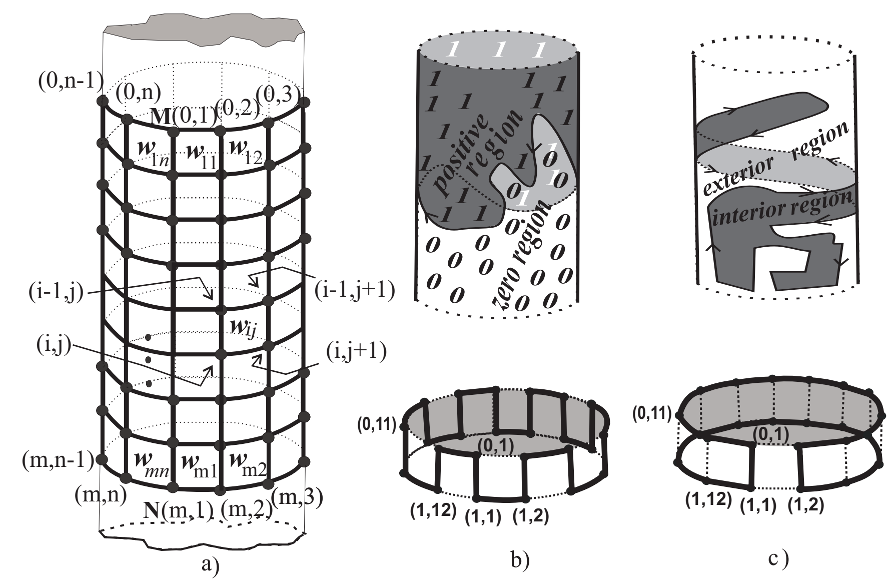

Figure 1: (a) The graph $P_{m+1} \times C_n$ with its cells (windows) labeled by $w_{i,j}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. (b) A non-contractible, closed Jordan curve on an infinite cylindrical surface. (c) A contractible, closed Jordan curve on an infinite cylindrical surface.

The paper is organised as follows: in Section 2, we examine HC ^{c}'s whose non-zero region is the exterior. Section 3 is devoted to HC ^{c}'s whose non-zero region is their interior.

The notations $h_m^{nc}(n)$ and $h_m^{c}(n)$ stand for the number of HC ^{nc}'s and HC ^{c}'s, respectively. Their respective generating functions are

$$\mathcal{H}_m^{nc}(x) \stackrel{\text{def}}{=} \sum_{n\geq 1} h_m^{nc}(n+1)x^n, \qquad \text{and} \qquad \mathcal{H}_m^{c}(x) \stackrel{\text{def}}{=} \sum_{n\geq 1} h_m^{c}(n+1)x^n.$$

The overall number of HC's in the thick grid cylinder graph $P_{m+1} \times C_n$ $(m \geq 1, n \geq 2)$ is denoted by $h_m(n)$. Clearly, $h_m(n) = h_m^{nc}(n) + h_m^{c}(n)$ and its generating function $\mathcal{H}_m(x) = \sum_{n\geq 1} h_m(n+1)x^n$ fulfills the equation $\mathcal{H}_m(x) = \mathcal{H}_m^{nc}(x) + \mathcal{H}_m^{c}(x)$.

The orientation of a HC ^{c} is determined in such a way that when traversing alongside the considered HC its interior region is always on the right-hand side (see Figure 1c). Recall an assertion from Theorem 1 of [**3**] concerning h_m^c:

(1) $$h_m^c(n) = 0 \text{ if and only if } m \text{ is even and } n \text{ is odd.}$$

Further, let us be reminded of a few additional definitions from Part I [**1**] needed hereinafter. All the rest may as well be found in the said paper, unless explicitly stated differently.

Definition 1. *Given an integer word* $d_1d_2\ldots d_m$*, its* **support** *is defined as the ternary word* $\bar{d}_1\bar{d}_2\ldots\bar{d}_m$*, where*

$$\bar{d}_i = \begin{cases} 1 & if \;\; d_i > 0, \\ 0 & if \;\; d_i = 0, \\ -1 & if \;\; d_i < 0. \end{cases}$$

The **support** *of an integer matrix* $[d_{i,j}]$ *is defined in a similar fashion.*

Definition 2. *The factor* u *of a word* v *is called a* b**-factor** *if it is a block of consecutive letters all of which are equal to* b*. A* b*-factor of* v *is said to be* **maximal** *if it is not a proper factor of another* b*-factor of* v*.*

Recall that the ***window lattice graph*** $W_{m,n}$, whose vertices are the square cells (or ***windows***) $w_{i,j}$ $(1 \leq i \leq m, \;\; 1 \leq j \leq n)$ of $P_{m+1} \times C_n$, is isomorphic to $P_m \times C_n$. For a HC ^{c}, the interior windows (marked with 0's as in Figure 3) form the ***interior tree (IT)*** in $W_{m,n}$. Nonetheless, the exterior windows form a forest of ***exterior trees (ET's)***. Note that only one ET from this forest contains exactly one window in the first as well as in the last (the m^{th}) row of $W_{m,n}$, called the ***up*** and ***down root***, respectively. We call this particular ET the ***split tree (ST)*** of the HC in question. Any other ET different from the split tree contains either exactly one down root or exactly one up root, but not both. The ET's with a down root are called the ***down trees (DT's)***, whereas the ET's with an up root are referred to as the ***up trees (UT's)***.

Example 1. For the purpose of illustration, take a look at the HC ^{c} depicted in Figures 3 and 4 whose split tree has the down root $w_{10,3}$, and the up root $w_{1,1}$. It also has one ET with the down root $w_{10,9}$ (hence a DT), and one ET with the up root $w_{1,7}$ (hence a UT); they are labeled by non-zero integers in Figure 3.

Note that, it suffices to examine only those HC ^{c}'s in $P_{m+1} \times C_n$ whose split tree has w_{11} for its up root. Let the number of such HC ^{c}'s be $\varphi_m^c(n-2)$, where $m \geq 1$, and $n \geq 2$, and let the associated generating function be:

$$\Phi_m^c(x) \stackrel{\text{def}}{=} \sum_{k \geq 0} \varphi_m^c(k) x^k.$$

This implies that the total number of HC ^{c}'s in $P_{m+1} \times C_n$ is given by

$$h_m^c(n) = n\varphi_m^c(n-2).$$

Consequently,

$$\mathcal{H}_m^c(x) = \sum_{n \geq 1} h_m^c(n+1)x^n = \sum_{n \geq 1}(n+1)\varphi_m^c(n-1)x^n$$

$$= \frac{d}{dx} \sum_{n \geq 1} \varphi_m^c(n-1)x^{n+1} = \frac{d}{dx}\left(x^2\, \Phi_m^c(x)\right).$$

There are two possible ways in which we code (or label) the windows with appropriate integers. The first, described in Section 2, is the one in which the windows of the IT are labeled with zeros, whilst the remaining windows are labeled with non-zero numbers. The second, which we deal with in Section 3, is the one in which the zero windows belong to the ET's, whereas the non-zero windows belong to the IT. This way, any HC^c can be viewed as a sequence of n columns comprising the coded windows. This sets up a one-to-one correspondence between the set of HC ^{c}'s and the set of sequences of n labeled columns.

Recall from [**1**] that $\mathcal{G}_m$ represents an infinite grid graph with vertices from the set $\{(i,j) \in \mathbb{Z}^2 \mid\ 0 \leq j \leq m\}$, in which the square cell determined by the points: $(j-1+kn, m-i)$, $(j+kn, m-i)$, $(j+kn, m-i+1)$ and $(j-1+kn, m-i+1)$, with $1 \leq i \leq m$ and $1 \leq j \leq n$, is labeled w_{ij}^k and is called a ***window***, too. We also say that w_{ij}^k belongs to the $(j+nk)^{\text{th}}$ column of $\mathcal{G}_m$. The set $\{w_{ij}^k \mid i,j,k \in \mathbb{Z}\ \wedge\ 1 \leq i \leq m\ \wedge\ 1 \leq j \leq n\}$ presents the set of vertices of another infinite grid graph denoted by $\mathcal{W}_m$.

Consider a HC ^{c} in the graph $P_{m+1} \times C_n$. Loosely speaking, a ***rolling imprint (RI)*** is a picture obtained as follows. First we "cut through" the surface of our graph $P_{m+1} \times C_n$ (with a HC in it) along the line which connects the vertices $M(0,1)$ and $N(m,1)$, see Figure 1a. Next, we unroll and flatten it; see the rectangle $\mathcal{R}_0 : M_0N_0N_1M_1$ in Figure 2a. Finally we produce many copies of the initial picture ($\mathcal{R}_{-1}$, $\mathcal{R}_{-2}$, $\mathcal{R}_{-3}, \ldots$ and $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3, \ldots$), and line them up to the left and to the right side accordingly; see Figure 2b. Since the HC is contractible, its RI is actually the graph $\mathcal{G}_m$ with infinitely many mutually congruent polygonal lines on it. These polygonal lines are the boundaries of the polygons consisting of all the vertices (w_{ij}^k) of $\mathcal{W}_m$ that correspond to the windows (w_{ij}) of $P_{m+1} \times C_n$ from the interior of the HC ^{c} (the white squares in Figure 2b or the gray squares in Figure 2c). That way parts of the interior and exterior trees that were initially broken by the process of "cutting" of the $W_{m,n}$ are now assembled again into the original forms, and multiplied in $\mathcal{W}_m$. What we obtain is a sequence of copies $\dots S^{-3}, S^{-2}, S^{-1}, S^0, S^1, S^2, \dots$ of a "new" split tree S^0, some sequences of copies $\dots T_s^{-3}, T_s^{-2}, T_s^{-1}, T_s^0, T_s^1, T_s^2, \dots$ of the "new" exterior trees $\dots T_s^0$ and a sequence of copies $\dots I^{-2}, I^{-1}, I^0, I^1, I^2, \dots$ of the "new" interior tree I^0.

At this point we need to modify a few definitions stated in [**1**], as follows:

Definition 3. *The* **basis of a rolling imprint (BRI)** *is the union of the vertex set of the split tree whose down root is in $\mathcal{R}_0$, the vertex sets of all the exterior trees (different from the split tree) each of which has its root in $\mathcal{R}_0$, and the vertex set of the interior tree whose leftmost window from the first row belongs to $\mathcal{R}_0$.*

Note that, in this way, we establish a bijection between the set of vertices in $V(W_{m,n})$ and the BRI (see Figure 2d).

The aforementioned coding of the windows is, in both cases, dealt with in two stages. In the first stage the graph $W_{m,n}$ is associated with the matrix $A^c = [a_{ij}]_{m\times n}$ whose entries are from $\{-1, 0, 1\}$. The windows $w_{i,j}$ are called the ***zero windows*** if and only if $a_{i,j} = 0$, otherwise they are named the ***non-zero windows***. The coding is done by associating the same number to each of the vertices of the same tree (be it a ST, ET or IT) during the first stage. For instance, in the first case all the vertices of the UT's were coded with -1, whilst all the vertices of the DT's and ST were coded with 1. Therefore, we say that the DT's and the ST are ***positive trees (PT)***, whereas for the UT's we say that they are ***negative trees (NT)***, or simply ***non-zero trees***, irrespective of the case. In the second case, the term ***positive tree (PT)*** or ***non-zero tree*** refers to the IT. The term ***zero tree*** is used in a similar manner. The roll number depends on the type of cell (zero or non-zero).

Definition 4. *The* **roll number** *(or simply* **roll***) of a window $w_{i,j} \in V(W_{m,n})$, denoted by $r(w_{i,j})$ (or simply r if the window is clear from the context) is a unique integer k for which w_{ij}^k belongs to a non-zero tree of the BRI, or in case w_{ij} is a zero window, we set $r = 0$. We shall also say that the window $w_{i,j}$* **belongs to roll** *r.*

Example 2. For the HC ^{c} whose BRI is presented in Figure 2d, the roll numbers of some specific windows are summarized below.

non-zero	coding	roll number									
tree(s)	method	$w_{9,12}$	$w_{7,12}$	$w_{3,11}$	$w_{5,11}$	$w_{1,2}$	$w_{1,n}$	$w_{8,11}$	$w_{8,8}$	$w_{4,12}$	$w_{2,2}$
IT	second	-2	-2	-1	-1	0	0	0	0	0	0
ETs	first	0	0	0	0	0	0	-1	0	1	2

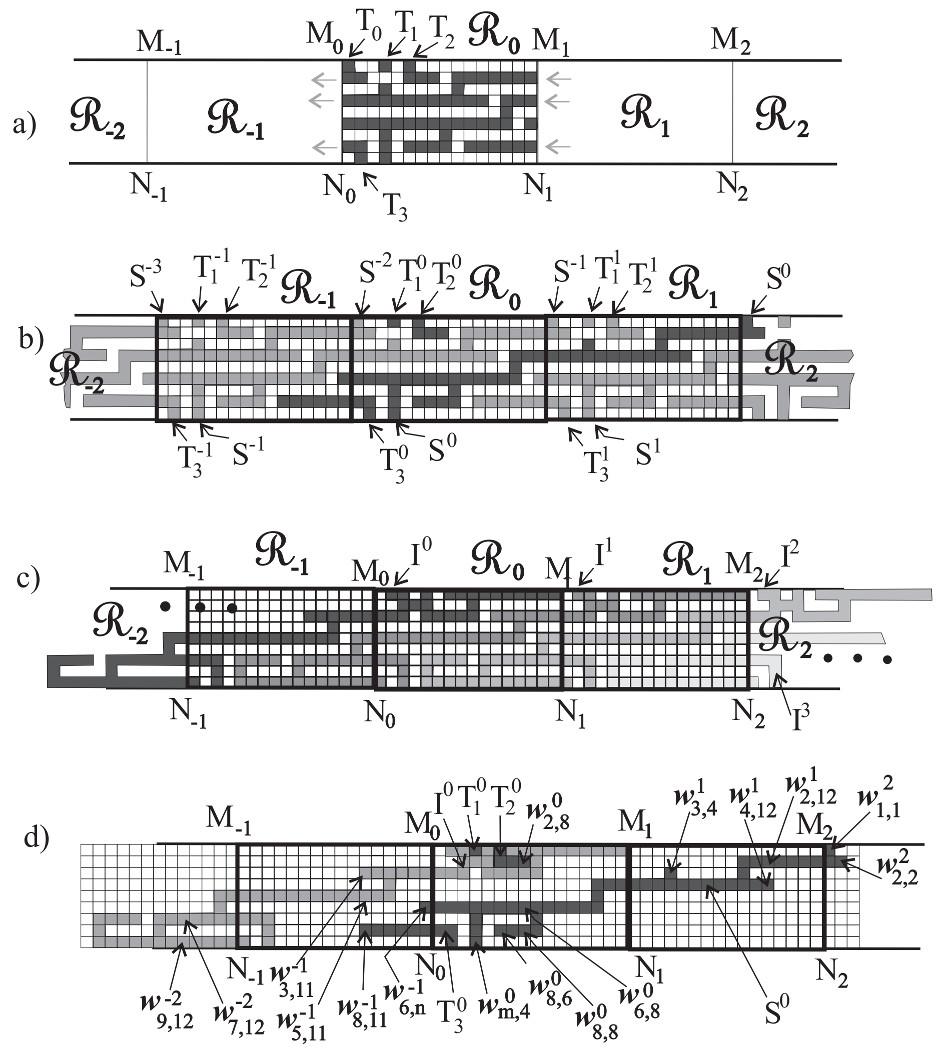

Figure 2: (a) Unrolling and flattening the cylindrical surface that contains a HC ^{c}. (b) The rolling imprint of a HC ^{c}, with the copies of the "new" split tree and ETs in gray. (c) The rolling imprint of a HC ^{c}, with the copies of the "new" interior tree in gray. (d) The basis of a rolling imprint (BRI) consists of the windows of all the "new" trees I^0, S^0, T_1^0, T_2^0 and T_3^0.

Definition 5. *Two non-zero vertices $w_{i,t}$ and $w_{j,s}$ of $W_{m,n}$ with $a_{i,t} = a_{j,s}$ are said to be* **joined at the k-th column with the roll number** *r, or simply* **k^r-joined**, *where $1 \le k \le n$, and $-\lfloor \frac{m}{2} \rfloor \le r \le \lfloor \frac{m}{2} \rfloor$, if and only if their corresponding windows in the BRI belong to the same component in the subgraph of $\mathcal{W}_m$ induced by the set of all non-zero windows $w^z_{x,y}$ from the BRI that satisfy both $a_{x,y} = a_{i,t} = a_{j,s}$, and either (i) $z = r(w_{xy}) < r$, or (ii) $z = r(w_{xy}) = r$ and $y \le k$.*

Example 3. Let us once again take a look at Figure 2d assuming the first way of coding ($a_{1,1} \neq 1$). There, windows $w_{6,8}$ and $w_{8,8}$ are not 8^0-joined, but instead are 9^0-joined. Also, $w_{2,12}$ is 12^1-joined with $w_{4,12}$.

