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ESTIMATES ON SOME QUADRATURE RULES VIA WEIGHTED HERMITE-HADAMARD INEQUALITY

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In this article new estimates on some quadrature rules are given using weighted Hermite-Hadamard inequality for higher order convex functions and weighted version of the integral identity expressed by w-harmonic sequences of functions. Obtained results are applied to weighted one-point formula for numerical integration in order to derive new estimates of the definite integral values.

1. INTRODUCTION

Weighted Hermite-Hadamard inequality for convex functions is given in the following theorem ([**4**], [**5**]).

Theorem A. *Let $p : [a, b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I, then we have*

$$f(\lambda) \leq \frac{1}{P(b)} \int_a^b p(x)f(x)\ dx \leq \frac{b-\lambda}{b-a} f(a) + \frac{\lambda - a}{b-a} f(b)$$

or

$$P(b)f(\lambda) \leq \int_a^b p(x)f(x)\ dx \leq P(b) \left[\frac{b-\lambda}{b-a} f(a) + \frac{\lambda - a}{b-a} f(b)\right], \tag{1}$$

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where

$$P(t) = \int_a^t p(x)\ dx \quad \text{and} \quad \lambda = \frac{1}{P(b)} \int_a^b p(x)x\ dx.$$

Various weighted versions of the general integral identities that are used for the approximation of an integral $\int_a^b f(t)\ dt$, using the harmonic sequences of polynomials and w-harmonic sequences of functions, are obtained in [**3**]. For introducing one of those identities let us consider subdivision $\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}$ of the segment $[a, b]$, $m \in \mathbb{N}$. Let $w : [a, b] \to \mathbb{R}$ be an arbitrary integrable function. For each segment $[x_{k-1}, x_k]$, $k = 1, ..., m$, we define w-harmonic sequences of functions $\{w_{kj}\}_{j=1,...,n}$ by:

$$(2) \qquad \begin{cases} w'_{k1}(t) = w(t), & t \in [x_{k-1}, x_k], \\ w'_{kj}(t) = w_{k,j-1}(t), & t \in [x_{k-1}, x_k],\ j = 2, 3, ..., n. \end{cases}$$

Also, we define function $W_{n,w}$ as follows

$$(3) \qquad W_{n,w}(t, \sigma) = \begin{cases} w_{1n}(t), & t \in [a, x_1], \\ w_{2n}(t), & t \in (x_1, x_2], \\ . & \\ . & \\ . & \\ w_{mn}(t), & t \in (x_{m-1}, b]. \end{cases}$$

Theorem B. *If* $g : [a, b] \to \mathbb{R}$ *is such that* $g^{(n)}$ *is a piecewise continuous on* $[a, b]$, *then the following identity holds*

$$(4) \qquad \begin{aligned} \int_a^b w(t)g(t)\ dt &= \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\ &+ \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\ &+ (-1)^n \int_a^b W_{n,w}(t, \sigma) g^{(n)}(t)\ dt. \end{aligned}$$

More recently obtained results on weighted versions of the general integral identities and harmonic sequences of polynomials or w-harmonic sequences of functions can be found in [**1**], [**2**], [**6**] and their references.

2. NEW RESULTS

In this section we derive Hermite-Hadamard's type inequalities using weighted version of the integral identity expressed by w-harmonic sequences of functions that is given in Theorem B.

Theorem 1. *Suppose $w : [a,b] \to \mathbb{R}$ is an arbitrary integrable function and w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\dots,n}$ are defined by* (2)*. Let the function $W_{n,w}$, defined by* (3)*, be nonnegative. Then,*

a) if $g : [a,b] \to \mathbb{R}$ is $(n+2)$-convex function, the following inequalities hold

$$\begin{aligned}(5)\qquad & (-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) \\ & \leq \int_a^b w(t)g(t)\, dt - \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\ & + \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\ & \leq (-1)^n \cdot P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda - a}{b-a} g^{(n)}(b) \right],\end{aligned}$$

where

$$\begin{aligned} P(b) = (-1)^n & \left[\frac{1}{n!} \int_a^b w(t) \cdot t^n \, dt - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+1)!} \right. \\ (6)\qquad & \left. \cdot \left(w_{mj}(b) b^{n-j+1} + \sum_{k=1}^{m-1} \left(w_{kj}(x_k) - w_{k+1,j}(x_k) \right) x_k^{n-j+1} - w_{1j}(a) a^{n-j+1} \right) \right] \end{aligned}$$

