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COMPLETE ASYMPTOTIC EXPANSIONS RELATED TO THE PROBABILITY DENSITY FUNCTION OF THE χ^2-DISTRIBUTION

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In this paper, we consider the function $f_p(t) = \sqrt{2p}\chi^2(\sqrt{2p}t + p; p)$, where $\chi^2(x; n)$ defined by $\chi^2(x; p) = \frac{2^{-p/2}}{\Gamma(p/2)}\, e^{-x/2} x^{p/2-1}$, is the density function of a χ^2-distribution with n degrees of freedom. The asymptotic expansion of $f_p(t)$ for $p \to \infty$, where p is not necessarily an integer, is obtained by an application of the standard asymptotics of $\ln\Gamma(x)$. Two different methods of obtaining the coefficients in the asymptotic expansion are presented, which involve the use of the Bell polynomials.

1. INTRODUCTION AND MOTIVATION

The density function of the χ^2-distribution with p degrees of freedom is given by

$$\chi^2(x;p) = \begin{cases} \dfrac{1}{2^{p/2}\Gamma(p/2)}\, e^{-\frac{x}{2}} x^{\frac{p}{2}-1} & (x > 0) \\ 0 & (x \leq 0). \end{cases}$$

Most often, the values of the parameter p are assumed to be positive integers. Here, we consider the probability density function $\chi^2(x; p)$ for all real $p > 0$.

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The density function $\chi^2(x;p)$ is asymmetrical. It is well known that

$$E(\chi^2) = p \qquad \text{and} \qquad \mathrm{Var}(\chi^2) = 2p.$$

By means of the following change of variable:

$$t = \frac{x-p}{\sqrt{2p}},$$

Chen and Wang [**4**] showed that

$$P\{\chi^2 < x\} = \int_{-\infty}^{x} \chi^2(x;p)\mathrm{d}x = \int_{-\infty}^{t} f_p(t)\mathrm{d}t,$$

where

$$f_p(t) = \sqrt{2p}\,\chi^2(\sqrt{2p}\,t + p; p),$$

that is,

$$f_p(t) = \begin{cases} \dfrac{\sqrt{2p}}{2^{p/2}\Gamma(p/2)}\ \exp\left(-\frac{\sqrt{2p}t+p}{2}\right)\left(\sqrt{2p}t+p\right)^{\frac{p}{2}-1} & \left(t > -\sqrt{\dfrac{p}{2}}\right) \\ 0 & \left(t \leq -\sqrt{\dfrac{p}{2}}\right). \end{cases} \tag{1}$$

Moreover, Chen and Wang [**4**] presented the following asymptotic formula in terms of $\frac{1}{n}$ (where $n \in \mathbb{N} := \{1,2,3,\cdots\}$):

$$\begin{aligned} f_n(t) = \varphi(t)\Bigg\{ & 1 + \sqrt{2}\left(\frac{1}{3}\,t^3 - t\right)\frac{1}{n^{1/2}} + \left(\frac{1}{9}\,t^6 - \frac{7}{6}\,t^4 + 2t^2 - \frac{1}{6}\right)\frac{1}{n} \\ & + \sqrt{2}\left(\frac{1}{81}\,t^9 - \frac{5}{18}\,t^7 + \frac{47}{30}\,t^5 - \frac{37}{18}\,t^3 + \frac{1}{6}\,t\right)\frac{1}{n^{3/2}} \\ & + \left(\frac{1}{486}\,t^{12} - \frac{13}{162}\,t^{10} + \frac{314}{360}\,t^8 - \frac{1031}{270}\,t^6 + \frac{151}{36}\,t^4 - \frac{1}{3}\,t^2 + \frac{1}{72}\right)\frac{1}{n^2} \\ & + O\left(\frac{1}{n^{5/2}}\right)\Bigg\} \end{aligned} \tag{2}$$

as $n \to \infty$, which leads us to the known result

$$\lim_{n\to\infty} f_n(t) = \varphi(t), \tag{3}$$

where

$$\varphi(t) = \frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}$$

is the probability density function of the standard normal distribution.