In Sections 2 and 3, as we said, we present two different characterizations of HC ^{c} where w_{11} is the up root of the split tree. Both of them allow for the use of the transfer matrix method with a view to obtaining the values of $h^c_m(n)$'s. In Section 4, we determine the upper bound of the so-called color^r words which appear in these procedures. Sections 5 and 6 contain comparative analysis of the numerical results obtained by using these two characterizations and some other conclusions including two new conjectures. Section 7 is devoted to closing remarks.

2. CODING THE EXTERIOR TREES BY NON-ZERO ENTRIES

2.1. The First Phase — the Matrix $A^{c,Ext}$

For any integer $m \ge 1$, we associate with each HC ^{c} in $P_{m+1} \times C_n$ with $w_{1,1}$ as the up root of the split tree a matrix $A^{c,Ext} = [a_{ij}]_{m\times n}$ whose entries are defined in the following way:

$$a_{i,j} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } w_{i,j} \text{ belongs to the IT,} \\ -1 & \text{if } w_{i,j} \text{ belongs to a UT,} \\ 1 & \text{if } w_{i,j} \text{ belongs to a DT or the ST.} \end{cases}$$

Obviously, $a_{1,1} \stackrel{\text{def}}{=} 1$, $a_{1,2} \stackrel{\text{def}}{=} 0, a_{1,n} \stackrel{\text{def}}{=} 0$, and $a_{2,1} \stackrel{\text{def}}{=} 1$. Note that $w_{1,1}$ is the only positive window in the first row (on the "negative coast"). We adopt the convention that $a_{i,n+1} \stackrel{\text{def}}{=} a_{i,1}$, and $a_{i,0} \stackrel{\text{def}}{=} a_{i,n}$, for $1 \le i \le m$.

Lemma 1. *Every HC ^{c} on the thick grid cylinder graph $P_{m+1} \times C_n$ (with $w_{1,1}$ as the up root of the split tree) determines a matrix $A^{c,Ext} = [a_{ij}]_{m\times n}$, with entries from the set $\{-1, 0, 1\}$, which satisfies the conditions below.*

1. ***First and Last Row Conditions*** *(FL ^{c,Ext}):*

   (a) $a_{1,1} = 1$, *and* $a_{1,2} = a_{1,n} = 0$.

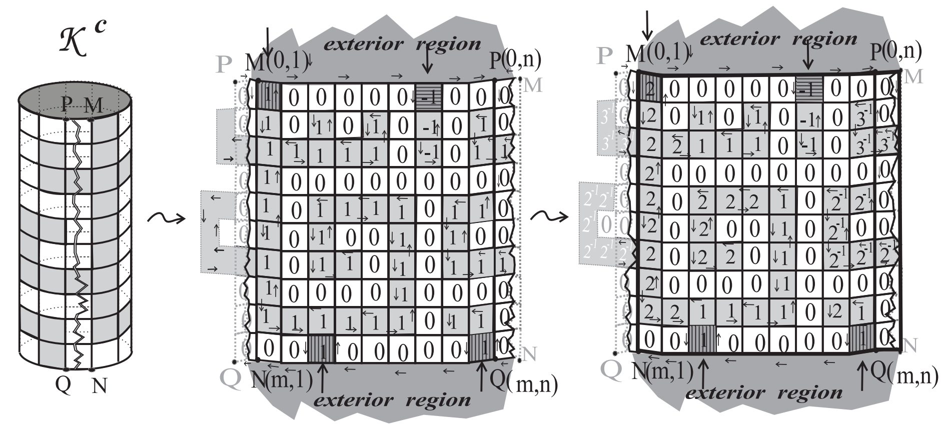

Figure 3: A contractible Hamiltonian cycle of $P_{11} \times C_{10}$ with the entries of the matrices $A^{c,Ext}$ (the first phase) and $B^{c,Ext}$ (the second phase) written on their windows as the cylindrical surface is drawn on a flat surface.

(b) *For* $2 \le j \le n$,

$$a_{1,j} \in \{0, -1\}, \quad \textit{and} \quad (a_{1,j}, a_{1,j+1}) \not\equiv (-1, -1).$$

(c) *For* $1 \le j \le n$,

$$a_{m,j} \in \{0, 1\}, \quad \textit{and} \quad (a_{m,j}, a_{m,j+1}) \not\equiv (1, 1).$$

2. ***Adjacency of Column Conditions*** *($AC^{c,Ext}$):*

(a) *For* $1 \le i \le m-1$ *and* $1 \le j \le n$,

$$\begin{aligned} &(|a_{i,j}|, |a_{i+1,j}|, |a_{i,j+1}|, |a_{i+1,j+1}|) \\ &\quad \notin \quad \{(1,1,1,1), (0,0,0,0), (0,1,1,0), (1,0,0,1)\}. \end{aligned}$$

(b) *For* $1 \le i \le m$ *and* $1 \le j \le n$,

$$a_{i,j} \cdot a_{i,j+1} \ne -1.$$

(c) *For* $1 \le i \le m-1$ *and* $1 \le j \le n$,

$$a_{i,j} \cdot a_{i+1,j} \ne -1.$$

3. ***Root Conditions*** *($RC^{c,Ext}$):*

(a) *Each connected component of the subgraph of the graph $W_{m,n}$ induced by the windows corresponding to the non-zero entries (± 1) of the matrix $A^{c,Ext}$ is a tree.*

(b) There exists exactly one such tree (the split tree) containing exactly one window from the first row ($a_{11} = 1$), and exactly one window from the last row of $W_{m,n}$.

(c) Each of the remaining trees, if any such exist, contains exactly one window from either the first or the last row of $W_{m,n}$.

Conversely, every matrix $[a_{ij}]_{m\times n}$ with entries from the set $\{-1, 0, 1\}$ which fulfills the conditions $FL^{c,Ext}$, $AC^{c,Ext}$ and $RC^{c,Ext}$, determines a unique HC^c on the graph $P_{m+1} \times C_n$ whose split tree contains the window $w_{1,1}$.

Proof. The necessity of all three imposed conditions is easily verifiable and is thus left to the reader. Therefore, we move on to the proof of their sufficiency. Let us observe all the regions determined by all the non-zero windows including the two half-cylinders from the one side (at this moment, we cannot assume that there exists a unique such region). The first two conditions are local conditions ensuring that the boundary of the said regions (the edges which belong to both non-zero window or the boundary of one of the two half-cylinders, and a zero window) determines a unique spanning 2-regular subgraph of $P_{m+1} \times C_n$, that is, a union of cycles.

The proof that this graph consists of only one component (consequently establishing the uniqueness of both zero and non-zero regions) can be derived constructively. The case $n = 2$ is trivial, so we can assume that $n \geq 3$. The condition AC ^{c,Ext} implies that each of the components of the subgraph of $W_{m,n}$ induced by the windows corresponding to the non-zero entries consists only of either 1-windows or (-1)-windows. Thus, we have justified the existence of both positive (DT's and ST) and negative trees (UT's).

Let $w_{m,p}$ be the down root of the unique positive tree T_0 (the split tree) with a window w_{11} in the first row in it and $T_1, T_2, \ldots, T_k$ be all the NT's (if any such tree exists at all) with the up roots $w_{1,j_1}, w_{1,j_2}, \ldots, w_{1,j_k}$, respectively, for which $3 \leq j_1 < j_2 < \ldots < j_k \leq n$. Let $T'_1, T'_2, \ldots, T'_l$ be all the PT's different from T_0 (if any such tree exists in the first place) with the down roots $w_{m,i_1}, w_{m,i_2}, \ldots, w_{m,i_l}$, respectively, for which $i_s \equiv i'_s \pmod n$, where $1 \leq s \leq l$ and $p + 2 \leq i'_1 < i'_2 < \ldots < i'_l \leq n+p-2$. Our task is to obtain the unique curve (the broken line) which separates the regions of the two kinds of windows (the zero and non-zero ones).

We can start from the point $M : (0, 1)$ (the upper-left point of the up root $w_{1,1}$ of T_0) and move to the lower-left point of $w_{m,p}$ using the edges of $P_{m+1} \times C_n$ that belong to the boundary of T_0. From there we continue towards the point $(m, i_l + 1)$, and then visit all the vertices on the boundary of T'_l finishing at the point (m, i_l). Next, we visit the boundary of T'_{l-1}, T'_{l-2}, ... After having visited the tree T'_1 we end up at the point (m, i_1). Then, we move further to the point $(m, p + 1)$ and continue towards the point $(0, 2)$ using the remaining edges of the boundary of ST. From there we similarly continue visiting the boundaries of trees $T_1, T_2, \ldots, T_k$, respectively, ending up at the point M again (see Figure 3). By doing so, we pass through all the edges on the boundary of these regions, obtaining a contractible HC. □

Recall that, for fixed values of k and r ($1 \leq k \leq n$, and $-\lfloor \frac{m}{2} \rfloor \leq r \leq \lfloor \frac{m}{2} \rfloor$), the relation k^r-joined represents an equivalence relation on the set of all non-zero windows $w_{x,y}$ that satisfy either (i) $r(w_{x,y}) < r$, or (ii) $r(w_{x,y}) = r$ and $y \leq k$ (that is, whose window from the BRI belongs to the $(y+nk)^{\text{th}}$ column of $\mathcal{G}_m$ or to the left of it). Furthermore, every equivalence class belongs to exactly one ET. Hence if this equivalence class belongs to a PT its windows can be k^r-joined with at most one down root. If it belongs to an NT its windows can be k^r-joined with at most one up (negative) root. Further, because an ST is a PT, we treat its down root as its main root and this is what we shall assume below. Note that the roll number of $w_{1,1}$ could be different from 0 (for example, the roll of $w_{1,1}$ in Figure 2d is 2). But, all the other roots of the ET's have their roll number equal to 0.

2.2. The First Characterization of HC ^{c} with w_{11} as the Up Root of the Split Tree

Let $\mathcal{C}^+ \stackrel{\text{def}}{=} \{2, 3, \ldots, \lfloor \frac{m}{2} \rfloor + 1\}$ and $\mathcal{C}^- \stackrel{\text{def}}{=} \{-2, -3, \ldots, -\lfloor \frac{m}{2} \rfloor - 1\}$. For each HC ^{c} with the window $w_{1,1}$ belonging to the split tree, we associate the matrix $A^{c,Ext} = [a_{i,j}]_{m \times n}$ with the matrix $B^{c,Ext} = [(b_{i,j}, r_{i,j})]_{m \times n}$, where $b_{i,j} \in \mathcal{C}^+ \cup \mathcal{C}^- \cup \{1, 0, -1\}$ and $-\lfloor \frac{m}{2} \rfloor \leq r_{i,j} \leq \lfloor \frac{m}{2} \rfloor$. The former of the two satisfies the conditions FL ^{c,Ext}, AC ^{c,Ext}, and RC ^{c,Ext}, whereas the latter is constructed in the following way:

1. Define $r_{i,j} = r(w_{i,j})$.
2. Set $b_{i,j} = a_{i,j} = 0$ if $w_{i,j}$ belongs to the IT.
3. If $w_{i,j}$, where $w_{i,j} \neq w_{1,1}$, is the (up) root of an NT (that is, $i = 1$ and $a_{i,j} = -1$) or the down root of a PT (that is, $i = m$ and $a_{i,j} = 1$), set $b_{i,j} = a_{i,j}$. If $w_{i,j}$ is neither the down root of a PT nor the up root of an NT, but it is j^r-joined with such a root, where $r = r(w_{i,j})$, set $b_{i,j} = a_{i,j}$.
4. For each fixed column, say column j:
   (a) Scan the remaining positive windows $w_{i,j}$ with the same roll number from bottom to top (that is, from $i = m$ to $i = 1$), and set $b_{i,j}$ to $z + 1$, where z is the ordinal number of the j^r-joined equivalence class, $r = r(w_{i,j})$, to which it belongs to (hence, the labels of the $b_{i,j}$'s start from 2).
   (b) Scan the remaining negative windows $w_{i,j}$ with the same roll number, from top to bottom (from $i = 1$ to $i = m$), and set $b_{i,j}$ to $z - 1$, where z is the negative value of the ordinal number of the j^r-joined equivalence class, $r = r(w_{i,j})$, to which it belongs to (hence, the labels of the $b_{i,j}$'s start from -2).

Example 4. In Figure 3, the entries in the matrix $B^{c,Ext}$ are written on their respective windows ($b_{ij}^{r_{ij}}$ stands in place of (b_{ij}, r_{ij}), or just b_{ij} if $r_{ij} = 0$). Note that in the 9^{th} column there exist three parts of the same PT (it is the split tree) with the same roll number -1, but the windows of only two of them are 9^{-1}-joined. Consequently, the same value is associated to their entries in the matrix $B^{\,c,Ext}$ ($b_{5,9} = b_{7,9} = 2$, and $b_{3,9} = b_{2,9} = 3$). Another example is shown in Figure 2d. There, the entries of the matrix $B^{\,c,Ext} = [(b_{i,j}, r_{i,j})]_{9\times 16}$ for windows $w_{1,1}$ and $w_{2,2}$ are $(1,2)$, for windows $w_{3,4}$, $w_{2,12}$ and $w_{4,12}$ are $(1,1)$, for windows $w_{m,2}$, $w_{m,4}$ and $w_{6,8}$ are $(1,0)$, for windows $w_{8,6}$ and $w_{8,8}$ are $(2,1)$, for 'windows $w_{1,4}$, $w_{1,6}$ and $w_{2,8}$ are $(-1,0)$ and for windows $w_{3,11}$, $w_{5,11}$, $w_{7,12}$ and $w_{9,12}$ are $(0,0)$.

Consider all the existing maximal b-factors, where $b > 0$ ($b < 0$), in the j^{th} column $v = b_{1,j}b_{2,j}\ldots b_{m,j}$, where $1 \le j \le n$, of the matrix $[b_{i,j}]_{m\times n}$ corresponding to the matrix $B^{\,c,Ext} = [(b_{i,j}, r_{i,j})]_{m\times n}$. Let them be, in their order of appearance (that is, from bottom to top for positive windows, but from top to bottom for negative windows), p_1-factor, p_2-factor, $\ldots$, p_k-factor, where $k \ge 1$, and $p_i \ge 1$ ($p_i \le -1$) for each i. In addition, let $r_1, r_2, \ldots, r_k$ denote the roll numbers associated with these maximal factors. The words $p_1p_2\ldots p_k$ and $r_1r_2\ldots r_k$ are called the ***positive (respectively, negative) truncated word*** and the ***positive (resp., negative) truncated roll word***, respectively, corresponding to the j^{th} column of $B^{\,c,Ext}$. A subsequence of a truncated word induced by the letters with the same roll number r is called a ***positive (resp., negative) color^r word***.

Example 5. For the 9^{th} column in Figure 3, the positive truncated word, the positive truncated roll word, the positive color^0 word, and the positive color^{-1} word are 1223, $0 \; -1 \; -1 \; -1$, 1 and 223, respectively. Note that, in general, $r_{1,1}$ need not be 0, and $b_{1,1}$ need not be 2.

2.3. Properties of the Matrix $B^{\,c,Ext}$

From the definition of the matrix $B^{\,c,Ext} = [(b_{i,j}, r_{i,j})]_{m\times n}$, we can easily obtain a number of properties expressed in the following theorem. Bear in mind that here $(b_{i,n+1}, r_{i,n+1}) \stackrel{\text{def}}{=} (b_{i,1}, r_{i,1})$, and $(b_{i,0}, r_{i,0}) \stackrel{\text{def}}{=} (b_{i,n}, r_{i,n})$.

Theorem 1. *The matrix $B^{\,c,Ext} = [(b_{i,j}, r_{i,j})]_{m\times n}$ satisfies the following conditions.*

1. ***Basic Properties***

   (a) *The support of the matrix $[b_{i,j}]_{m\times n}$, that is, the matrix $[a_{i,j}]_{m\times n}$, satisfies the conditions $FL^{\,c,Ext}$ and $AC^{\,c,Ext}$.*

   (b) Harmonization of the adjacent entries which have the same sign*: For $2 \le i \le m$, and $1 \le k \le n$, if $a_{i-1,k} = a_{i,k}$, then $(b_{i-1,k}, r_{i-1,k}) = (b_{i,k}, r_{i,k})$.*

   (c) *For $1 \le i \le m$, and $1 \le j \le n$, if $a_{i,j} = 0$, then $r_{i,j} = 0$.*

   (d) *For $3 \le j \le (n-1)$, if $a_{1,j} \ne 0$, then $(b_{1,j}, r_{1,j}) = (-1, 0)$.*
   For $1 \le j \le n$, if $a_{m,j} \ne 0$, then $(b_{m,j}, r_{m,j}) = (1, 0)$.