and

$$\begin{aligned} \lambda = (-1)^n & \left[\frac{1}{(n+1)! P(b)} \int_a^b w(t) \cdot t^{n+1} \, dt - \frac{1}{P(b)} \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+2)!} \right. \\ (7)\qquad & \left. \cdot \left(w_{mj}(b) b^{n-j+2} + \sum_{k=1}^{m-1} \left(w_{kj}(x_k) - w_{k+1,j}(x_k) \right) x_k^{n-j+2} - w_{1j}(a) a^{n-j+2} \right) \right], \end{aligned}$$

b) if g is $(n+2)$-concave function, then (5) *holds with the reversed sign of inequalities.*

Proof. a) Inequality (5) follows from the weighted Hermite-Hadamard inequality (1) substituting nonnegative function p by nonnegative function $W_{n,w}$ and convex function f by convex function $g^{(n)}$. To obtain desired result, we need to calculate the values of $P(b)$ and λ in the terms of new substitutions.

The value of $P(b)$ will be obtained from (4) taking $g(t) = \frac{t^n}{n!}$. Then, $g^{(n)}(t) = 1$ and, regarding the formula of $P(t)$ from Theorem A, we get

$$
\begin{aligned}
P(b) &= \int_a^b W_{n,w}(t,\sigma)\ dt \\
&= (-1)^n \int_a^b w(t)\frac{t^n}{n!}\ dt - (-1)^n \sum_{j=1}^{n} (-1)^{j-1} \Bigg[w_{mj}(b)\frac{1}{(n-j+1)!} b^{n-j+1} \\
&\quad + \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k)\right] \frac{1}{(n-j+1)!} x_k^{n-j+1} \\
&\quad - w_{1j}(a)\frac{1}{(n-j+1)!} a^{n-j+1} \Bigg].
\end{aligned}
$$

Rearranging above equality we get (6).

Applying our substitutions to the formula of λ from Theorem A we get $\lambda = \frac{1}{P(b)} \int_a^b W_{n,w}(t,\sigma) \cdot t\ dt$. The value of this integral follows from (4) taking $g(t) = \frac{t^{n+1}}{(n+1)!}$. Then, $g^{(n)}(t) = t$ and $g^{(j-1)}(t) = \frac{(n+1)n \cdot \dots \cdot (n-j+3)}{(n+1)!} \cdot t^{n-j+2} = \frac{1}{(n-j+2)!} \cdot t^{n-j+2}$. Equality (7) follows.

Now, applying inequality (1) to function $W_{n,w}$ instead of p, and function $g^{(n)}$ instead of f, and replacing $(-1)^n \int_a^b W_{n,w}(t,\sigma) g^{(n)}\ dt$ by the expression from the identity (4), we get inequality (5).

b) If $W_{n,w}(t,\sigma) \leq 0$ for all $t \in [a,b]$, then $-W_{n,w}(t,\sigma) \geq 0$, $t \in [a,b]$, so applying the step $a)$ of this proof, we get the required result.

c) If g is $(n+2)$-concave function, i.e. $-g^{(n+2)} \geq 0$, then $-g^{(n)}$ is a convex function so applying weighted Hermite-Hadamard inequalities on convex function $-g^{(n)}$ we get reversed inequlities in (5).

□

In order to obtain our next result, let us expand w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\dots,n}$ by $w_{k,n+1}$, such that $w'_{k,n+1}(t) = w_{k,n}(t)$ for $t \in [x_{k-1}, x_k]$.

Now, function $W_{n+1,w}$ has the following form.

$$W_{n+1,w}(t,\sigma) = \begin{cases} w_{1,n+1}(t), & t \in [a, x_1], \\ w_{2,n+1}(t), & t \in (x_1, x_2], \\ . \\ . \\ . \\ w_{m,n+1}(t), & t \in (x_{m-1}, b]. \end{cases} \tag{8}$$