This paper is essentially a sequel to the earlier work [**3**]. We here establish complete asymptotic expansions for the probability density function:

$$f_p(t) = \sqrt{2p}\chi^2(\sqrt{2p}t + p; p)$$

as $p \to \infty$. More precisely, we prove an explicit formula for determining the coefficients $a_j \equiv a_j(t)$ $(j \in \mathbb{N})$ (see Theorem 1) such that

$$f_p(t) \sim \varphi(t) \exp\left(\sum_{j=1}^{\infty} \frac{a_j}{p^{j/2}}\right) \qquad (p \to \infty).$$

Based upon the obtained result, we give a recurrence relation (see Theorem 2) and an explicit formula (see Theorem 3) for determining the coefficients $b_j \equiv b_j(t)$ $(j \in \mathbb{N})$ such that

$$f_p(t) \sim \varphi(t) \left(1 + \sum_{j=1}^{\infty} \frac{b_j}{p^{j/2}}\right) \qquad (p \to \infty), \tag{4}$$

which further develops the Chen-Wang result (2) in order to produce a complete asymptotic expansion.

2. THE ASYMPTOTIC EXPANSION OF $f_p(t)$

In this section, we first state and prove Theorem 1 below.

Theorem 1. *The density function $f_p(t)$, defined by* (1), *has the following asymptotic expansion*:

$$f_p(t) \sim \varphi(t) \exp\left(\sum_{j=1}^{\infty} \frac{a_j}{p^{j/2}}\right) \qquad (p \to \infty), \tag{5}$$

with the coefficients $a_j \equiv a_j(t)$ $(j \in \mathbb{N})$ given by

$$a_{2j-1} = \frac{(\sqrt{2}t)^{2j+1}}{2(2j+1)} - \frac{(\sqrt{2}t)^{2j-1}}{2j-1}, \ a_{2j} = -\frac{(-1)^{j+1}2^j B_{j+1}}{j(j+1)} - \frac{(\sqrt{2}t)^{2j+2}}{2(2j+2)} + \frac{(\sqrt{2}\,t)^{2j}}{2j}, \tag{6}$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are the Bernoulli numbers defined by the following generating function:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi). \tag{7}$$

Proof. The logarithm of the (Euler's) gamma function $\Gamma(x)$ has the following asymptotic expansion (see, for example, [**8**, p. 32]; see also [**12**, p. 24]):

$$\text{(8)}\quad \ln\Gamma(x) \sim \left(x-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}}{n(n+1)}\,\frac{1}{x^n} \qquad (x\to\infty),$$

where B_n denotes the Bernoulli numbers defined by the generating function: (7). Using (8) and the Taylor-Maclaurin expansion of $\ln(1+t)$:

$$\ln(1+t) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\,t^j \qquad (-1<t\le 1),$$

we obtain

$$\begin{aligned}
\ln\big(f_p(t)\big) &= \frac{1}{2}\ln(2p) - \frac{1+\ln 2}{2}p - \frac{t}{\sqrt{2}}\sqrt{p} - \ln\Gamma\left(\frac{p}{2}\right)\\
&\quad + \left(\frac{p}{2}-1\right)\ln p + \left(\frac{p}{2}-1\right)\ln\left(1+\frac{\sqrt{2}t}{\sqrt{p}}\right)\\
&\sim \frac{1}{2}\ln(2p) - \frac{1+\ln 2}{2}p - \frac{t}{\sqrt{2}}\sqrt{p}\\
&\quad - \left[\left(\frac{p}{2}-\frac{1}{2}\right)\ln\left(\frac{p}{2}\right) - \frac{p}{2} + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}}{j(j+1)}\,\frac{2^j}{p^j}\right]\\
&\quad + \left(\frac{p}{2}-1\right)\ln p + \left(\frac{p}{2}-1\right)\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\left(\frac{\sqrt{2}t}{\sqrt{p}}\right)^j
\end{aligned}$$