2. ***Column Properties***

For $1 \le k \le n$, the k-th column $[(b_{1,k}, r_{1,k}), (b_{2,k}, r_{2,k}), \dots, (b_{m,k}, r_{m,k})]^T$ of the matrix $B^{\,c,Ext}$ satisfies these conditions:

   (a) *If there exists an entry (s, r) in the k^{th} column of the matrix $B^{\,c,Ext}$, where $s \ge 3$, then for each $\ell \in \{2, 3, \dots, s-1\}$, at least one copy of the entry (ℓ, r) must appear after the last appearance of the entry (s, r). Likewise, if there exists an entry (s, r) in the k^{th} column of the matrix $B^{\,c,Ext}$, where $s \le -3$, then for each $\ell \in \{-2, -3, \dots, s+1\}$, at least one copy of the entry (ℓ, r) must appear before the first appearance of the entry (s, r).*

   (b) *For $1 \le i \le m$, if $b_{i,k} \in \{-1, 1\}$, then $r_{i,k} \ge 0$.*

   (c) *If there exists an entry $(2, r)$ with $r \ge 1$ in the k^{th} column of the matrix $B^{\,c,Ext}$, then at least one entry $(1, r)$ must exist in the same column. Likewise, if there exists an entry $(-2, r)$ with $r \ge 1$ in the k^{th} column of the matrix $B^{\,c,Ext}$, then at least one entry $(-1, r)$ must exist in the same column.*

   (d) *If the negative (positive) truncated roll word of the k^{th} column of the matrix $B^{\,c,Ext}$ is not an empty word, it begins (ends) with an element from $\{-1, 0, 1\}$.*

3. ***Adjacency Properties***

For $1 \le k \le n$, the k^{th} column of $B^{\,c,Ext}$ satisfies these conditions.

   (a) *For $1 \le i \le m$ and $2 \le k \le n$, if $a_{i,k-1} = a_{i,k} \neq 0$, then $r_{i,k-1} = r_{i,k}$.*

   (b) *For $1 \le i \le m$, if $b_{i,k-1} = 1$, then $b_{i,k} \in \{0, 1\}$, and if $b_{i,k-1} = -1$, then $b_{i,k} \in \{-1, 0\}$.*

   (c) *For each ordered pair (b, r) with $b \ge 2$ ($b \le -2$) which appears in the $(k-1)^{st}$ column, there must be an index i for which $(b_{i,k-1}, r_{i,k-1}) = (b, r)$, and $b_{i,k} \in C^+ \cup \{1\}$ ($b_{i,k} \in C^- \cup \{-1\}$).*

   (d) *For $1 \le i, j \le m$, where $i \neq j$, if $(b_{i,k-1}, r_{i,k-1}) = (b_{j,k-1}, r_{j,k-1})$ and $a_{i,k} = a_{j,k} = a_{i,k-1} = a_{j,k-1} \neq 0$, then $b_{i,k} = b_{j,k}$.*

   (e) *For $1 \le i, j \le m$, where $i \neq j$, if $(b_{i,k-1}, r_{i,k-1}) = (b_{j,k-1}, r_{j,k-1})$, $a_{i,k} = a_{j,k} = a_{i,k-1} = a_{j,k-1} \neq 0$, and $b_{i,k} = b_{j,k} = b$, then there is no b-factor in the word $b_{1,k} b_{2,k} \dots b_{m,k}$ which contains both $b_{i,k}$ and $b_{j,k}$.*

   (f) *For every maximal 1-factor (respectively, (-1)-factor) v in the word $b_{1,k} b_{2,k} \dots b_{m,k}$, exactly one of the following three conditions is fulfilled:*

      i. *v either contains the letter $b_{m,k}$ (resp., $b_{1,k}$), or*

      ii. *in the $(k-1)^{st}$ column there is exactly one letter $b_{i,k-1} = 1$ (resp., $b_{i,k-1} = -1$) for which $b_{i,k} \in v$, or*

iii. *there exists exactly one sequence* $v = v_1, v_2, \ldots, v_p$, *where* $p > 1$, *of different maximal 1-factors (respectively,* (-1)*-factors) in the word* $b_{1,k}b_{2,k}\ldots b_{m,k}$ *satisfying the following conditions:*

- *For every* i $(1 \leq i \leq p-1)$ *in the word* $b_{1,k-1}b_{2,k-1}\ldots b_{m,k-1}$, *there is exactly one letter* $b_{j_i,k-1} \in \mathcal{C}^+$ *(resp.,* $b_{j_i,k-1} \in \mathcal{C}^-$*) for which* $b_{j_i,k} \in v_i$, *and there is exactly one letter* $b_{s_{i+1},k-1} \in \mathcal{C}^+$ *(resp.,* $b_{s_{i+1},k-1} \in \mathcal{C}^-$*) for which* $b_{s_{i+1},k} \in v_{i+1}$,

$$(b_{j_i,k-1}, r_{j_i,k-1}) = (b_{s_{i+1},k-1}, r_{s_{i+1},k-1}),$$

and $j_i \neq s_i$ *for* $1 < i < p$.

- *The factor* v_p *contains either the letter* $b_{m,k}$ *(resp.,* $b_{1,k}$*), or in the* $(k-1)^{st}$ *column there exists exactly one letter* $b_{i,k-1} = 1$ *(resp.,* $b_{i,k-1} = -1$*) for which* $b_{i,k} \in v_p$.

(g) *For* $b \notin \{-1, 0, 1\}$, *if* v *and* u *represent two different maximal* b*-factors in the word* $b_{1,k}b_{2,k}\ldots b_{m,k}$ *with the same roll number, then there is a unique sequence* $v = v_1, v_2, \ldots, v_p = u$, *where* $p > 1$, *of distinct maximal* b*-factors for which it is true that:*

- *For every* i, *where* $1 \leq i \leq p-1$, *there is exactly one* $b_{j_i,k-1}$ *with* $a_{j_i,k-1} = a_{j_i,k} = 1$ $(a_{j_i,k-1} = a_{j_i,k} = -1)$, *such that* $b_{j_i,k} \in v_i$, *and there is a unique* $b_{s_{i+1},k-1}$ *with* $a_{s_{i+1},k-1} = a_{s_{i+1},k} = 1$ $(a_{s_{i+1},k-1} = a_{s_{i+1},k} = -1)$ *such that* $b_{s_{i+1},k} \in v_{i+1}$,

$$(b_{j_i,k-1}, r_{j_i,k-1}) = (b_{s_{i+1},k-1}, r_{s_{i+1},k-1}),$$

and $j_i \neq s_i$.

4. ***Buckle Properties*** *(Specific Properties of the First, Second, and Last Columns)*

(a) *We have* $b_{1,1} > 0$, $b_{2,1} > 0$, $(b_{1,n}, r_{1,n}) = (b_{1,2}, r_{1,2}) = (0,0)$.

(b) *For* $2 \leq i \leq m$, *if* $a_{i,n} = a_{i,1} \neq 0$, *then* $r_{i,1} = r_{i,n} + 1$.

(c) *If there exists* $i \in \{1, 2, \ldots, m\}$ *such that* $r_{i,1} < 0$, *then there exists* $j \in \{3, 4, \ldots, m\}$ *such that* $j \neq i$, $a_{j,1} = a_{i,1}$, *and* $r_{j,1} = 0$.

(d) *If there exists a maximal 1-factor* $b_{i_1,1}\ldots b_{i_2,1}$, *where* $1 \leq i_1 \leq i_2 \leq m$, *with* $r_{i_1,1} = \ldots = r_{i_2,1} = 0$, *then*

- $i_2 = m$, *or*
- *there exists* j_1 *with* $i_2 \leq j_1 < m$ *such that the word* $b_{j_1,1}\ldots b_{m,1}$ *is a maximal 1-factor with* $r_{j_1,1} = r_{j_1+1,1} = \ldots = r_{m,1} = 0$, *and there exist* $i, j \in \{1, 2, \ldots, m\}$ *such that* $i_1 \leq i \leq i_2 < j_1 \leq j < m$, $b_{i,n} = b_{j,n}$, *and* $r_{i,n} = r_{j,n} = -1$.

In addition, the first column does not contain any of the -1*-factor* $b_{i_1,1}\ldots b_{i_2,1}$, *where* $1 \leq i_1 \leq i_2 \leq m$, *with* $r_{i_1,1} = \ldots = r_{i_2,1} = 0$.

(e) *If the last column of* $B^{\,c,Ext}$ *contains the entry* $(2,0)$*, then it must contain the entry* $(1,0)$ *just as well. Similarly, if the last column of* $B^{\,c,Ext}$ *contains the entry* $(-2,0)$*, then it must contain the entry* $(-1,0)$*, too.*

5. ***Topological Properties***

(a) *For* $1 < i_1 < j_1 < i_2 < j_2 < m$*, if* $b_{i_1,k} = b_{i_2,k} < -1$*,* $b_{j_1,k} = b_{j_2,k} < -1$*, and* $r_{i_1,k} = r_{i_2,k} = r_{j_1,k} = r_{j_2,k}$*, then* $b_{i_1,k} = b_{j_1,k}$*. Likewise, for* $1 \le i_1 < j_1 < i_2 < j_2 < m$*, if* $b_{i_1,k} = b_{i_2,k} > 1$*,* $b_{j_1,k} = b_{j_2,k} > 1$*, and* $r_{i_1,k} = r_{i_2,k} = r_{j_1,k} = r_{j_2,k}$*, then* $b_{i_1,k} = b_{j_1,k}$*.*

(b) *For* $1 \le i_1 < j < i_2 < m$*, if* $b_{i_1,k} = b_{i_2,k} \le -1$*,* $b_{j,k} = -1$*, and* $r_{i_1,k} = r_{i_2,k} = r_{j,k}$*, then* $b_{i_1,k} = b_{i_2,k} = -1$*. Likewise, for* $1 \le i_1 < j < i_2 \le m$*, if* $b_{i_1,k} = b_{i_2,k} \ge 1$*,* $b_{j,k} = 1$*, and* $r_{i_1,k} = r_{i_2,k} = r_{j,k}$*, then* $b_{i_1,k} = b_{i_2,k} = 1$*.*

(c) *Assume* $1 \le i < j \le m$*, if* $b_{i,k} = b_{j,k} = -1$ *and* $r_{j,k} \ne r_{i,k}$*, then we must have* $r_{i,k} < r_{j,k}$*. Likewise, if* $b_{i,k} = b_{j,k} = 1$ *and* $r_{j,k} \ne r_{i,k}$*, then we must have* $r_{i,k} > r_{j,k}$*.*

(d) *The absolute value of the difference between two adjacent letters in the negative (or positive) truncated roll word (unless it is an empty word) corresponding to the* k^{th} *column of the matrix* $B^{\,c,Ext}$ *is at most 1.*

(e) *For* $1 \le i, j \le m$*, if* $b_{i,k} = -1$ *and* $b_{j,k} = 1$*, then* $i < j$*.*

(f) *If the word* $b_{1,k} \dots b_{m,k}$ *does not contain 1 or* -1*, with the exception of eventual roots (*$b_{1,k} = -1$ *and/or* $b_{m,k} = 1$*), and if among all the entries of the* k^{th} *column of* $B^{\,c,Ext}$ *with the same fixed roll number* r *(note that* $r \le 0$*) there exist both negative* $b_{i,k}$*, where* $1 < i < m$*, and positive* $b_{j,k}$*, where* $1 \le j < m$*, then the first occurrence of the entry* $(b_{i,k}, r)$ *in the column with the smallest negative number* $b_{i,k}$*, such that* $r_{i,k} = r$*, must appear before (when viewed from the top row to the bottom row) the last occurrence of the entry* $(b_{j,k}, r)$ *with the largest positive number* $b_{j,k}$ *such that* $r_{j,k} = r$*.*

Proof. If we were to compare the statements of this Theorem, except for 5(e) and 5(f), to the corresponding ones in Theorem 4 of [**1**], which relate to HC ^{nc}, we would find their formulations fairly similar to one another. The proofs of them are thus analogous to their counterparts, and shall not be restated. Instead, we move on to the two remaining exceptional cases.

Proof of 5(e): Suppose, on the contrary, that $i > j$. Then, the shortest path in the IR from the window $w^0_{i,k}$ to its root (a part of an NT) must cross the shortest path in the IR from the window $w^0_{j,k}$ to its root (a part of a PT), which is impossible.

Proof of 5(f): Suppose, on the contrary, that $i > j$. Then, the shortest path in the BRI from the positive window $w^r_{j,k}$ to its root (located to the right and below the window $w^r_{j,k}$) must cross the shortest path in the BRI from the negative window $w^r_{i,k}$ to its root (located to the right and above the window $w^r_{i,k}$), which is impossible. □

Having Part I in mind, it now comes as no surprise that Properties 1–4 are sufficient when it comes to determining a unique HC ^{c}. Again, the following proof is analogous to its counterpart from Part I. Nevertheless, in order to make this paper as self-contained as possible, we will still provide a rough sketch of the proof.

Theorem 2. *Every matrix* $B^{\,c,Ext} = [(b_{i,j}, r_{i,j})]_{m\times n}$ *with entries from* $\big(\mathcal{C}^+ \cup \{1,0,-1\} \cup \mathcal{C}^-\big) \times \big\{ -\lfloor \frac{m}{2} \rfloor, \dots, \lfloor \frac{m}{2} \rfloor \big\}$ *which satisfies Properties 1–4 determines a unique HC*^{c} *on the graph* $P_{m+1} \times C_n$.

Proof. The support of matrix $[b_{i,j}]_{m\times n}$, namely matrix $[a_{i,j}]_{m\times n}$, satisfies the conditions FL ^{c,Ext} and AC ^{c,Ext} (Property 1(a) of Theorem 1). We will prove that RC ^{c,Ext} holds, through a set of claims. But first, take the set of all windows of $\mathcal{G}_m$ into consideration and divide them into positive, negative and zero ones, in accordance with the sign of the value corresponding to $b_{i,j}$. Note that the window corresponding to 1 can not be adjacent to a window corresponding to -1 because of Property 1(a).

The edges of $\mathcal{G}_m$ which belong to different kinds of windows (zero and non-zero ones), together with the edges of the zero windows belonging to lines M_0M_1 and N_0N_1, determine a spanning 2-regular subgraph of $\mathcal{G}_m$. Adding the lines M_0M_1 and N_0N_1 to it gives way to a clear distinction between the positive, negative and zero regions. The first, of course, being determined by $b_{i,j} > 0$, the second by $b_{i,j} < 0$, and the last one by $b_{i,j} = 0$. However, instead of focusing on these regions per say, we can observe the components of the subgraph $\mathcal{W}_m$ induced by the windows of the same kind (positive, negative or zero). We will refer to them as the ***positive, negative*** or ***zero*** regions "induced by the positive, negative or zero entries of the matrix $[b_{i,j}]_{m\times n}$".

Note that every entry (b_{ij}, r_{ij}) of the matrix $B^{\,c,Ext}$ is assigned to *exactly one* window $w^r_{i,j}$, where $r = r_{ij}$. In other words, $w^r_{i,j}$ belongs to the $(j+nr)^{\text{th}}$ column of $\mathcal{G}_m$ (the square $M_rN_rN_{r+1}N_{r+2}$), although there are infinitely (countably) many vertices of $\mathcal{W}_m$ corresponding to this b_{ij}. If we collect all the positive and negative windows assigned to entries of the matrix $B^{\,c,Ext}$ we will obtain a finite number of completely fulfilled regions, as the claim below shows.

Claim 1. *Every window from any positive or negative region that contains* $w^r_{i,j}$, *where* $r = r_{ij}$, *is assigned to an entry of the matrix* $B^{\,c,Ext}$.