Theorem 2. *Let $g : [a, b] \to \mathbb{R}$ be $(n+2)$-convex on $[a, b]$. Suppose $w : [a, b] \to \mathbb{R}$ is an arbitrary integrable function and $\{w_{kj}\}_{j=1,\ldots,n+1}$ are w-harmonic sequences of functions. Let the function $W_{n+1,w}$, defined by* (8)*, be nonnegative. Then, inequality* (5) *is valid for*

$$P(b) = w_{m,n+1}(b) + \sum_{k=1}^{m-1} [w_{k,n+1}(x_k) - w_{k+1,n+1}(x_k)] - w_{1,n+1}(a)$$

and

$$\begin{aligned} \lambda \quad = \quad & \frac{1}{P(b)} \Bigg[b w_{m,n+1}(b) - a w_{1,n+1}(a) \\ & + \sum_{k=1}^{m-1} \left(x_k w_{k,n+1}\left(x_k\right) - x_k \cdot w_{k+1,n+1}\left(x_k\right) \right) - w_{m,n+2}(b) \\ & - \sum_{k=1}^{m-1} \left(w_{k,n+2}\left(x_k\right) - w_{k+1,n+2}\left(x_k\right) \right) + w_{1,n+2}(a) \Bigg]. \end{aligned} \tag{9}$$

If $W_{n,w}(t,\sigma) \le 0$ or g is $(n+2)$-concave function, then (5) *holds with the reversed sign of inequalities.*

Proof. We calculate only $P(b)$ and λ. Replacing, in Theorem A, p by $W_{n,w}$ and f by $g^{(n)}$, where $g(t) = \frac{t^{n+1}}{(n+1)!}$, we get

$$\begin{aligned} P(b) \quad &= \quad \int_a^b W_{n,w}(t,\sigma)\, dt \\ &= \quad \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w_{k,n}(t)\, dt \\ &= \quad \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w'_{k,n+1}(t)\, dt \\ &= \quad w_{m,n+1}(b) + \sum_{k=1}^{m-1} [w_{k,n+1}(x_k) - w_{k+1,n+1}(x_k)] - w_{1,n+1}(a) \end{aligned}$$

and

$$\begin{aligned}
\lambda &= \frac{1}{P(b)} \int_a^b W_{n,w}(t,\sigma) \cdot t \, dt \\
&= \frac{1}{P(b)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w_{k,n}(t) \cdot t \, dt \\
&= \frac{1}{P(b)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w'_{k,n+1}(t) \cdot t \, dt \\
&= \frac{1}{P(b)} \sum_{k=1}^{m} \left(x_k \cdot w_{k,n+1}(x_k) - x_{k-1} \cdot w_{k,n+1}(x_{k-1}) - \int_{x_{k-1}}^{x_k} w_{k,n+1}(t) \, dt \right) \\
&= \frac{1}{P(b)} \left[b w_{m,n+1}(b) - a w_{1,n+1}(a) + \right. \\
&\left. \sum_{k=1}^{m-1} \left(x_k w_{k,n+1}(x_k) - x_k \cdot w_{k+1,n+1}(x_k) \right) - \int_a^b W_{n+1,w}(t,\sigma) \, dt \right]. \qquad (10)
\end{aligned}$$

Adding the function $w_{k,n+2}$ to the w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n+1}$, such that $w'_{k,n+2}(t) = w_{k,n+1}(t)$, $t \in [x_{k-1}, x_k]$ and rewriting the last integral in (10) in the following sense

$$\begin{aligned}
\int_a^b W_{n+1,w}(t,\sigma) \, dt &= \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w_{k,n+1}(t) \, dt \\
&= \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w'_{k,n+2}(t) \, dt \\
&= \sum_{k=1}^{m} \left(w_{k,n+2}(x_k) - w_{k,n+2}(x_{k-1}) \right) \\
&= w_{m,n+2}(b) + \sum_{k=1}^{m-1} \left(w_{k,n+2}(x_k) - w_{k+1,n+2}(x_k) \right) \\
&\quad - w_{1,n+2}(a),
\end{aligned}$$

identity (9) is obtained. □

3. ONE-POINT FORMULA

In this section we apply obtained results of previous section to weighted one-point formula for numerical integration. We observe function $g : [a,b] \to \mathbb{R}$, integrable function $w : [a,b] \to \mathbb{R}$ and w-harmonic sequences of functions $\{w_{kj}\}_{j=0,1,..,n}$ on $[x_{k-1}, x_k]$, where $k = 1,2$. We consider subdivision $\sigma = \{a = x_0 < x_1 = x < x_2 = b\}$ of the segment $[a,b]$ and we assume $w_{1j}(a) = 0$ and $w_{2j}(b) = 0$, for $j = 1,...,n$. In [**3**] authors proved the following theorem.