as $p\to\infty$. After some elementary transformations, we find that

$$\begin{aligned}
\ln\big(f_p(t)\big) &\sim -\ln(\sqrt{2\pi}) - \frac{t^2}{2} - \sum_{j=1}^{\infty}\frac{(-1)^{j+1}2^jB_{j+1}}{j(j+1)p^j}\\
&\quad + \sum_{j=1}^{\infty}(-1)^{j-1}\left(\frac{(\sqrt{2}t)^{j+2}}{2(j+2)} - \frac{(\sqrt{2}\,t)^j}{j}\right)\frac{1}{p^{j/2}}
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
\ln\big(f_p(t)\big) &\sim -\ln(\sqrt{2\pi}) - \frac{t^2}{2} - \sum_{j=1}^{\infty}\left(\frac{(-1)^{j+1}2^jB_{j+1}}{j(j+1)} + \frac{(\sqrt{2}t)^{2j+2}}{2(2j+2)} - \frac{(\sqrt{2}t)^{2j}}{2j}\right)\frac{1}{p^j}\\
&\quad + \sum_{j=1}^{\infty}\left(\frac{(\sqrt{2}\,t)^{2j+1}}{2(2j+1)} - \frac{(\sqrt{2}t)^{2j-1}}{2j-1}\right)\frac{1}{p^{(2j-1)/2}},
\end{aligned}$$

which can be written as follows

$$\ln\big(f_p(t)\big) \sim -\ln(\sqrt{2\pi}) - \frac{t^2}{2} + \sum_{j=1}^{\infty} \frac{a_j}{p^{j/2}}, \tag{9}$$

where the coefficients a_j $(j \in \mathbb{N})$ are given by

$$a_{2j-1} = \frac{(\sqrt{2}t)^{2j+1}}{2(2j+1)} - \frac{(\sqrt{2}t)^{2j-1}}{2j-1}, \; a_{2j} = -\frac{(-1)^{j+1}2^j B_{j+1}}{j(j+1)} - \frac{(\sqrt{2}t)^{2j+2}}{2(2j+2)} + \frac{(\sqrt{2}t)^{2j}}{2j}.$$

Clearly, (9) can be written as (5). The proof of Theorem 1 is now completed. □

We find from (6) that the first few coefficients a_j are given by

$$a_1 = \sqrt{2}\left(\frac{1}{3}\,t^3 - t\right), \; a_2 = -\frac{1}{2}\,t^4 + t^2 - \frac{1}{6}, \; a_3 = 2\sqrt{2}\left(\frac{1}{5}\,t^5 - \frac{1}{3}\,t^3\right), \; a_4 = -\frac{2}{3}\,t^6 + t^4. \tag{10}$$

We thus obtain the following explicit asymptotic expansion:

$$\begin{aligned} f_p(t) \sim \varphi(t)\exp\Bigg\{ & \sqrt{2}\left(\frac{1}{3}\,t^3 - t\right)\frac{1}{p^{1/2}} + \left(-\frac{1}{2}\,t^4 + t^2 - \frac{1}{6}\right)\frac{1}{p} \\ & + 2\sqrt{2}\left(\frac{1}{5}\,t^5 - \frac{1}{3}\,t^3\right)\frac{1}{p^{3/2}} + \left(-\frac{2}{3}\,t^6 + t^4\right)\frac{1}{p^2} + \cdots \Bigg\} \quad (p \to \infty). \end{aligned}$$

Theorem 2 below gives a recurrence relation for determining the coefficients b_j in (4), based upon the Bell polynomials. The Bell polynomials, named in honor of Eric Temple Bell (1883-1960), are a triangular array of polynomials given by (see, for example, Comtet [**5**, pp. 133–134] and Cvijović [**6**])

$$\begin{aligned} & B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) \\ & \quad = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \end{aligned}$$

where the sum is taken over all sequences $j_1, j_2, j_3, \cdots, j_{n-k+1}$ of non-negative integers such that

$$j_1 + j_2 + \cdots + j_{n-k+1} = k \qquad \text{and} \qquad j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

The following sum:

$$B_n(x_1, x_2, x_3, \cdots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, x_3, \cdots, x_{n-k+1})$$

is sometimes called the nth complete Bell polynomial. The complete Bell polynomials satisfy the following identity:

$$B_n(x_1, x_2, x_3, \cdots, x_n) \\ = \begin{vmatrix} x_1 & \binom{n-1}{1}x_2 & \binom{n-1}{2}x_3 & \binom{n-1}{3}x_4 & \cdots & \cdots & x_n \\ -1 & x_1 & \binom{n-2}{1}x_2 & \binom{n-2}{2}x_3 & \cdots & \cdots & x_{n-1} \\ 0 & -1 & x_1 & \binom{n-3}{1}x_2 & \cdots & \cdots & x_{n-2} \\ 0 & 0 & -1 & x_1 & \cdots & \cdots & x_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \cdots & x_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & x_1 \end{vmatrix}. \tag{11}$$

In order to contrast them with complete Bell polynomials, the polynomials $B_{n,k}$ defined above are sometimes called partial Bell polynomials. The complete Bell polynomials appear in the exponential of a formal power series:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n!} u^n\right) = \sum_{n=0}^{\infty} \frac{B_n(x_1, \cdots, x_n)}{n!} u^n. \tag{12}$$

The Bell polynomials are quite general polynomials and they have been found in many applications in combinatorics. In his monograph, Comtet [**5**] devoted much to a thorough presentation of the Bell polynomials in the chapter on identities and expansions. For more results, see the works by Charalambides [**2**, Chapter 11] and Riordan [**10**, Chapter 5].