Proof. Since there is a path between any two windows in the considered regions, this comes as a consequence of Properties 1(b), 3(a) and 4(b). □

As a result, every positive or negative region is bounded, and there are infinitely (countably) many regions congruent to it. The regions described in the previous lemma will be called the ***basis positive regions*** or the ***basis negative***

regions and its each of its windows w_{ij}^r's for which $(b_{ij}, r) = (b, r)$ a b-***window***, where $b \neq 0$. If any such window belongs to the last (that is, the m^{th}) row and $b \neq 0$, it must be a 1-window with $r = 0$ (Property 1(d)); it will be called the ***down root***. If any such window belongs to the first row and $b < 0$, it must be a -1-window with $r = 0$ (Property 1(d)); it will be called the ***up root***. The window w_{11}^r, where $r = r_{11}$ is the only ***up root*** which is a positive window. Recall that in case the path which connects the window $w_{i,j}^r$ to another window consists only of windows from its column (the $(j + nr)^{\text{th}}$ column) or/and those to the left of it, we call this path the ***left path for the window*** $w_{i,j}^r$.

Claim 2. *For any two windows $w_{i,j}^r$ and $w_{i',j}^r$, where $i < i'$, from the same basis positive (or negative) region and the same column (the $(j + nr)^{th}$ column) for which there exists a left path for and between them, the following must be fulfilled: $b_{i,j} = b_{i',j}$.*

Proof. This can be proved by strong induction on the length l of the considered path using Properties 1(b) and 3(d), in the exact same way we did in the proof of Lemma 2 in [**1**]. □

Claim 3. *The subgraph of $W_{m,n}$ induced by positive (or negative) entries of matrix $B^{c,Ext}$ has a forest structure.*

Proof. Assuming the opposite holds, that there exists a cycle in a basis positive region, then in the rightmost column of its windows once we apply Claim 2 we reach a contradiction with either Property 3(e), 3(f) or 3(g) (compare with the proof of Lemma 3 in [**1**]). □

Claim 4. *Let $w_{i_1,j}^r$ and $w_{i_2,j}^r$ be any two windows from the same basis positive region, which belong to the same column (the $(j + nr)^{th}$ column), with $b_{i_1,j} = b_{i_2,j} = b$ and $|\, b \,| > 1$ $(r_{i_1,j} = r_{i_2,j} = r)$. Then, there exists a unique left path for and between them in this region.*

Proof. The existence of such a left path is proved by induction on $j + nr$ using Property 3(g) and 3(b). The base case deals with the leftmost windows of the considered region, whereas Claim 3 implies its uniqueness. □

Claim 5. *For every 1-window (resp., (-1)-window) $w_{i,j}^r$, where $r = r_{ij}$, there exists a unique left path for it which connects it to a down root (resp., an up root).*

Proof. The proof can be obtained by induction on $j + nr$. If the window $w_{i,j}^r$ belongs to the leftmost windows in the considered region (the base case of the induction), the letter $b_{i,j}$ and $b_{m,j}$ (resp., $b_{1,j}$) must belong to the same 1-factor (resp., (-1)-factor) (Property 3(f)i). If it is not the case, from Property 3(f) and Claim 4 we conclude that either there is a unique left path for and from it to $w_{m,j}^r$ $(w_{1,j}^r)$ which is a down root (resp., an up root), or there is a unique left path for and from it to a unique 1-window (resp., (-1)-window) from the previous column (the $(j+nr-1)^{\text{th}}$ column). In the second case, we apply the induction hypothesis to the newly obtained 1-window (resp., (-1)-window) instead of to $w_{i,j}^{r}$. □

Claim 6. *Every positive region has a unique down root, whereas every negative region has a unique up root.*

Proof. Property 3(c) implies that every rightmost window of any basis positive (resp., negative) region is a 1-window (resp., (-1)-window). By applying Claim 5 to these windows we obtain the desired statement. □

Now we can finish the proof of the main statement. Claims 6 and 3 together with Property 3(c) imply that the $\mathrm{RC}^{c,Ext}$ is satisfied. By applying Lemma 1 we finally obtain the existence and uniqueness of a HC ^{c} on the graph $P_{m+1}\times C_n$ whose split tree contains the window $w_{1,1}$. □

For each integer $m \geq 1$, we will create an auxiliary digraph whose role will be to enumerate the number of HC ^{c}'s in $P_{m+1}\times C_n$. Here is how we intend to do that. At first, let $\mathcal{F}_m = \mathcal{F}_m^{c,Ext}$ denote the set of all the possible first columns of $B^{c,Ext}$, and $\mathcal{D}_m^{c,Ext}$ a digraph with the vertex set $V(\mathcal{D}_m^{c,Ext})$ which consists of all the possible remaining columns of the same matrix. For any $v,u \in V(\mathcal{D}_m^{c,Ext})$, there exists an arc from v to u if and only if the vertex

$$v = [(b_{1,k},r_{1,k}),(b_{2,k},r_{2,k}),\ldots,(b_{m,k},r_{m,k})]^T$$

may appear as a column preceding the vertex

$$u = [(b_{1,k+1},r_{1,k+1}),(b_{2,k+1},r_{2,k+1}),\ldots,(b_{m,k+1},r_{m,k+1})]^T,$$

for $2 \leq k \leq n-1$. Note that the vertices of the disjoint sets $\mathcal{F}_m$ and $V(\mathcal{D}_m^{c,Ext})$ are in both cases the column vectors of the form $[(b_1,r_1),(b_2,r_2),\ldots,(b_m,r_m)]^T$ with entries from $(\mathcal{C}^-\cup\{-1,0,1\}\cup\mathcal{C}^+)\times\{-\lfloor\frac{m}{2}\rfloor,\ldots,\lfloor\frac{m}{2}\rfloor\}$. The difference between the two is that $b_1=b_2>0$ for the vertices from $\mathcal{F}_m$, whereas $b_1\in\{-1,0\}$ and $r_1=0$ for the vertices in $V(\mathcal{D}_m^{c,Ext})$.

Let $\mathcal{S}_m,\mathcal{L}_m \subseteq V(\mathcal{D}_m^{c,Ext})$ denote the set of all possible second and last (that is, the n^{th}) columns of the matrix $B^{c,Ext}$, respectively. Also, let $\mathcal{LFS}_m$ denote the set of all possible ordered triples $(l,f,s)\in\mathcal{L}_m\times\mathcal{F}_m\times\mathcal{S}_m$ of columns which can appear as the last (n^{th}), first and second column, respectively, in $B^{c,Ext}$. The aforementioned auxiliary diagraph from the previous paragraph will be denoted by $\overline{\mathcal{D}}_m^{c,Ext}$. Its set of vertices will be $V(\overline{\mathcal{D}}_m^{c,Ext}) = \mathcal{F}_m\cup V(\mathcal{D}_m^{c,Ext})$ and its set of edges

$$\begin{aligned} & E(\overline{\mathcal{D}}_m^{c,Ext}) \\ & \quad = \quad E(\mathcal{D}_m^{c,Ext})\cup\{(u,v)\mid(\exists w)(u,v,w)\in\mathcal{LFS}_m\vee(\exists w)(w,u,v)\in\mathcal{LFS}_m\}. \end{aligned}$$

Note that all the vertices of this graph do abide by the Basic and Column properties (as well as by the Topological properties). Additionally, the arcs of the digraph

$\mathcal{D}_m^{c,Ext}$ abide by the Adjacency properties, whereas the arcs coming out of the set $\mathcal{L}_m$ and into the vertices from the set $\mathcal{F}_m$ satisfy both the Adjacency and Buckle properties. The same goes for the arcs spanning from the set $\mathcal{F}m$ and into the vertices from the set $\mathcal{S}_m$

For example, when $m = 2$, the digraph $\overline{\mathcal{D}}_2^{c,Ext}$ has four vertices, and the set $\mathcal{LFS}_m$ consists of just one triplet (v_1, f_1, v_1) (see Figure 5). When $m = 3$, the digraph $\overline{\mathcal{D}}_3^{c,Ext}$ has fourteen vertices, 3 of which belong to the set $\mathcal{F}_3 = \{f_1, f_2, f_3\}$; whereas 11 of them as in $V(\mathcal{D}_3^{c,Ext})$. At the same time, there exist precisely two arcs from each of the vertices from the set $\mathcal{F}_3$ into the set $\mathcal{S}_3 = \{v_1, v_2, v_3, v_4, v_5\}$. Also, there exist two arcs per every vertex of the set $\mathcal{F}_3$ to which they point from the set $\mathcal{L}_3 = \{v_1, v_2, v_3, v_{10}\}$ thus forming 12 triplets - elements of the set $\mathcal{LFS}_3$ (see Subsection 5.3).

In this way, the enumeration of HC ^{c}s on $P_{m+1} \times C_n$ is reduced to the enumeration of oriented walks of length $n-2$ in the digraph $\mathcal{D}_m^{c,Ext}$ with the pairs of initial and final vertices which are respectively the third and first coordinates of the triplets from the set $\mathcal{LFS}_m$. In other words, this enumeration is reduced to the enumeration of closed oriented walks of length n in the digraph $\overline{\mathcal{D}}_m^{c,Ext}$ for which it holds that they both start and finish in the same vertex from the set $\mathcal{F}_m$ and no other vertex from the set $\mathcal{F}_m$ belongs to them. Finally, this number $\varphi_m^{c,Ext}(n-2)$, where $n \geq 2$, needs to be multiplied by n so as to obtain the correct number of HC ^{c} of $P_{m+1} \times C_n$.

3. CODING THE INTERIOR TREE BY NON-ZERO ENTRIES

3.1. The First Phase — the Matrix $A^{c,Int}$

Here, the zero windows belong to the exterior trees and w_{11} remains the up root of the split tree. To put it differently, $w_{1,2}^0$ is the leftmost window from the first row of the interior region in the BRI.

Each HC ^{c} on $P_{m+1} \times C_n$, where $m \geq 1$, with the window $w_{1,1}$ as the up root of the split tree can be encoded by a $(0,1)$-matrix $A^{c,Int} = [a_{i,j}]_{m \times n}$ where

$$a_{i,j} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } w_{i,j} \text{ belongs to the interior of HC}^{\,c}, \\ 0 & \text{otherwise.} \end{cases}$$

By doing so, we obtain one positive region and one or more zero regions in the BRI. Note that in Figure 4 almost all the windows of the IT are in $\mathcal{R}_0$ except for the two windows which are in $\mathcal{R}_1$. On the other hand, in Figure 2 the windows of the IT belong to the rectangles $\mathcal{R}_{-2}, \mathcal{R}_{-1}$ and $\mathcal{R}_0$.

Lemma 2. *Every HC^c on the thick grid cylinder graph $P_{m+1} \times C_n$ with the window $w_{1,1}$ as the up root of the split tree determines a $(0,1)$-matrix $A^{c,Int} = [a_{ij}]_{m \times n}$ that satisfies the following conditions ($a_{i,n+1} \stackrel{\text{def}}{=} a_{i,1}$, and $a_{i,0} \stackrel{\text{def}}{=} a_{i,n}$ for $1 \leq i \leq m$).*

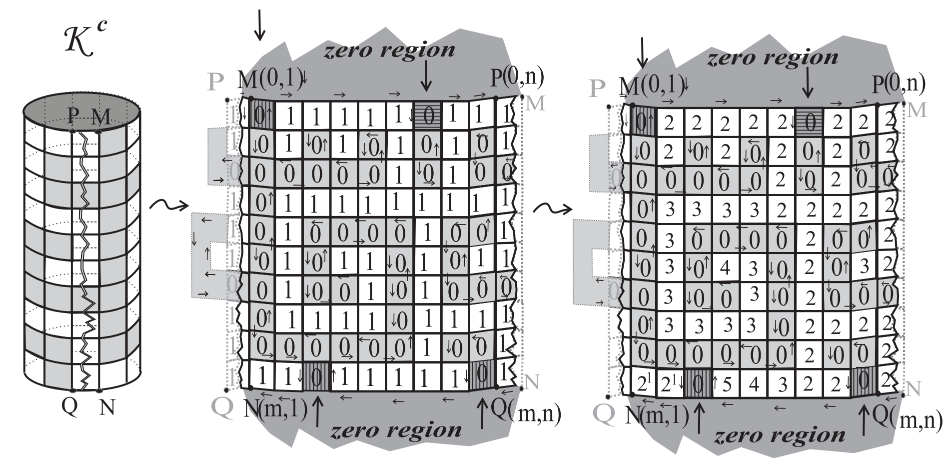

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Figure 4: A contractible HC of $P_{11} \times C_{10}$ with the entries of $A^{\,c,Int}$ (the first phase) and $B^{\,c,Int}$ (the second phase) inscribed on its windows, once the cylindrical surface was represented on a flat surface.

1. ***First and Last Row Conditions*** *($FL^{\,c,Int}$):*
   - *(a)* $a_{1,1} = 0$.
   - *(b) For* $1 \leq j \leq n$, $(a_{1,j}, a_{1,j+1}) \not\equiv (0,0)$.
   - *(c) For* $1 \leq j \leq n$, $(a_{m,j}, a_{m,j+1}) \not\equiv (0,0)$.
2. ***Adjacency Conditions*** *($AC^{c,Int}$): For* $1 \leq i \leq m-1$, *and* $1 \leq j \leq n$,

   $$(a_{i,j}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}) \notin \{(1,1,1,1), (0,0,0,0), (0,1,1,0), (1,0,0,1)\}.$$

3. ***Tree Condition*** *($TC^{c,Int}$): The vertices of $W_{m,n}$ corresponding to 1's in $A^{\,c,Int}$ induce a unique tree in $W_{m,n}$.*

Conversely, every $(0,1)$-matrix $[a_{ij}]_{m\times n}$ which satisfies the conditions $FL^{\,c,Int}$, $AC^{c,Int}$, and $TC^{c,Int}$ determines a unique HC^c on the thick grid cylinder graph $P_{m+1} \times C_n$ with the window $w_{1,1}$ as the up root of its split tree.

Proof. The first two conditions provide the local whereas the third one provides the global aspect of hamiltonicity and their necessity is easily verifiable (note that FL ^{c,Int} implies that $a_{1,2} = a_{1,n} = 1$). With the intention of showing that all the above mentioned conditions are sufficient as well, note the following. The first two conditions ensure that the set of edges belonging to both a zero and a positive window or to both a positive window and one of the lines M_0M_1 or N_0N_1 determines a unique 2-regular spanning subgraph of $P_{m+1} \times C_n$, that is, a union of cycles. The third condition implies that there exists a unique cycle — the boundary of the positive region (IT).

Let us walk from the upper horizontal edge of the window w_{12} (the windows on the right-hand side belong to the IT), walking in accordance with the aforementioned boundary. As there are no consecutive zeros in the m^{th} row of $A^{\,c,Int}$, there exists at least one window corresponded to 1 in that row. Find the last lower horizontal edge of some window from the m^{th} row through which we pass along our walk. If we denote that window by w_{mk} (in Figure 4, we have $k = 4$), then $w_{m,k-1}$ represents the down root of the split tree. The reason behind that is that the rest of our walk consists of edges which belong to zero windows that are connected to both $w_{m,k-1}$ and w_{11}, with the latter of which we end our walk as it is. Therefore, the constructed HC is contractible and has $w_{1,1}$ for the up root of the split tree. □

3.2. The Second Characterization of HC ^{c} with w_{11} as the Up Root of the Split Tree

For each HC ^{c} with $w_{1,1}$ in the split tree, we associate the matrix $A^{\,c,Int} = [a_{i,j}]_{m\times n}$ to the matrix $B^{\,c,Int} = [(b_{i,j}, r_{i,j})]_{m\times n}$. The first matrix satisfies the conditions FL ^{c,Int}, AC ^{c,Int}, and TC ^{c,Int}. The second matrix with $b_{i,j} \in \mathcal{C}^{+} \cup \{0,1\}$ and $-\left\lfloor \frac{m}{2} \right\rfloor \le r_{i,j} \le \left\lfloor \frac{m}{2} \right\rfloor$ is constructed in the following way:

1. Define $r_{i,j} = r(w_{i,j})$.
2. Set $b_{i,j} = a_{i,j} = 0$ if $w_{i,j}$ belongs to an ET.
3. For each fixed column j, partition the positive windows from the j^{th} column with the same roll number into j^r-joined equivalence classes. Then, label all the windows within each equivalence class with 2, 3 . . ., according to the order in which the equivalence classes first appear within the j^{th} column, from top to bottom.