Theorem C. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function and $x \in [a,b]$. Further, let us suppose $\{w_{kj}\}_{j=1,..,n}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1,2$ and some $n \in \mathbb{N}$, defined by the following relations*:

$$w_{1j}(t) = \frac{1}{(j-1)!} \int_a^t (t-s)^{j-1} w(s)\, ds, \quad t \in [a,x],$$

$$w_{2j}(t) = \frac{1}{(j-1)!} \int_b^t (t-s)^{j-1} w(s)\, ds, \quad t \in (x,b],$$

for $j = 1,...,n$. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous function, then we have

$$\int_a^b w(t) g(t)\, dt = \sum_{j=1}^{n} A_j(x) g^{(j-1)}(x) + (-1)^n \int_a^b W_{n,w}(t,x) g^{(n)}(t)\, dt, \tag{11}$$

where for $j = 1,...n$

$$A_j(x) = \frac{(-1)^{j-1}}{(j-1)!} \int_a^b (x-s)^{j-1} w(s)\, ds \tag{12}$$

and

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} w(s)\, ds, & t \in [a,x] \\ w_{2n}(t) = \frac{1}{(n-1)!} \int_b^t (t-s)^{n-1} w(s)\, ds, & t \in (x,b]. \end{cases} \tag{13}$$

Using integral identity (11), in the following theorem we obtain new estimates of the definite integral as a special case of the Theorem 1.

Theorem 3. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function and $x \in [a,b]$ fixed. Suppose $\{w_{kj}\}_{j=1,..,n}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k =$*

$1,2$ and $n \in \mathbb{N}$, defined in Theorem C. Let the function $W_{n,w}$, defined by (13), be nonnegative. If $g : [a,b] \to \mathbb{R}$ is $(n+2)$-convex function, then

$$\begin{aligned}(-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) &\\ \leq \int_a^b w(t)g(t)\,dt - \sum_{j=1}^{n} A_j(x) \cdot g^{(j-1)}(x) &\\ \leq (-1)^n \cdot P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda - a}{b-a} g^{(n)}(b)\right], &\end{aligned} \tag{14}$$

where

$$P(b) = (-1)^n \left[\frac{1}{n!}\int_a^b w(t) \cdot t^n\,dt - \sum_{j=1}^{n} \frac{x^{n-j+1}}{(n-j+1)!} \cdot A_j(x)\right],$$

$$\lambda = \frac{(-1)^n}{P(b)}\left[\frac{1}{(n+1)!}\int_a^b w(t) \cdot t^{n+1}\,dt - \sum_{j=1}^{n} \frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x)\right]$$

and A_j is defined as in Theorem C. If $W_{n,w}(t,\sigma) \leq 0$ or g is $(n+2)$-concave then (14) holds with the reversed sign of inequalities.

Proof. Inequality (14) follows directly from (1) replacing nonnegative function p by nonnegative function $W_{n,w}$ and convex function f by convex function $g^{(n)}$ and then applying identity (11) on $(-1)^n \int_a^b W_{n,w}(t,x) g^{(n)}(t)\,dt$.

Now,we calculate $P(b)$ and λ using formulas from Theorem 1 and the facts that in new subdivision $\sigma = \{a = x_0 < x_1 = x < x_2 = b\}$ of the segment $[a,b]$ we have: $m = 2$ and $x_1 = x$.

$$\begin{aligned} P(b) &= (-1)^n \left[\frac{1}{n!}\int_a^b w(t)t^n\,dt \right.\\ &\quad - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+1)!} \cdot \big(w_{2j}(b) b^{n-j+1} + w_{1j}(x) \cdot x^{n-j+1} \\ &\quad \left. - w_{2j}(x) \cdot x^{n-j+1} - w_{1j}(a) a^{n-j+1}\big)\right]. \end{aligned}$$

By the assumptions from the begining of this section: $w_{1j}(a) = 0$ and $w_{2j}(b) = 0$, for $j = 1, ..., n$. Then,

$$\begin{aligned} P(b) &= (-1)^n \left[\frac{1}{n!}\int_a^b w(t)t^n\,dt \right.\\ &\quad \left. - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+1)!} \cdot (w_{1j}(x) - w_{2j}(x)) \cdot x^{n-j+1}\right]. \end{aligned}$$