Example 6. In Figure 4, the values (b_{ij}, r_{ij}) of $B^{\,c,Int}$ are inscribed on the windows as $b_{ij}^{r_{ij}}$ or just as b_{ij} if $r_{ij} = 0$. In the 9^{th} column there are four parts of the IT that belong to the roll 0. Three of them, $w_{1,9}$, $w_{4,9}$, and $w_{8,9}$, are 9^0-joined. Consequently, the same b-value is assigned to them in $B^{\,c,Int}$. More specifically, we have $b_{1,9} = b_{4,9} = b_{8,9} = 2$. The fourth window, $w_{6,9}$, while still belonging to the roll 0, belongs to a different equivalent class. Hence, $b_{6,9} = 3$. In the second column, there are three windows that belong to the IT. Two of them belong to the roll 0, but they belong to two different equivalence classes with respect to the relation 2^0-joined. The last window, $w_{10,2}$, belongs to roll 1; thus, $(b_{10,2}, r_{10,2}) = (2, 1)$.

In an arbitrary column of the matrix $[b_{i,j}]_{m\times n}$ that corresponds to $B^{\,c,Ext} = [(b_{i,j}, r_{i,j})]_{m\times n}$, we consider all the maximal b-factors (if it exists), where $b > 0$. Let them be, in their order of appearance, that is, from top to bottom, p_1-factor,

p_2-factor, ..., p_k-factor, where $k \geq 1$, and $p_i \geq 2$ for each i. In addition, let $r_1, r_2, \ldots, r_k$ denote the roll numbers associated with these maximal factors. The words $p_1 p_2 \ldots p_k$ and $r_1 r_2 \ldots r_k$ are called the ***truncated word*** and the ***truncated roll word***, respectively. A subsequence of a truncated word induced by the letters with the same roll number r is called a ***color^r word***.

Example 7. For the second column in Figure 4, the truncated word, the truncated roll word, the color^0 word, and the color^1 word are: 232, 001, 23, and 2, respectively. The color^0 word for the fourth column is 23435.

3.3. Properties of $B^{c,Int}$

The properties listed below follow straightforwardly from the definition of the matrix $B^{c,Int} = [(b_{i,j}, r_{i,j})]_{m\times n}$. Here, $(b_{i,n+1}, r_{i,n+1}) \stackrel{\text{def}}{=} (b_{i,1}, r_{i,1})$, and $(b_{i,0}, r_{i,0}) \stackrel{\text{def}}{=} (b_{i,n}, r_{i,n})$.

Theorem 3. *The matrix $B^{c,Int} = [(b_{i,j}, r_{i,j})]_{m\times n}$ satisfies the following conditions.*

1. ***Basic Properties***
   - (a) *The support of the matrix $[b_{i,j}]_{m\times n}$, that is, the matrix $[a_{i,j}]_{m\times n}$, satisfies the $FL^{c,Int}$ and $AC^{c,Int}$.*
   - (b) Harmonization of the adjacent entries having the same sign*: For $2 \leq i \leq m$ and $1 \leq k \leq n$, if $a_{i-1,k} = a_{i,k}$, then $(b_{i-1,k}, r_{i-1,k}) = (b_{i,k}, r_{i,k})$.*
   - (c) *For $1 \leq i \leq m$ and $1 \leq j \leq n$, if $a_{i,j} = 0$, then $r_{i,j} = 0$.*
2. ***Column Properties***

   For $1 \leq k \leq n$, the k^{th} column $[(b_{1,k}, r_{1,k}), (b_{2,k}, r_{2,k}), \ldots, (b_{m,k}, r_{m,k})]^T$ of the matrix $B^{c,Int}$ satisfies these conditions:
   - (a) *If there exists an entry (s, r) in the k^{th} column of $B^{c,Int}$, where $s \geq 3$, then for each $\ell \in \{2, 3, \ldots, s-1\}$, at least one copy of the entry (ℓ, r) must appear before the first appearance of the entry (s, r).*
   - (b) *If the truncated roll word of the first column of $B^{c,Int}$ is not an empty word, it begins with 0 or 1. The truncated roll word of the k^{th} column of $B^{c,Int}$ for $k \neq 1$ is non-empty and begins with 0.*
3. ***Adjacency Properties***

   For $1 \leq k \leq n$, the k^{th} column of $B^{c,Int} = [(b_{i,j}, r_{i,j})]_{m\times n}$ satisfies these conditions:
   - (a) *For $1 \leq i \leq m$ and $2 \leq k \leq n$, if $a_{i,k-1} = a_{i,k} = 1$, then $r_{i,k-1} = r_{i,k}$.*
   - (b) *For each ordered pair (b, r) that appears in the k^{th} column, with the exception of the following two cases:*

- *$b = 2$, $r > 0$ is the maximal roll in this column, and there is no occurrence of $(3, r)$ in this column;*
- *$k = n$, $b = 2$, $r = 0$ is the maximal roll in this column, and there is no occurrence of $(3, 0)$ in this column;*

there must exist an index i for which $(b_{i,k}, r_{i,k}) = (b, r)$, and $b_{i,k+1} \neq 0$.

(c) For $1 \leq i, j \leq m$, where $i \neq j$, if $(b_{i,k-1}, r_{i,k-1}) = (b_{j,k-1}, r_{j,k-1})$ and $a_{i,k} = a_{j,k} = a_{i,k-1} = a_{j,k-1} = 1$, then $b_{i,k} = b_{j,k}$.

(d) For $1 \leq i, j \leq m$, where $i \neq j$, if $(b_{i,k-1}, r_{i,k-1}) = (b_{j,k-1}, r_{j,k-1})$, $a_{i,k} = a_{j,k} = a_{i,k-1} = a_{j,k-1} = 1$, and $b_{i,k} = b_{j,k} = b$, then there is no b-factor in the word $b_{1,k}b_{2,k}\dots b_{m,k}$ which contains both $b_{i,k}$ and $b_{j,k}$.

(e) For $b \neq 0$, if v and u represent two different maximal b-factors in the word $b_{1,k}b_{2,k}\dots b_{m,k}$ with the same roll number, then there is a unique sequence $v = v_1, v_2, \dots, v_p = u$ $(p > 1)$ of distinct maximal b-factors for which it is true that:

- *For every i, where $1 \leq i \leq p-1$, there is exactly one $b_{j_i,k-1}$ with $a_{j_i,k-1} = a_{j_i,k} = 1$, such that $b_{j_i,k} \in v_i$, and there is a unique $b_{s_{i+1},k-1}$ with $a_{s_{i+1},k-1} = a_{s_{i+1},k} = 1$ such that $b_{s_{i+1},k} \in v_{i+1}$, $(b_{j_i,k-1}, r_{j_i,k-1}) = (b_{s_{i+1},k-1}, r_{s_{i+1},k-1})$, and $j_i \neq s_i$.*

(f) For $r > 0$, if the ordered pair $(2, r)$ appears in the k^{th} column, then there must exist an index i for which

- *If $k = 1$, then $r_{i,n} = r - 1$, and $a_{i,1} = a_{i,n} = 1$.*
- *If $k > 1$, then $r_{i,k-1} = r$, and $a_{i,k-1} = a_{i,k} = 1$.*

*4. **Buckle Properties** (Specific Properties of the First and Last Columns)*

(a) $(b_{1,1}, r_{1,1}) = (b_{2,1}, r_{2,1}) = (0, 0)$, and $(b_{1,n}, r_{1,n}) = (2, 0)$.

(b) For $2 < i \leq m$, if $a_{i,n} = a_{i,1} = 1$, then $r_{i,1} = r_{i,n} + 1$.

*5. **Topological Properties***

(a) For $1 < i_1 < j_1 < i_2 < j_2 \leq m$, if $b_{i_1,k} = b_{i_2,k} > 1$, $b_{j_1,k} = b_{j_2,k} > 1$, and $r_{i_1,k} = r_{i_2,k} = r_{j_1,k} = r_{j_2,k}$, then $b_{i_1,k} = b_{j_1,k}$.

(b) The absolute value of the difference between two adjacent letters in the non-empty truncated roll word corresponding to the k^{th} column of $B^{\,c,Int}$ is at most 1.

Proof. We shall omit the proofs of those items that we consider to be fairly straightforward, due to their similarity to the ones in Theorem 1 or Theorem 4 of [**1**]. However, we shall discuss the remaining items.

Column Property 2(a): It is a trivial consequence of the chosen method of coding, that is, of the way in which $B^{\,c,Int}$ is formed.

Column Property 2(b): Notice at first that using the definition of the matrix $B^{\,c,Int}$ and the definition of BRI, we have that $b_{1,1} = b_{2,1} = 0$, $b_{1,2} = b_{1,n} = 2$ and

$r_{1,2} = r_{1,n} = 0$. The number $r_{1,n}$ must be zero because, assuming the contrary, the path which connects the windows $w^0_{1,2}$ and $w^{r_{1,n}}_{1,n}$ would encompass the split tree and so it could not reach its down root.

The truncated roll word of the k^{th} column where $k > 1$ is a non-empty word because $b_{1,k}b_{2,k}\dots b_{m,k}$ can be zero word just for $k = 1$. Now, suppose to the contrary, that the truncated roll word of the k^{th} column, where $k > 1$, of $B^{\,c,Int}$ begins with $r \geq 1$. Let $w^r_{i,k}$ be a positive window from the BRI, for which $b_{lk} = 0$, where $1 \leq l \leq i-1$. Let P^0 denote the unique path from $w^r_{i,k}$ to $w^0_{1,2}$. This path must cross the $(k+n(r-1))^{\text{th}}$ column of the RI in a window $w^{r-1}_{s,k}$ for some integer $s > i$. But, in that case, the copy P^{-1} of the path P^0 in the RI meets P^0, which is impossible. This proves that $r < 1$. Similarly, we conclude that $r > -1$, observing the path from $w^r_{i,k}$ to $w^0_{1,n}$ instead of the path from $w^r_{i,k}$ to $w^0_{1,2}$. Together, we conclude that $r = 0$.

Further we show that the first column of $B^{\,c,Int}$ possesses the property that a non-empty truncated roll word begins with $r_{i,1} = 0$ or $r_{i,1} = 1$ $(3 \leq i \leq m)$. Namely, assuming the opposite, we would have $r_{i,1} > 1$ or $r_{i,1} < 0$. If $r_{i,1} > 1$, then the unique path P^0 from $w^0_{1,n}$ to $w^{r_{i,1}}_{i,1}$ must cross the $(n+1)^{\text{st}}$ column of the RI in a window $w^1_{s,1}$ for some integer $s > i$. It implies that the copy P^{-1} of the path P^0 in the RI meets P^0, which is impossible. Similarly, we may consider the case when $r_{i,1} < 0$ and reach a contradiction again.

Adjacency Properties 3(b): Assume the rightmost positive windows in the BRI correspond to column $\tilde{k}$. They have maximal roll in this column, say $\tilde{r}$, which is either (i) $\tilde{r} > 0$, or (ii) $\tilde{r} = 0$ and $\tilde{k} = n$. Since there exists exactly one $\tilde{k}^{\tilde{r}}$-joined equivalence class for these windows, all corresponding entries of the matrix $B^{\,c,Int}$ are $(2, \tilde{r})$. For any other window $w^r_{j,k}$ in the BRI which corresponds to the pair (b, r) there is a path from it to one of these rightmost positive windows in the BRI over some window $w^r_{i,k+1}$ from the next, $(rn + (k+1))^{\text{th}}$ column. Clearly, $(b_{i,k}, r_{i,k}) = (b, r)$.

Additionally, note that if the pair $(b_{i,j}, r_{i,j}) = (2, r)$, where $r > 0$ is the maximal roll in the j^{th} column of $B^{\,c,Int}$, and there is no occurrence of $(3, r)$ in this column, then the window $w^r_{i,j}$ can be (but need not be) one of the rightmost windows in the considered region (see, as an example, the first and the second column of $B^{\,c,Int}$ in Figure 4).

Adjacency Properties 3(f): If the ordered pair $(2, r)$ with $r > 0$ appears in the k^{th} column, then the corresponding window belongs to the $(rn + k)^{th}$ column in BRI. Since there exists a path from it to the window $w_{1,2}$, it must pass through the previous column. $\square$

Properties 1–4 are sufficient for determining a unique HC ^{c}.

Theorem 4. *Every matrix* $B^{\,c,Int} = [(b_{i,j}, r_{i,j})]_{m\times n}$ *with entries from* $\big(\mathcal{C}^+ \cup \{1,0\}\big) \times \big\{ -\lfloor \frac{m}{2} \rfloor, \dots, \lfloor \frac{m}{2} \rfloor \big\}$ *which satisfies Properties 1–4 of Theorem 3 determines a unique* HC^c *on the graph* $P_{m+1} \times C_n$.

Proof. By applying Property 1(a), we see that the support of the matrix $[b_{i,j}]_{m\times n}$, in other words, the matrix $[a_{i,j}]_{m\times n}$, satisfies the FL ^{c,Int} and AC ^{c,Int} of Lemma 2. In order to prove that the TC ^{c,Int} of Lemma 2 is satisfied as well, observe the set of all non-zero windows $w_{i,j}^k$ $(k \in \mathbb{Z})$ of $\mathcal{W}_m$ — the windows corresponding to the entries $(b_{i,j}, r_{i,j})$ of $B^{\,c,Int}$ where $b_{i,j} > 0$. These windows determine the *non-zero regions*. The union of the boundaries of these regions present a spanning 2-regular subgraph of $\mathcal{G}_m$. If we consider only the non-zero windows $w_{i,j}^k$ where $k = r_{i,j}$, applying Properties 1(b), 3(a) and 4(b), we can conclude that they cover up completely one or more of those regions. To distinguish these regions from their copies we shall call them the ***basis regions (BR)*** (although there is just one such region, as will be shown later).

Recall that in case a path in $\mathcal{W}_m$, which connects the window $w_{i,j}^r$ to another window, consists only of the windows from its column (the $(j+nr)^{\text{th}}$ column) or/and those to the left of it, we call it the ***left path for the window*** $w_{i,j}^r$. The path visiting only the windows from the same column which are assigned to the entry (b,r) of $B^{c,Int}$ is called a ***b-factor***. Properties 1(b) and 3(c) imply the next statement.

Claim 7. *If there exists a left path for and between the two windows $w_{i,j}^r$ and $w_{i',j}^r$, where $i < i'$, from the same basis region, then $b_{i,j} = b_{i',j}$.*

The proof of this statement can be obtained by strong induction on the length l of the considered path in a similar fashion by which Claim 2 was proven. Further, using Claim 7, Properties 3(d) and 3(e), analogously as in the proof of Claim 3, the following statement can be shown.

Claim 8. *The subgraph of $W_{m,n}$ induced by the windows determined by the positive entries of $B^{c,Int}$ has a forest structure.*

The next claim can be proved by induction on $j + nr$ (with the base case which refers to the leftmost windows of the considered region), whilst at the same time relying upon Property 3(e).

Claim 9. *For any two windows $w_{i_1,j}^r$ and $w_{i_2,j}^r$ from the same basis region and with $b_{i_1,j} = b_{i_2,j}$ there exists a left path for and between them.*

The uniqueness of this path is a consequence of the forest structure of the subgraph of W_m induced by the windows belonging to the considered region.

It remains to prove that the subgraph of W_m induced by the positive windows which belong to the basis regions has just one component (hence it is a connected graph). We will prove that there exists a path for and between an arbitrary such window and $w_{1,n}^0$ (Property 4(a) guaranties the existence of such a path in the considered region).

Note that Property 3(b) implies that all the rightmost positive windows in the BR correspond to either the pair $(2,0)$ and belong to the n^{th} column (Case I), or to the pair $(2,r)$ where $r > 0$ (Case II). Claim 9 indicates that all positive windows of the last column of the BR are connected (with some paths) to each other.

We begin our considerations for Case I, first. Property 3(b) and Claim 9 imply that for every window $w_{i,j}^r$ from BR, with the exception of the windows from the last (the n^{th}) column (that is, from the $(rn+j)^{\text{th}}$ column, where $rn+j<n$), there exists a path which connects it with a window from the next (the $(rn+j+1)^{\text{th}}$) column. Consequently, every positive window in the BR is connected to a window from the last column. This implies that all these windows belong to the same component.

As for Case II, let $rn+k$, where $r>0$ and $1 \le k \le n$, be the ordinal number of the last column of the BR (whose windows are all assigned to the pair $(2,r)$). Property 3(f) implies the existence of the positive window from the BR in the previous $((rn+k-1)^{\text{th}})$ column which is connected to the positive windows from the $(rn+k)^{\text{th}}$ column. If all the windows from this previous column are assigned to the pair $(2,r)$, then we use Claim 9. Otherwise we use Property 3(b) to conclude that all the windows from this column are connected to the windows of the last column. Now, we can obtain the same conclusion by using the $(rn+k-2)^{\text{th}}$ and $(rn+k-1)^{\text{th}}$ columns instead of the $(rn+k-1)^{\text{th}}$ and $(rn+k)^{\text{th}}$ columns. We continue this procedure till we reach the rectangle $\mathcal{R}_0$, that is, finishing with the n^{th} and $(n+1)^{st}$ columns. This way all the windows from the n^{th} column and to right of it are connected. For the rest of the windows (from the columns to the left of the n^{th} column) we use Property 3(b) and Claim 9 similarly as in Case I. Consequently, the subgraph of $W_{m,n}$ induced by $B^{c,Int}$'s positive entries has a tree structure. □

Let $\mathcal{F}_m$ denote the set of all possible first columns of $B^{\,c,Int}$. We already know that the set of all possible columns of the matrix $B^{\,c,Int}$ (as defined by Properties 1–4 or 1–5 above) forms the vertex set of a digraph $\mathcal{D}_m^{c,Int}$. Note that $\mathcal{F}_m \subseteq V(\mathcal{D}_m^{c,Int})$. Furthermore, the directed edges are determined by the adjacency conditions. Let $\mathcal{FL}_m$ denote the subset of $V(\mathcal{D}_m^{c,Int}) \times V(\mathcal{D}_m^{c,Int})$ consisting of all possible pairs of first and last columns (which we call the ***fl-pairs***) of $B^{\,c,Int}$ determined by the specific properties of the first and last columns. The enumeration of HC ^{c}'s on $P_{m+1} \times C_n$ basically comes down to the enumeration of oriented walks of length $n-1$ in the digraph $\mathcal{D}_m^{c,Int}$ with the initial and last vertices from the set $\mathcal{FL}_m$. For $m=2$, see Figure 6. Finally, this number $\varphi_m^{c,Int}(n-1)$ should be multiplied by n to obtain the correct number of HC ^{c}.

Note that the size of $\mathcal{D}_m^{c,Int}$ depends on whether we have imposed the additional conditions from Property 5 on the vertices and edges. The previously mentioned properties are quite handy, particularly when it comes to generating the set of vertices of $\mathcal{D}_m^{c,Int}$. Owing to them, it is possible to reduce the number of edges in the said diagraph. In other words, we are actually able to exclude the superfluous edges.

4. THE NUMBER OF Color^r WORDS OF FIXED LENGTH AND CATALAN NUMBERS

A color word was defined in [**4**] as a word of length k over the alphabet $\{2, 3, \ldots, k+1\}$ with the following properties:

- *P1*: If the letter $s \geq 3$ appears in a word, then each letter from the set $\{2, 3, \ldots, s-1\}$ must appear at least once prior to the first occurrence of s. Consequently, if i_s denotes the position at which the first occurrence of s can be found, then we must have $i_2 < i_3 < i_4 < \cdots$.
- *P2*: If $abab$ is a subword of a word, then $a = b$. In every other word in which $a \neq b$, $abab$ cannot appear as subword.

The number of color words of length k is determined by the k^{th} Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$ [**4**].

In [**1**] and in the previous sections we have introduced the notions of a *positive (resp., negative)* or just *truncated word* and a *positive (resp., negative)* or just *color^r word*. These depend on the type of HC in question (HC ^{nc} or HC ^{c}) as well as on the type of coding applied (HC ^{nc}, HC ^{c,Ext},HC ^{c,Int}). Throughout the whole process of generating the vertices of $\mathcal{D}_m^{c,Ext}$ and $\mathcal{D}_m^{c,Int}$, as well as of $\mathcal{D}_m^{nc}$ (which was described in PART I) we need to construct the set of all (positive/negativ/-) color^r words of length k. Now, we want to find the upper bound of this set's cardinality.

In case of $B_m^{c,Int}$ (for all possible r), this set is determined by P1 (in accordance with Property 2 (a)) and P2 (in accordance with Property 5(a)). As a result, the upper bound of this set's cardinality is precisely C_k.

Proposition 1. *The upper bound of the cardinality of color^r words of length k in case of $B_m^{c,Int}$ (for all possible r) is the k^{th} Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$.*

Exactly the same situation occurs in B^{nc} and $B_m^{c,Ext}$ when $r < 0$. In the latter one we use the term "positive" or "negative color^r words" in place of the "color^r word" term. This is in accordance with Properties 2 (a) and 5 (a) for both matrix B^{nc} and $B_m^{c,Ext}$.

If $r = 0$ and a color^0 word (or a positive/negative color^0 word) of length k is not assigned to the last (n^{th}) column of B_m^{nc} ($B_m^{c,Ext}$), then this word is a word of length k over the alphabet $\{1, 2, 3, \ldots, k+1\}$ having an additional property, apart from P1 and P2:

- *P3*: If $a1a$ is a subword of the word of length k, then $a = 1$.

This is in accordance with Property 5 (b) of the matrix B^{nc} ($B_m^{c,Ext}$). If we add 1 in front of each considered word, then P3 can be interpreted as P2, but for an augmented alphabet. Therefore, the upper bound of the cardinality of color^0 words of length k in these cases is C_{k+1}.

Last but not least, if $r > 0$ (or $r = 0$ and the word is assigned to the last (n^{th}) column of B_m^{nc} ($B_m^{c,Ext}$)), then the set of all color^r words (positive color^r

words) of length k can be described as the subset of the set of all words from the alphabet $\{1, 2, \ldots, k+1\}$ that contain at least one letter 1, and satisfy P1, P2, and P3. This is in accordance with Properties 2 (c) and 4 (d) of the matrix B^{nc} (or in accordance with Properties 2 (c) and 4 (e) of the matrix $B_m^{c,Ext}$). Note that the number of all the negative color^r words of length k is equal to the number of all the positive color^r words of length k. Therefore, from the previous two cases, we determine that the upper bound of these color^r words' cardinality in this case is $C_{k+1} - C_k$. This way we have proved the following:

Proposition 2. *The upper bound of the cardinality of color^r words (positive or negative color^r words) of length k belonging to the l^{th} column of B^{nc} ($B_m^{c,Ext}$) is*

$$\begin{cases} C_k & \text{if } r < 0, \\ C_{k+1} & \text{if } r = 0 \text{ and } l \neq n, \\ C_{k+1} - C_k & \text{if } r > 0 \text{ or } (r = 0 \text{ and } l = n). \end{cases}$$

The words which satisfy P1 and P2 are called the non-interlocking and non-skipping columns in [**11**]; the interpretation of $C_{k+1} - C_k$, in the same paper, provides an alternative proof for the case of $r > 0$.

5. COMPUTATIONAL RESULTS

The technique we use to compute $\mathcal{H}_m^c(x) \stackrel{\text{def}}{=} \sum_{n \geq 1} h_m^c(n+1)x^n$, the generating function for the contractible HC's is, technically speaking, essentially the same as the one utilized in Part I. For that reason, we shall only discuss a few dissimilarities here, from the data obtained through the use of a computer.

The primary goal of Topological Properties is to shorten the search process throughout the digraph. Note that they are, in fact, not necessary for the determination of HC ^{c}'s or HC ^{nc}'s. However, their importance role is to reduce the digraph's dimension to a reasonable size by eliminating all the irrelevant vertices and edges that cannot occur in generating any HC ^{c}.

Based on all of the above theory and considerations, we wrote computer programs to generate the matrices $M_m^{c,Ext}$ and $M_m^{c,Int}$, together with the adjacency matrices of the digraphs $\mathcal{D}_m^{c,Ext}$ and $\mathcal{D}_m^{c,Int}$. The dimensions of $\mathcal{D}_m^{c,Ext}$ and $\mathcal{D}_m^{c,Int}$ are collected in Tables 1, for some reasonable values of m.

The computation was performed on a personal computer equipped with an Intel(R) Core (TM) i7-4712MQ processor (running at a speed of 2.30GHz) with 6.00 GB of RAM, and run on a 64-bit operating system.

Similar to the case of the HC ^{nc}'s, for the HC ^{c}'s by coding the interior tree we find that the $\mathcal{F}_m = \mathcal{F}_m^{c,Int} \subseteq V(\mathcal{D}_m^{c,Int})$. However, when coding the exterior trees, we came to realise that $\mathcal{F}_m^{c,Ext} \cap V(\mathcal{D}_m^{nc}) = \emptyset$. The reason behind it is that the first row of the matrix $[b_{i,j}]_{m \times n}$ has only one positive number which must be the entry $b_{1,1}$.

m	2	3	4	5	6	7	8	9
$\lvert V(\mathcal{D}_m^{c,Ext})\rvert$	3	11	44	174	644	2488	-	-
$\lvert \mathcal{F}_m^{c,Ext}\rvert$	1	3	7	28	92	341	-	-
$\lvert E(\mathcal{D}_m^{c,Ext})\rvert$	4	24	123	677	3446	18569	-	-
$\lvert \mathcal{LFS}_m\rvert$	1	12	49	406	2461	19913	-	-
$\lvert V(\mathcal{D}_m^{c,Int})\rvert$	4	10	33	104	318	985	3121	9943
$\lvert E(\mathcal{D}_m^{c,Int})\rvert$	5	23	96	423	1792	7857	34505	153500
$\lvert \mathcal{FL}_m\rvert$	1	6	18	80	325	1413	6083	26583

Table 1: The characteristics of digraphs $\mathcal{D}_m^{c,Ext}$ and $\mathcal{D}_m^{c,Int}$.

Our findings for $m \leq 4$ were confirmed by manual computations. The results displayed below agree with the values of $h_m^c(n)$ for $m \leq 9$ and $n \leq 10$ obtained in [**3**], as well as with the values $h_m(n) = h_m^{nc}(n) + h_m^c(n)$ for $m \leq 9$ and $11 \leq n \leq 22$ in [**8**].

Recall that the number $\varphi_m^c(k)$ represents the number of HC ^{c}'s in $P_{m+1} \times C_{k+2}$ with w_{11} as the up root of the split tree. But, in the case of coding the exterior trees, it represents the number of oriented walks of length k in the digraph $\mathcal{D}_m^{c,Ext}$ with the pairs of initial and final vertices which are respectively the third and first coordinates of the special triples. Similarly, when coding the interior tree, it represents the number of oriented walks of length $k+1$ in the digraph $\mathcal{D}_m^{c,Int}$ with the initial and last vertices from some special sets. Hence, we label the coefficients $\varphi_m^c(k)$, where $k \geq 0$, of the generating function $\Phi_m^c(x) \stackrel{\text{def}}{=} \sum_{k\geq 0} \varphi_m^c(k)x^k$ with $\varphi_m^{c,Ext}(k)$ in the coding by exterior trees, and with $\varphi_m^{c,Int}(k+1)$ in the coding with the interior tree. Generating the digraphs $\mathcal{D}_7^{c,Int}$, $\mathcal{D}_8^{c,Int}$, and $\mathcal{D}_9^{c,Int}$ requires 7 seconds, 2 minutes, and 39 minutes, respectively.

5.1. Thick Cylinder $P_2 \times C_n$ $(m = 1)$

For $m = 1$, it is easy to show that $h_1^c(n) = n$ for all $n \geq 1$. Since

$$h_1^{nc}(n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

we can write $h_1(n) = n + 1 + (-1)^n$.

5.2. Thick Cylinder $P_3 \times C_n$ $(m = 2)$

The digraph $\mathcal{D}_2^{c,Ext}$ is displayed in Figure 5.

The incidence matrix $M_2^{c,Ext}$ of the corresponding digraph $\mathcal{D}_2^{c,Ext}$ is of order 3. The set of all possible triplets (l, f, s) has only one element (v_1, f_1, v_1). Using

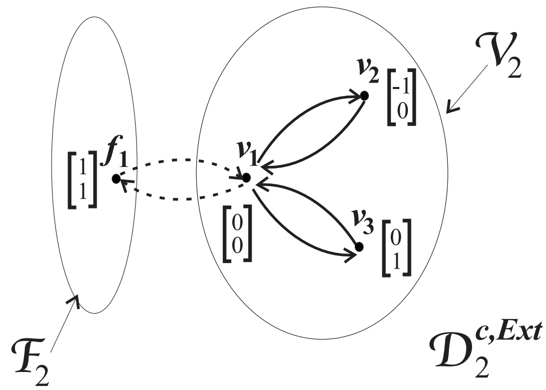

Figure 5: The digraph $\mathcal{D}_2^{c,Ext}$ and the corresponding set $\mathcal{F}_2$.

a similar technique as in the case of the NC-type of HC's, we obtain

$$h_2^c(n) = \frac{n}{4}\sqrt{2}^{\,n}[1 + (-1)^n]. \tag{2}$$

Since $h_2^{nc}(n) = 2^n - 2$ [**1**] (Part I), from (2), we determine that

$$h_2(n) = 2^n - 2 + \frac{n}{4}\sqrt{2}^{\,n}[1 + (-1)^n]$$

for all integers $n \geq 1$. Identical results can be obtained from $M_2^{c,Int}$ (see Figure 6).

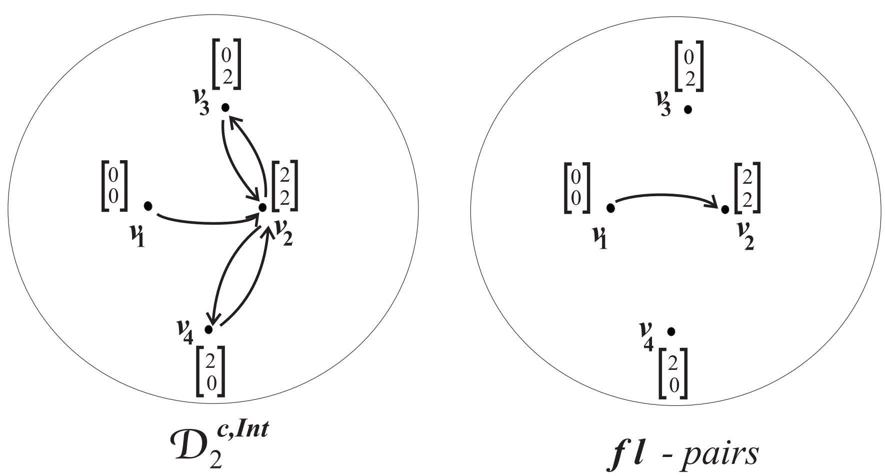

Figure 6: The digraph $\mathcal{D}_2^{c,Int}$ and the corresponding set of pairs $\mathcal{F}_2\mathcal{L}_2$.

5.3. Thick Cylinder $P_4 \times C_n$ ($m = 3$)

In this subsection, we provide a detailed discussion for the case of $m = 3$. We study the HC's of type C, with coding carried out on the exterior region first; and then we move on to the coding of the interior region.

Coding the Exterior Region

We find $V(\mathcal{D}^{\,c,Ext}) = \{v_1, v_2, \dots, v_{11}\}$; the vertices and the adjacency matrix $M_3^{c,Ext} = [m_{i,j}^{c,Ext}]$ are listed below:

$$
\begin{array}{l}
v_1 = (0^0, 0^0, 0^0) \\
v_2 = (0^0, 1^0, 0^0) \\
v_3 = (0^0, 1^0, 1^0) \\
v_4 = (0^0, 2^0, 0^0) \\
v_5 = (0^0, 1^1, 0^0) \\
v_6 = (-1^0, -1^0, 0^0) \\
v_7 = (-1^0, 0^0, 1^0) \\
v_8 = (0^0, -2^{-1}, 0^0) \\
v_9 = (0^0, -2^0, 0^0) \\
v_{10} = (0^0, 2^{-1}, 0^0) \\
v_{11} = (0^0, -1^0, 0^0)
\end{array}
\qquad
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

We also find $\mathcal{F}_3 = \{f_1, f_2, f_3\}$, $|\mathcal{LFS}_3| = 12$,

$$
\begin{aligned}
\mathcal{LFS}_3 \;=\; & \{(v_{10}, f_1, v_1), (v_{10}, f_1, v_2), (v_1, f_1, v_1), (v_1, f_1, v_2), \\
& (v_{10}, f_2, v_4), (v_{10}, f_2, v_3), (v_1, f_2, v_4), (v_1, f_2, v_3), \\
& (v_3, f_3, v_1), (v_3, f_3, v_5), (v_2, f_3, v_1), (v_2, f_3, v_5)\},
\end{aligned}
$$

where $f_1 = (1^0, 1^0, 1^0)$, $f_2 = (2^0, 2^0, 0^0)$, $f_3 = (1^1, 1^1, 0^0)$. The characteristic polynomial of $M_3^{c,Ext}$ is

$$
P_3^{c,Ext}(x) = -x^2 + 7x^3 - 22x^4 + 38x^5 - 34x^6 + 6x^7 + 18x^8 - 18x^9 + 7x^{10} - x^{11}.
$$

This implies that the sequence $\varphi_3^{c,Ext}(n)$ satisfies a recurrence relation of order 9.