Using definitions of $\{w_{kj}\}$ from Theorem C, we calculate:

$$w_{1j}(x) - w_{2j}(x) = \frac{1}{(j-1)!} \int_a^b (x-s)^{j-1}\, w(s)\; ds.$$

Now, it follows

$$\begin{aligned} P(b) &= (-1)^n \left[\frac{1}{n!} \int_a^b w(t) t^n\; dt \right. \\ &\quad \left. - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+1)!} \cdot \frac{x^{n-j+1}}{(j-1)!} \int_a^b (x-s)^{j-1}\, w(s)\; ds \right] \\ &= (-1)^n \left[\frac{1}{n!} \int_a^b w(t) t^n\; dt - \sum_{j=1}^{n} \frac{A_j(x)}{(n-j+1)!} \cdot x^{n-j+1} \right]. \end{aligned}$$

Similary, using Theorem 1 for subdivision $\sigma = \{a = x_0 < x_1 = x < x_2 = b\}$, $m = 2$ and $x_1 = x$, we calculate λ.

$$\lambda = \frac{(-1)^n}{P(b)} \left[\frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1}\; dt - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+2)!} \right.$$
$$\left. \cdot \left(w_{2j}(b) b^{n-j+2} + w_{1j}(x) \cdot x^{n-j+2} - w_{2j}(x) \cdot x^{n-j+2} \right) - w_{1j}(a) a^{n-j+2} \right) \Big]$$

Since, $w_{1j}(a) = 0$ and $w_{2j}(b) = 0$, for $j = 1, ..., n$, it follows

$$\lambda = \frac{(-1)^n}{P(b)} \left[\frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1}\; dt - \sum_{j=1}^{n} \frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x) \right].$$

□

Following the reasoning of Theorem 2 now we expand w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$ by $w_{k,n+1}$ and $w_{k,n+2}$ such that $w'_{k,n+1}(t) = w_{k,n}(t)$ and $w'_{k,n+2}(t) = w_{k,n+1}(t)$, $t \in [x_{k-1}, x_k]$. For a new subdivision $\sigma = \{a = x_0 < x_1 = x < x_2 = b\}$ of the segment $[a, b]$ and the values $w_{1j}(a) = 0$ and $w_{2j}(b) = 0$, for $j = 1, ..., n+2$, we obtain the following result.

Theorem 4. *Let $w : [a, b] \to \mathbb{R}$ be an integrable function and $x \in [a, b]$ fixed. Suppose $\{w_{kj}\}_{j=1,..,n+2}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, $k = 1, 2$, $n \in \mathbb{N}$, defined in Theorem C. Let the function $W_{n,w}$, defined by* (13)*, be* *nonnegative. If $g : [a,b] \to \mathbb{R}$ is $(n+2)$-convex function then inequality* (14) *is valid for*

$$P(b) = w_{1,n+1}(x) - w_{2,n+1}(x)$$

and

$$\lambda = x - \frac{1}{P(b)}\left(w_{1,n+2}(x) - w_{2,n+2}(x)\right).$$

If $W_{n,w}(t,\sigma) \le 0$ or g is $(n+2)$-concave function then (14) *holds with the reversed sign of inequalities.*

Proof. The values of $P(b)$ and λ follow from the proof of Theorem 2, since $m = 2$, $x_1 = x$, $w_{1j}(a) = 0$ and $w_{2j}(b) = 0$, for $j = 1, ..., n+2$ and from the definitions of $\{w_{kj}\}$ from Theorem C. $\square$

Using integral mean value theorem to the $\int_a^b W_{2n,w}(t,x) g^{(2n)}(t)\ dt$, where $g : [a,b] \to \mathbb{R}$ is such that $g^{(2n)}$ is a continuous function, in [**3**, Theorem 5] authors proved that there exists $\nu \in (a,b)$ such that

$$\int_a^b w(t) g(t)\ dt - \sum_{j=1}^{2n} A_j(x) g^{(j-1)}(x) = A_{2n+1}(x) g^{(2n)}(\nu). \tag{15}$$

Applying this integral identity to our result in inequality (14), we obtain the following theorem.