Coding the Interior Region

We find $V(\mathcal{D}^{\,c,Int}) = \{v_1, v_2, \dots, v_{10}\}$; the vertices and the adjacency matrix $M_3^{c,Int} = [m_{i,j}^{c,Int}]$ are listed below:

$$\begin{array}{l} v_1 = (0^0, 0^0, 0^0) \\ v_2 = (0^0, 0^0, 2^0) \\ v_3 = (0^0, 0^0, 2^1) \\ v_4 = (2^0, 0^0, 2^{-1}) \\ v_5 = (2^0, 0^0, 3^0) \\ v_6 = (2^0, 2^0, 2^0) \\ v_7 = (0^0, 2^0, 0^0) \\ v_8 = (2^0, 0^0, 0^0) \\ v_9 = (2^0, 0^0, 2^0) \\ v_{10} = (2^0, 0^0, 2^1) \end{array} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

We also find $\mathcal{F}_3\mathcal{L}_3 = \{(v_1, v_6), (v_1, v_9), (v_2, v_4), (v_2, v_8), (v_3, v_6), (v_3, v_9)\}$. The characteristic polynomial for ${M_3}^{c,Int}$ is

$$ {P_3}^{c,Int}(x) = -x^4(1-x)^2(1-2x+2x^2+2x^3-x^4). $$

Common Results for Both Types of Coding

Recall that $h_3^c(n) = n{\varphi_3}^{c,Ext}(n-2) = n{\varphi_3}^{c,Int}(n-1)$,

$$ {\Phi_3}^{c,Ext}(x) = \sum_{n\geq 0} {\varphi_3}^{c,Ext}(n)x^n, \qquad \text{and} \qquad {\Phi_3}^{c,Int}(x) = \sum_{n\geq 0} {\varphi_3}^{c,Int}(n)x^n. $$

The first few nonzero values are listed below.

$$\begin{array}{l}
h_3^c(2) = 2 \cdot {\varphi_3}^{c,Ext}(0) = 2 \cdot {\varphi_3}^{c,Int}(1) = 2 \cdot 1 = 2 \\
h_3^c(3) = 3 \cdot {\varphi_3}^{c,Ext}(1) = 3 \cdot {\varphi_3}^{c,Int}(2) = 3 \cdot 4 = 12 \\
h_3^c(4) = 4 \cdot {\varphi_3}^{c,Ext}(2) = 4 \cdot {\varphi_3}^{c,Int}(3) = 4 \cdot 12 = 48 \\
h_3^c(5) = 5 \cdot {\varphi_3}^{c,Ext}(3) = 5 \cdot {\varphi_3}^{c,Int}(4) = 5 \cdot 32 = 160 \\
h_3^c(6) = 6 \cdot {\varphi_3}^{c,Ext}(4) = 6 \cdot {\varphi_3}^{c,Int}(5) = 6 \cdot 83 = 498 \\
h_3^c(7) = 7 \cdot {\varphi_3}^{c,Ext}(5) = 7 \cdot {\varphi_3}^{c,Int}(6) = 7 \cdot 212 = 1484 \\
h_3^c(8) = 8 \cdot {\varphi_3}^{c,Ext}(6) = 8 \cdot {\varphi_3}^{c,Int}(7) = 8 \cdot 540 = 4320 \\
h_3^c(9) = 9 \cdot {\varphi_3}^{c,Ext}(7) = 9 \cdot {\varphi_3}^{c,Int}(8) = 9 \cdot 1372 = 12348 \\
h_3^c(10) = 10 \cdot {\varphi_3}^{c,Ext}(8) = 10 \cdot {\varphi_3}^{c,Int}(9) = 10 \cdot 3485 = 34850 \\
h_3^c(11) = 11 \cdot {\varphi_3}^{c,Ext}(9) = 11 \cdot {\varphi_3}^{c,Int}(10) = 11 \cdot 8848 = 97328 \\
h_3^c(12) = 12 \cdot {\varphi_3}^{c,Ext}(10) = 12 \cdot {\varphi_3}^{c,Int}(11) = 12 \cdot 22464 = 269568
\end{array}$$

The generating functions are obtained for both cases in the usual way:

$$ {\Phi_3}^{c,Int}(x) = x\,{\Phi_3}^{c,Ext}(x) = \frac{x(1+x)}{(1-x)(1-2x-2x^2+2x^3-x^4)}. $$

Then

$$\begin{aligned} \mathcal{H}_3^c(x) &= \frac{d}{dx}\left(x^2\,{\Phi_3}^{c,Ext}(x)\right) = \frac{d}{dx}\left(x\,{\Phi_3}^{c,Int}(x)\right) \\ &= \frac{2x(1-3x^2-2x^3+3x^4-x^6)}{(1-x)^2(1-2x-2x^2+2x^3-x^4)^2}. \end{aligned}$$

From (3) in [**1**] and the above equality we obtain

$$\begin{aligned}\mathcal{H}_3(x) &= \mathcal{H}_3^{nc}(x) + \mathcal{H}_3^{c}(x)\\ &= 2x(2 - 6x + 3x^2 - 36x^3 + 97x^4 + 96x^5 - 372x^6\\ &\quad + 96x^7 + 280x^8 - 142x^9 + 64x^{10} - 252x^{11} + 132x^{12}\\ &\quad + 168x^{13} - 193x^{14} + 64x^{15} - 11x^{16} + 6x^{17} - 2x^{18})\ /\\ &\quad [(1-x)^2(1+x-x^2)(1-x-x^2)(1-2x-2x^2+2x^3-x^4)^2\\ &\quad (1-x-3x^2-x^3+x^4)(1+x-3x^2+x^3+x^4)].\end{aligned}$$

Its power series expansion is

$$\mathcal{H}_3(x) = 4x + 24x^2 + 306x^3 + 850x^4 + 7010x^5 + 18452x^6 + 126426x^7 + \cdots.$$

5.4. Thick Cylinder $P_5 \times C_n$ $(m = 4)$

For Hamiltonian cycles of type C, the degrees of the characteristic polynomials for $M_4^{c,Ext}$ and $M_4^{c,Int}$ are 44 and 33, respectively; they determine recursions of order 28 and 16, respectively. However, their generating functions $\Phi_4^{c,Ext}(x)$ and $\Phi_4^{c,Int}(x)$ indicate the same recursion of order 12 (which was expected) for the sequences $\varphi_4^{c,Ext}(n)$ and $\varphi_4^{c,Int}(n)$. Since $\varphi_4^{c,Ext}(n-2) = \varphi_4^{c,Int}(n-1) = h_4^c(n)/n$, it is clear that

$$\mathcal{H}_4^c(x) = \frac{d}{dx}\left(x^2\,\Phi_4^{c,Ext}(x)\right) = \frac{d}{dx}\left(x\,\Phi_4^{c,Int}(x)\right),$$

where

$$\Phi_4^c(x) = \frac{x(1+16x^2-48x^4-8x^6+77x^8-8x^{10}+2x^{12})}{(1-x)(1+x)(1-3x^2)^2(1-11x^2-2x^6)}.$$

Thus,

$$\begin{aligned}\mathcal{H}_4^c(x) &= 2x(1 + 35x^2 - 419x^4 + 791x^6 + 1251x^8 - 6807x^{10} + 9747x^{12}\\ &\quad - 5055x^{14} + 1032x^{16} + 168x^{18} + 36x^{20} - 12x^{22})\ /\\ &\quad [(1-x)^2(1+x)^2(1-3x^2)^3(1-11x^2-2x^6)^2]\\ &= 2x + 136x^3 + 2832x^5 + 44288x^7 + 621720x^9 + 8268432x^{11}\\ &\quad + 106467592x^{13} + 1341213504x^{15} + 16625223000x^{17}\\ &\quad + 203511990480x^{19} + 2466221656712x^{21} + 29639129297760x^{23}\\ &\quad + 353729229308728x^{25} + 4196610165544048x^{27}\\ &\quad + 49534151824335720x^{29} + \cdots.\end{aligned}$$

From $\mathcal{H}_4(x) = \mathcal{H}_4^{nc}(x) + \mathcal{H}_4^c(x)$, we determine

$$\begin{aligned}\mathcal{H}_4(x) \quad = \quad & 2x(2 - 4x - x^2 - 399x^3 + 264x^4 + 12003x^5 - 17018x^6 - 132589x^7 \\ & + 242972x^8 + 741418x^9 - 1592134x^{10} - 2249196x^{11} + 5298156x^{12} \\ & + 3895631x^{13} - 6944234x^{14} - 7550348x^{15} - 9067499x^{16} \\ & + 32222212x^{17} + 44031695x^{18} - 104031851x^{19} - 46708066x^{20} \\ & + 189953465x^{21} - 27366464x^{22} - 195625961x^{23} + 111621212x^{24} \\ & + 106177438x^{25} - 112507558x^{26} - 19954314x^{27} + 54710202x^{28} \\ & - 6461551x^{29} - 11552268x^{30} + 1712058x^{31} - 156451x^{32} \\ & + 2433592x^{33} - 1096202x^{34} - 349808x^{35} + 355012x^{36} + 47076x^{37} \\ & - 57312x^{38} + 6792x^{39} - 5340x^{40} + 2208x^{41} - 72x^{42} + 96x^{44}) \; / \\ & [(1-x)^2(1+x)^2(1-2x)(1+2x)(1-2x^2)(1-3x^2)^2 \\ & (1-3x+x^3)(1-3x^2+x^3-x^4)(1-4x^2+2x^3-2x^4-x^5) \\ & (1-5x+2x^2+8x^3-8x^4+x^5-x^6)(1+11x^2+2x^6)^2].\end{aligned}$$

Upon expansion, we obtain

$$\begin{aligned}\mathcal{H}_4(x) = {} & 4x + 24x^2 + 306x^3 + 850x^4 + 7010x^5 + 18452x^6 + 126426x^7 + 351258x^8 \\ & + 2127332x^9 + 6355404x^{10} + 35085590x^{11} + 112481980x^{12} + 577875650x^{13} \\ & + 1970896234x^{14} + 9576146794x^{15} + 34373120896x^{16} + 160047128522x^{17} \\ & + 598167523522x^{18} + 2697774177200x^{19} + 10398653965136x^{20} \\ & + 45813998934398x^{21} + 180683364527008x^{22} + 782729112571558x^{23} \\ & + 3138757868554550x^{24} + 13435232382112114x^{25} + 54519162573345144x^{26} \\ & + 231410726096158954x^{27} + 946929235189639806x^{28} \\ & + 3995898137059583288x^{29} + 16446553366281600876x^{30} + \cdots .\end{aligned}$$

5.5. Thick Cylinder $P_6 \times C_n$ $(m = 5)$

For the HC's of type C, the characteristic polynomial for $M_5^{c,Ext}$ (of order 174) yields a recurrence of order 140, and the characteristic polynomial for $M_5^{c,Int}$ (of order 104) determines a recurrence of order 68. The generating functions $\Phi_5^{c,Ext}(x)$ and $\Phi_5^{c,Int}(x)$ indicate a recursion of order 48.

$$\begin{aligned}\Phi_5^c(x) = {} & x(1 + 6x - 18x^2 - 414x^3 + 848x^4 + 6554x^5 - 16045x^6 - 37690x^7 \\ & + 103281x^8 + 128504x^9 - 265355x^{10} - 672050x^{11} + 502008x^{12} + 3340076x^{13} \\ & - 2448123x^{14} - 9954494x^{15} + 10693383x^{16} + 18205338x^{17} - 29135762x^{18} \\ & - 20019118x^{19} + 53993028x^{20} + 8041536x^{21} - 72266964x^{22} + 14531980x^{23} \\ & + 71713080x^{24} - 34388270x^{25} - 52305506x^{26} + 39831436x^{27} + 26241113x^{28} \\ & - 30511612x^{29} - 7246130x^{30} + 16104692x^{31} - 373346x^{32} - 5790448x^{33} \\ & + 1221241x^{34} + 1368750x^{35} - 510697x^{36} - 198552x^{37} + 111455x^{38} + 14804x^{39} \\ & - 13910x^{40} - 84x^{41} + 945x^{42} - 72x^{43} - 27x^{44} + 4x^{45}) \; /\end{aligned}$$

$$
\begin{aligned}
&[(1-x)(1-x-x^2)(1-2x-x^2+x^3)(1+2x-x^2-x^3)\\
&(1-2x-x^2+2x^3-x^4)(1-x-7x^2+2x^3-2x^4)(1-2x^2+3x^3+x^4-x^5)\\
&(1-4x+x^2+6x^3-4x^4-2x^5+x^6)(1+4x+x^2-6x^3-4x^4+2x^5+x^6)\\
&(1-5x-14x^2+63x^3-12x^4-90x^5+35x^6+66x^7-118x^8+8x^9\\
&\ +82x^{10}-42x^{11}-28x^{12}+4x^{13}-2x^{14})]
\end{aligned}
$$

Below is the power series expansion of $\mathcal{H}_5^c(x)$.

$$
\begin{aligned}
\mathcal{H}_5^c(x) = {}& 2x+48x^2+612x^3+4520x^4+35964x^5+229698x^6+1575288x^7+9806292x^8\\
&+62999960x^9+387822094x^{10}+2411860680x^{11}+14706401372x^{12}\\
&+89805227764x^{13}+542922027450x^{14}+3277207263040x^{15}+19667429401654x^{16}\\
&+117755148280932x^{17}+702348082721928x^{18}+4179389353497440x^{19}\\
&+24800669448996294x^{20}+146859912517712812x^{21}+867755187436181848x^{22}\\
&+5117982982251905808x^{23}+30131949609066739700x^{24}\\
&+177121683074273170272x^{25}+1039599437405096152836x^{26}\\
&+6093526747267596471744x^{27}+35670915471779662426386x^{28}\\
&+208566648331009396119000x^{29}+\cdots.
\end{aligned}
$$

5.6. Thick Cylinder $P_7 \times C_n$ $(m = 6)$

We obtain

$$
\begin{aligned}
\mathcal{H}_6^c(x) = {}& 2x+2032x^3+263736x^5+22337664x^7+1641664580x^9+113092326312x^{11}\\
&+7512031798348x^{13}+487293888097600x^{15}+31078838281479156x^{17}\\
&+1956749096194717760x^{19}+121942699478516467980x^{21}\\
&+7535939697350674950480x^{23}+462464193503836875708188x^{25}\\
&+28212097969001607154778424x^{27}+1712255987823212304590396640x^{29}+\cdots.
\end{aligned}
$$

which was derived from

$$
\begin{aligned}
\Phi_6^c(x) = {}& x(1+306x^2-40690x^4+2088888x^6-59854356x^8+1041724854x^{10}\\
&-9963350575x^{12}-1561595514x^{14}+1764222372901x^{16}-33359541871130x^{18}\\
&+391283632798625x^{20}-3409940836072834x^{22}+23527202411977523x^{24}\\
&-132804217617691704x^{26}+626327321659400394x^{28}-2507190853725762016x^{30}\\
&+8634200715329254103x^{32}-25906190273641652336x^{34}\\
&+68559708278394067292x^{36}-161854074407081021262x^{38}\\
&+343843110397151257836x^{40}-660259656542874312136x^{42}\\
&+1145535722603029360938x^{44}-1788319405800757683806x^{46}\\
&+2496733228293042684759x^{48}-3090833148880542271276x^{50}\\
&+3332184138797783431832x^{52}-2977551413530470915268x^{54}\\
&+1874479143252245895283x^{56}-135583843251370310752x^{58}\\
&-1725296888982641415559x^{60}+2978301300202755230596x^{62}\\
&-3196573579330611658014x^{64}+2594775826885761943406x^{66}
\end{aligned}
$$

$$
\begin{aligned}
&- 1764806452649464365958x^{68} + 1132024373770684603442x^{70}\\
&- 747388500864816970949x^{72} + 488605725204679503240x^{74}\\
&- 281985452668526323045x^{76} + 129099929765096588046x^{78}\\
&- 40724401950514905533x^{80} + 4766321154300357988x^{82}\\
&+ 3107539531302455678x^{84} - 2141576362869785724x^{86} + 699992921850375378x^{88}\\
&- 164891856789160528x^{90} + 42040968674824716x^{92} - 11653173730620232x^{94}\\
&+ 2342301220645576x^{96} - 366020508217376x^{98} + 106155952612144x^{100}\\
&- 34874926374880x^{102} + 6594270112768x^{104} - 673590339328x^{106}\\
&+ 37441754880x^{108} + 169198080x^{110} - 292896768x^{112} + 15998976x^{114})) \,/\\
&[(1-2x)(1+2x)(1-2x^2)(1-4x^2+2x^4)(1-8x^2+14x^4)\\
&(1-27x^2+225x^4-641x^6+659x^8-227x^{10}+46x^{12}-123x^{14}\\
&+169x^{16}-49x^{18}+4x^{20})(1-37x^2+397x^4-1681x^6+2639x^8\\
&-903x^{10}-56x^{12}+307x^{14}+525x^{16}-209x^{18}-8x^{20})\\
&(1-35x^2+322x^4-1485x^6+4262x^8-7682x^{10}+7755x^{12}\\
&-10671x^{14}+18616x^{16}-9492x^{18}-1484x^{20}+1589x^{22}-62x^{24}-40x^{26})\\
&(1-85x^2+1932x^4-20403x^6+116734x^8-386724x^{10}+815141x^{12}-1251439x^{14}\\
&+1690670x^{16}-2681994x^{18}+4008954x^{20}-3390877x^{22}+1036420x^{24}\\
&+178842x^{26}-92790x^{28}-17732x^{30}+5972x^{32}-1728x^{34}-144x^{36})],
\end{aligned}
$$

5.7. Thick Cylinder $P_{m+1} \times C_n$ $(7 \leq m \leq 9)$

For the sake of brevity, we only display the power series expansion of the generating functions $\mathcal{H}_m^c(x)$ for $7 \leq m \leq 9$.

$$
\begin{aligned}
\mathcal{H}_7^c(x) = {}& 2x + 192x^2 + 8192x^3 + 127860x^4 + 2779014x^5 + 35663964x^6 + 605992784x^7\\
&+ 7769376972x^8 + 116791523380x^9 + 1519170232976x^{10} + 21412201037580x^{11}\\
&+ 280509236582900x^{12} + 3817205794180856x^{13} + 50048772776920380x^{14}\\
&+ 667452277157951872x^{15} + 8730496956098122924x^{16}\\
&+ 114990875591325208344x^{17} + 1498721829346080971718x^{18}\\
&+ 19577329280144309140500x^{19} + 254184297263298653321994x^{20}\\
&+ 3300736306177174727889026x^{21} + 42700068205017640140982112x^{22}\\
&+ 551992937500828921720192872x^{23} + 7117443280523917273056970850x^{24}\\
&+ 91678074802674943650656279184x^{25} + 1178664397321648769186649515370x^{26}\\
&+ 15136943041102404084253082680484x^{27}\\
&+ 194109908825815965787089284625154x^{28}\\
&+ 2486557768079418847989177095267850x^{29} + \cdots,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{H}_8^c(x) = {} & 2x + 29104x^3 + 22869384x^5 + 10215798448x^7 + 3817933082020x^9 \\
& + 1320093157541136x^{11} + 437662770447567560x^{13} + 141338955368771390016x^{15} \\
& + 44820915345596090414880x^{17} + 14022558891295056887443160x^{19} \\
& + 4341003538573245627716733276x^{21} + 1332438402563600063493162541728x^{23} \\
& + 406097228507066527913083107762828x^{25} \\
& + 123030579109675576558337448581510896x^{27} \\
& + 37082080183727206133637758977416982500x^{29} + \cdots ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_9^c(x) = {} & 2x + 768x^2 + 112164x^3 + 3616880x^4 + 222067212x^5 + 5539931796x^6 \\
& + 242178636928x^7 + 6169925169414x^8 + 224360971248960x^9 \\
& + 5973677282007402x^{10} + 195609021230822100x^{11} + 5368583261802264972x^{12} \\
& + 165442535132471644292x^{13} + 4616256789338403997830x^{14} \\
& + 137248345001943810512192x^{15} + 3858464231851630072287480x^{16} \\
& + 112245666066341094211887474x^{17} + 3163164190641471563556546716x^{18} \\
& + 90762230874863948167367645720x^{19} + 2557054414248684611758303990008x^{20} \\
& + 72707096815758305550335466105864x^{21} \\
& + 2045202670958705026338344754333024x^{22} \\
& + 57786674199846212563657479095447016x^{23} \\
& + 1622114630916418603623686588180361000x^{24} \\
& + 45620599345582502819377370379402448790x^{25} \\
& + 1277757171356042779682960928336763812048x^{26} \\
& + 35808036061847453918421598630326003072512x^{27} \\
& + 1000745362534879987116390583909098103232022x^{28} \\
& + 27965040033048966560373497047404511628553450x^{29} + \cdots .
\end{aligned}$$

6. ASYMPTOTIC VALUES — A SUMMARY OF RESULTS

For type C Hamiltonian cycles, our computational data confirm that for $2 \leq m \leq 6$, the characteristic polynomials of $M^{\,c,Ext}$ and $M^{\,c,Int}$ have only one (and the same) simple real positive dominant characteristic root $\theta_{m,c}$, see Table 3 (for even m, there are two dominant characteristic roots $\theta_{m,c}$ and $-\theta_{m,c}$; whereas for odd m, there is a unique dominant simple characteristic root $\theta_{m,c}$).

Note that the sum of the degrees of the denominators of $\mathcal{H}_m^{nc}$ and $\mathcal{H}_m^{c}$ does not exceed the degree of the denominator of $\mathcal{H}_m$ which determine the order of the recurrence relation of the sequence $h_m(n)$. This goes in favour of our decision to split our work in two — the problem of determining the HC ^{nc}'s and the HC ^{c}'s. However, this was not the case for thin cylinders [**3**].

The denominators of the generating functions $\Phi_m^{c,Ext}(x)$ (or $\Phi_m^{c,Int}(x)$) and $\mathcal{H}_m^c(x)$ have the radius of convergence $1/\theta_{m,c}$. For $\mathcal{H}_m^c(x)$, the dominant root of

m	$d.d.(\Phi_m^{c,Int}) = d.d.(\Phi_m^{c,Ext})$	$d.d.(\mathcal{H}_m^{c,Int}) = d.d.(\mathcal{H}_m^{c,Ext})$	$d.d.(\mathcal{H}_m^{nc})$	$d.d.(\mathcal{H}_m)$
2	2	4	2	5
3	5	10	12	22
4	12	22	26	44
5	48	96	84	180
6	114	228	-	-

Table 2: The degree of denominators ($d.d.$) of $\Phi_m^{c,Int}$ ($\Phi_m^{c,Ext}$), $\mathcal{H}_m^{c,Int}$ ($\mathcal{H}_m^{c,Ext}$), $\mathcal{H}_m^{nc}$ and $\mathcal{H}_m$ for $2 \le m \le 6$.

the denominator has multiplicity 2 because

$$\mathcal{H}_m^c(x) = \frac{d}{dx}\left(x^2\,\Phi_m^{c,Ext}(x)\right) = \frac{d}{dx}\left(x\,\Phi_m^{c,Int}(x)\right),$$

which is deduced from

$$h_m^c(n) = n\varphi_m^{c,Int}(n-1) = n\varphi_m^{c,Ext}(n-2). \tag{3}$$

Let $a_{m,c}$ denote the coefficient of $n\theta_{m,c}^n$ in the explicit expression for $h_m^c(n)$ derived from the recurrence relation. From (3) and (1), we conclude that the coefficient for $\theta_{m,c}^n$ is 0, when m is odd; and for even m, the coefficients of $n(-\theta_{m,c})^n$, $\theta_{m,c}^n$, and $(-\theta_{m,c})^n$ are $a_{m,c}$, 0, and 0, respectively. All this is neatly summarized below:

$$h_m^c(n) \sim \begin{cases} a_{m,c} n\theta_{m,c}^n & \text{if } m \text{ is odd,} \\ 2a_{m,c} n\theta_{m,c}^n & \text{if } m \text{ is even and } n \text{ is even,} \\ 0 & \text{if } m \text{ is even and } n \text{ is odd.} \end{cases}$$

Example 8. The number $h_5^c(250)$ has 190 digits:

$h_5^c(250) =$**5315308482081368176130135765364458028442269812845**829751132
2824493660377059160812449663786164807652523348584626304505471427 28
7078322603370886758945517424366777432362736327602269517441303980 00

as well as $a_{5,c} \cdot 250 \cdot \theta_{5,c}^{250}$, and their first 49 digits are identical.

Example 9. The number $h_8^c(100)$ has 124 digits:

$h_8^c(100) =$**650572515095530**4765274909197134354116977319209669418653015912
0966061950648692459066633776606313737469111316745172668642246 00

as well as $2a_{8,c} \cdot 100 \cdot \theta_{8,c}^{100}$, and their first 15 digits are identical.

With the results and conjecture of Part I [**1**] in mind, together with the assumptions about the positive dominant characteristic root $\theta_{m,c}$, we may now

m	$\theta_{m,nc}$	$\theta_{m,c}$
2	2	$\sqrt{2}$
3	2.36920540709246654628	2.53861576354917625747
4	4.16748148276892815337	3.31910824039947675342
5	5.34684254175541433292	5.65205864851675849429
6	$\approx_{(100)}$ 8.908937311	7.52634546292690578713
7	$\approx_{(100)}$ 11.8249316	$\approx_{(100)}$ 12.382351641593
8	$\approx_{(70)}$ 19.17	$\approx_{(100)}$ 16.77216819355
9	$\approx_{(30)}$ 26	$\approx_{(30)}$ 27

m	$a_{m,c}$
2	0.25
3	0.31357228606585772287
4	0.19324623166497686532
5	0.18876590435542745301
6	0.14384483205795162266
7	$\approx_{(100)}$ 0.13626186172698
8	$\approx_{(100)}$ 0.11306933143427
9	$\approx_{(30)}$ 0.1053

Table 3: The approximate values of $\theta_{m,nc}$ (with $a_{m,nc} = 1$), $\theta_{m,c}$ and $a_{m,c}$ for $2 \leq m \leq 9$, where $\approx_{(n)}$ means the estimate based on the first n entries of the sequence.

make a conjecture regarding the behaviour of the number of all Hamiltonian cycles $h_m(n)$ in the graph $P_{m+1} \times C_n$, when m is fixed and $n \to \infty$, as below:

$$h_m(n) = h_m^c(n) + h_m^{nc}(n) \sim \begin{cases} a_{m,c} n \theta_{m,c}^n + (1 + (-1)^n)\theta_{m,nc}^n & \text{if } m \text{ is odd,} \\ a_{m,c} n (1 + (-1)^n)\theta_{m,c}^n + \theta_{m,nc}^n & \text{if } m \text{ is even .} \end{cases}$$

When $m \leq 9$ the data shows that $\theta_{m,c} > \theta_{m,nc}$ for odd m; whereas $\theta_{m,nc} > \theta_{m,c}$ for even m. Assuming that the same holds for all the values of m, we propose

Conjecture 1.

$$h_m(n) = h_m^c(n) + h_m^{nc}(n) \sim \begin{cases} a_{m,c} n \theta_{m,c}^n & \textit{if } m \textit{ is odd,} \\ \theta_{m,nc}^n & \textit{if } m \textit{ is even .} \end{cases}$$

Example 10. For $m = 5$ and $n = 250$, $h_5^{nc}(250) \sim 2.1153 \cdot 10^{182}$, whilst $h_5^c(250) \sim 5.3153 \cdot 10^{189}$, and so $h_5(250) = h_5^c(250) + h_5^{nc}(250) \sim 5.3153 \cdot 10^{189} \sim h_5^c(250)$.

Example 11. For $m = 6$ and $n = 100$, $h_6^{nc}(100) = 96070554870981782995827137142128630238785786563044765223962050649940105158800796411036738881670 \sim 9,6071 \cdot 10^{94}$, while $h_6^c(100) = 131020464919763494924652519229638201495869414699961724074530458968504938173958572002437400 \sim 1,3102 \cdot 10^{89}$. Thus, $h_6(100) \sim 9,6071 \cdot 10^{94} \sim \theta_{6,nc}^{100}$.

From the results obtained for $m \leq 6$ we have spotted that the positive dominant characteristic root $\theta_{m,c}$ of $h^c_m(n)$ corresponding to $P_{m+1} \times C_n$ is the same as the positive dominant characteristic root of the same sequence associated to $P_{m+1} \times P_n$ [**2**]. Observe that the polynomial $1 - 2x - 2x^2 + 2x^3 - x^4$, being in the denominator of $\Phi_3^{c,Ext}(x)$ and $\Phi_3^{c,Int}(x)$ (or $\mathcal{H}_3^c(x)$ and $\mathcal{H}_3(x)$) is also in the denominator of the generating function of sequence corresponding to $P_4 \times P_n$ [**5**]. The same phenomenon occurs for $4 \leq m \leq 6$, as well. The obtained approximate values of the dominant characteristic root for $7 \leq m \leq 9$ speak in favour of the same conclusion. That brings us to our next conjecture:

Conjecture 2. *Let $r_m(n)$ $(m \geq 1)$ be the number of HC's in $P_{m+1} \times P_n$ and $h^c_m(n)$ be the number of contractible HC's in $P_{m+1} \times C_n$ $(m \geq 1)$. Then*

$$\lim_{n\to\infty} \frac{r_m(n)}{r_m(n-1)} = \lim_{n\to\infty} \frac{h^c_m(n)}{h^c_m(n-1)} \text{ for odd } m$$

and

$$\lim_{n\to\infty} \frac{r_m(2n)}{r_m(2n-2)} = \lim_{n\to\infty} \frac{h^c_m(2n)}{h^c_m(2n-2)} \text{ for even } m.$$

If the above conjecture holds, then using merely the data acquired from the sequence of $r_m(n)$'s we could conclude that $\theta_{10,c} \sim 37.03764916$, $\theta_{11,c} \sim 58.75$, $\theta_{12,c} \sim 81.366569$ and $\theta_{13,c} \sim 127.7$. In other words, we would not require the exact value of $h^c_m(n)$ to do so.

7. CLOSING REMARKS AND FURTHER RESEARCH

For the purpose of enumerating Hamiltonian cycles on $P_{m+1} \times C_n$ we have provided one characterization of the non-contractible HC's in Part I, and two characterizations of the contractible HC's with fixed up root of the split tree in w_{11}.

1. Confirmation of the old data and the process of obtaining the new ones

Both of the computer programs dealing with the HC^c case have provided the same number for $h^c_m(n)$, when $m \leq 7$, which agrees with the corresponding values of [**3**]. The latter holds for $h^c_m(n)$ as well, where $8 \leq m \leq 9$, when obtained in the act of coding the interior. The sum of sequences obtained in all the three programs, i.e. the numerical values of $h_m(n) = h^{nc}_m(n) + h^c_m(n)$ agree with the ones obtained earlier in [**3**], [**9**] and [**8**], for $m \leq 9$ and $n \leq 22$. We have derived new data for $m \leq 9$ and $n \geq 23$.

2. The advantage of coding the interior over coding the exterior

Comparing the number of vertices of the digraphs $\overline{\mathcal{D}}_m^{c,Ext}$ and $\mathcal{D}_m^{c,Int}$ (see Table 1) one can come to a conclusion that coding the windows of the interior region is much more efficient than the process of coding the windows of the exterior region of a HC ^{c}.

3. The advantage of coding the regions over coding the vertices

For the purpose of obtaining the total number of HC's, in case of thin cylinder $C_m \times P_n$, coding the vertices has proven itself to be a better approach. Namely, for a fixed m, the number of vertices of the assigned digraph in the aforementioned approach [**2**] turned out to be smaller than the number obtained when coding the regions [**3**]. Additionally, the order of recursion of the total number of HC's for thin cylinders is smaller than for special HC's, i.e. HC ^{nc} and HC ^{c}. The results show that the opposite is true for thick cylinders. This supports the choice of our approach when tackling the thick cylinders, although it has to be split into parts. It goes without saying, that further research in this direction, would be nice. Particularly, it would be a good idea to utilize the approach with coding the vertices so as to be able to reach a precise conclusion regarding the pros and cons of coding the regions, by a direct comparison of the number of vertices of the assigned digraphs.

4. Open questions

For the initial values of m we have come to notice that the numbers of HC ^{nc}'s are the dominant ones for even m, whereas the numbers of HC ^{c}'s are such for odd m. That prompted us to make a conjecture about the asymptotic behaviour of the total number of HC's in the graph $P_{m+1} \times C_n$. Moreover, certain matchings between the dominant characteristic roots of the sequences the numbers of HC ^{c}'s in $P_{m+1} \times C_n$ and $P_{m+1} \times P_n$ for small values of m are noticed. This way, we have come to yet another conjecture regarding the asymptotic behaviour of the entries of these two sequences.

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