Theorem 5. *Assume w and $\{w_{kj}\}$ satisfies the conditions of Theorem 4 for $j = 1, .., 2n+1$. Let A_j be defined as in* (12). *If $g : [a,b] \to \mathbb{R}$ is $(2n+2)$- convex, then there exists $\nu \in (a,b)$ such that*

$$\begin{aligned} & P(b) \cdot g^{(2n)}(\lambda) \\ & \le \frac{g^{(2n)}(\nu)}{(2n)!} \int_a^b (x-s)^{2n} \cdot w(s)\ ds \\ & \le P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda - a}{b-a} g^{(2n)}(b) \right], \end{aligned} \tag{16}$$

where

$$P(b) = \frac{1}{(2n)!} \int_a^b w(t) \cdot t^{2n}\ dt - \sum_{j=1}^{2n} \frac{x^{2n-j+1}}{(2n-j+1)!} \cdot A_j(x)$$

and

$$\lambda = \frac{1}{P(b)} \left[\frac{1}{(2n+1)!} \int_a^b w(t) \cdot t^{2n+1}\ dt - \sum_{j=1}^{2n} \frac{x^{2n-j+2}}{(2n-j+2)!} \cdot A_j(x) \right].$$

Proof. Inequality (16) follows directly from (14) replacing it's middle term by $A_{2n+1}(x)g^{(2n)}(\nu)$, according to the integral identity (15), and then applying (12) to A_{2n+1}. □

4. SPECIAL CASES

Taking some special cases of the weight function w, in our results of the previous section, we obtain following estimates for the definite integral.

Example 1. *Let us assume that $w(t) = 1$, $t \in [a,b]$.*
Now, from Theorem C, we calculate

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) = \frac{(t-a)^n}{n!}, & t \in [a,x] \\ w_{2n}(t) = \frac{(t-b)^n}{n!}, & t \in (x,b] \end{cases}$$

and

$$A_j(x) = \frac{1}{j!}\left[(b-x)^j - (a-x)^j\right].$$

In order to apply new estimates from Theorem 3 to the function $w(t) = 1$, $t \in [a,b]$, we will replace n, in the definition of the $W_{n,w}$, by $2n$ to provide the nonnegativity of $W_{n,w}$ and we will assume that $g : [a,b] \to \mathbb{R}$ is $(2n+2)$-convex since then $g^{(2n)}$ is also convex function. Now, according to (14)*, we get*

$$\begin{aligned} & P(b) \cdot g^{(2n)}(\lambda) \\ & \le \int_a^b g(t)\, dt - \sum_{j=1}^{2n} \frac{g^{(j-1)}(x)}{j!}\left[(b-x)^j - (a-x)^j\right] \\ & \le P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda-a}{b-a} g^{(2n)}(b)\right], \end{aligned}$$

where

$$P(b) = \frac{b^{2n+1} - a^{2n+1}}{(2n+1)!} - \sum_{j=1}^{2n} \frac{x^{2n-j+1}}{j!(2n-j+1)!} \cdot \left((b-x)^j - (a-x)^j\right) \tag{17}$$

and

$$\lambda = \frac{1}{P(b)}\left[\frac{b^{2n+2} - a^{2n+2}}{(2n+2)!} - \sum_{j=1}^{2n} \frac{x^{2n-j+2}}{j!(2n-j+2)!} \cdot \left((b-x)^j - (a-x)^j\right)\right]. \tag{18}$$

Values of $P(b)$ and λ for the function $w(t) = 1$, $t \in [a,b]$, can also be calculated using the results of Theorem 4 as follows.

$$\begin{aligned} P(b) &= w_{1,2n+1}(x) - w_{2,2n+1}(x) \\ &= \frac{1}{(2n+1)!}\left[(x-a)^{2n+1} - (x-b)^{2n+1}\right]. \end{aligned}$$

$$\begin{aligned} \lambda &= x - \frac{1}{P(b)}\left[w_{1,2n+2}(x) - w_{2,2n+2}(x)\right] \\ &= x - \frac{1}{(2n+2)!P(b)}\left[(x-a)^{2n+2} - (x-b)^{2n+2}\right]. \end{aligned}$$

If the assumptions of Theorem 5 hold, for $w(t) = 1$, $t \in [a, b]$, we get

$$\begin{aligned} P(b) \cdot g^{(2n)}(\lambda) &\leq \frac{g^{(2n)}(\nu)}{(2n+1)!}\left[(x-a)^{2n+1} - (x-b)^{2n+1}\right] \\ &\leq P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda - a}{b-a} g^{(2n)}(b)\right], \end{aligned}$$

where $P(b)$ and λ have the same values as in (17) *and* (18) *respectively.*

Example 2. *Suppose that $w(t) = (b-t)^{\alpha} \cdot (t-a)^{\beta}$, $t \in [a, b]$, $\alpha, \beta > -1$. From Theorem C, taking substitution $x = \frac{b-s}{b-a}$ in the definition of Beta function, we get*

$$W_{n,w}(t,x) = \begin{cases} \frac{(b-a)^{\alpha}(t-a)^{n+\beta}}{(n-1)!} B(\beta+1, n) \\ \quad \cdot F\left(-\alpha, \beta+1, \beta+n+1; \frac{t-a}{b-a}\right), & t \in [a, x] \\ (-1)^n \frac{(b-a)^{\beta}(b-t)^{n+\alpha}}{(n-1)!} B(\alpha+1, n) \\ \quad \cdot F\left(-\beta, \alpha+1, \alpha+n+1; \frac{b-t}{b-a}\right), & t \in (x, b] \end{cases}$$

and

$$A_j(x) = \begin{cases} \frac{(a-x)^{j-1}(b-a)^{\alpha+\beta+1}}{(j-1)!} B(\alpha+1, \beta+1) \\ \quad \cdot F\left(1-j, \beta+1, \alpha+\beta+2; \frac{b-a}{x-a}\right), & x \neq a, \\ \frac{(b-a)^{\alpha+\beta+j}}{(j-1)!} B(\alpha+1, \beta+j), & x = a, \end{cases}$$

where

$$B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1}\, dx$$

is the Beta function and

$$F(\alpha, \beta, \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\, dt$$

is the hypergeometric function for $\gamma > \beta > 0$ *and* $z < 1$.

If the assumptions of Theorem 3 hold, according to (14), *we get*

$$\begin{aligned} &(-1)^n P(b) \cdot g^{(n)}(\lambda) \\ &\leq \int_a^b (b-t)^{\alpha} \cdot (t-a)^{\beta} g(t)\, dt - \sum_{j=1}^{n} A_j(x) \cdot \frac{g^{(j-1)}(x)}{j!} \\ &\leq (-1)^n P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda - a}{b-a} g^{(n)}(b) \right], \end{aligned}$$

where

$$\begin{aligned} P(b) &= (-1)^n \cdot \left[\frac{(b-a)^{\alpha+\beta+1}}{n!} B(\alpha+1, \beta+1) \right. \\ &\quad \left. \cdot F\left(-n, \alpha+1, \alpha+\beta+2; \frac{(1-t)(b-a)}{b-t}\right) - \sum_{j=1}^{n} \frac{x^{n-j+1}}{(n-j+1)!} \cdot A_j(x) \right] \end{aligned}$$

and

$$\begin{aligned} \lambda &= \frac{(-1)^n}{P(b)} \cdot \left[\frac{(b-a)^{\alpha+\beta+1}}{(n+2)!} B(\alpha+1, \beta+1) \right. \\ &\quad \left. \cdot F\left(-n-1, \alpha+1, \alpha+\beta+2; \frac{(1-t)(b-a)}{b-t}\right) - \sum_{j=1}^{n} \frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x) \right]. \end{aligned}$$

Using the same integral calculations similar results can be obtained under the conditions of Theorem 4 and Theorem 5.

REFERENCES

1. A. Aglić Aljinović, A. Čivljak, S. Kovač, J. Pečarić, M. Ribičić Penava: *General Integral Identities and Related Inequalities.* Element, Zagreb, 2013.
2. I. Franjić, J. Pečarić: *Hermite-Hadamard type inequalities for higher order convex functions and various quadrature rules.* Acta Mathematica Vietnamica, **37 (2012), 109–120.**
3. S. Kovač, J. Pečarić: *Weighted version of general integral formula of Euler type.* Mathematical Inequalities and Applications, **13** (2010), 579–599.
4. J. Pečarić, I. Perić: *Refinements of the integral form of Jensen's and the Lah-Ribarič inequalities and applications for Csiszar divergence.* J. Inequal. Appl. **2020** (2020).
5. J. E. Pečarić, F. Proschan, Y. L. Tong: *Convex Functions, Partial Orderings and Statistical Applications.* Academic Press, Inc., San Diego, 1992.

6. J. Pečarić, S. Varošanec: *Harmonic Polynomials and Generalization of Ostrowski Inequality with Applications in Numerical Integration.* Nonlinear Analysis Series A: Theory, Methods and Applications, **47** (2001), 2365–2374.

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