We now state and prove our second main result as Theorem 2.

Theorem 2. *The density function $f_p(t)$, defined by* (1), *has the following asymptotic expansion*:

$$f_p(t) \sim \varphi(t)\left(1 + \sum_{j=1}^{\infty} \frac{b_j}{p^{j/2}}\right) \qquad (p \to \infty), \tag{13}$$

with the coefficients $b_j \equiv b_j(t)$ $(j \in \mathbb{N})$ given by

$$b_0 = 1, \quad b_j = \sum_{\ell=0}^{j-1} \frac{j-\ell}{j} a_{j-\ell} b_\ell \qquad (j \in \mathbb{N}), \tag{14}$$

where a_j $(j \in \mathbb{N})$ are given in (6).

Proof. Replacing p by p^2 in (5), we get

$$f_{p^2}(t) \sim \varphi(t) \exp\left(\sum_{j=1}^{\infty} \frac{a_j}{p^j}\right) \qquad (p \to \infty). \tag{15}$$

By using (15) and (12), we find that

$$\frac{f_{p^2}(t)}{\varphi(t)} \sim \exp\left(\sum_{n=1}^{\infty} \frac{n! a_n}{n!} \frac{1}{p^n}\right) = \sum_{n=0}^{\infty} \frac{b_n}{p^n},$$

where

$$b_n = \frac{B_n\Big(1!\, a_1, 2!\, a_2, \cdots, n!\, a_n\Big)}{n!}. \tag{16}$$

Bulò *et al.* [**1**, Theorem 1] proved that the complete Bell polynomials can be expressed using the following recursive formula:

$$B_n(x_1, x_2, \cdots, x_n) = \begin{cases} \sum\limits_{\ell=0}^{n-1} \binom{n-1}{\ell} x_{n-\ell} B_\ell(x_1, x_2, \cdots, x_\ell) & (n > 0) \\ \\ 1 & (\text{otherwise}). \end{cases}$$

Thus, clearly, the formula (16) can be rewritten as follows:

$$\begin{aligned} b_0 &= 1 \quad \text{and} \\ b_n &= \frac{1}{n!} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (n-\ell)! a_{n-\ell} B_\ell\Big(1!\, a_1, 2!\, a_2 \cdots, \ell!\, a_\ell\Big) \\ &= \frac{1}{n!} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (n-\ell)! a_{n-\ell} \ell! b_\ell \\ &= \sum_{\ell=0}^{n-1} \frac{n-\ell}{n}\, a_{n-\ell} b_\ell \qquad (n \in \mathbb{N}). \end{aligned}$$

The proof of Theorem 2 is thus completed. □

We now give explicit numerical values of the first few b_j by using the recurrence relation (14). Noting that (10) holds true, we have

$$
\begin{aligned}
b_0 &= 1,\\
b_1 &= a_1 b_0 = \sqrt{2}\left(\frac{1}{3}\,t^3 - t\right),\\
b_2 &= \frac{1}{2} a_1 b_1 + a_2 b_0 = \frac{1}{9}\,t^6 - \frac{7}{6}\,t^4 + 2t^2 - \frac{1}{6},\\
b_3 &= \frac{1}{3}\,a_1 b_2 + \frac{2}{3}\,a_2 b_1 + a_3 b_0 = \sqrt{2}\left(\frac{1}{81}\,t^9 - \frac{5}{18}\,t^7 + \frac{47}{30}\,t^5 - \frac{37}{18}\,t^3 + \frac{1}{6}\,t\right),\\
b_4 &= a_4 b_0 + \frac{3}{4}\,a_3 b_1 + \frac{1}{2}\,a_2 b_2 + \frac{1}{4}\,a_1 b_3\\
&= \frac{1}{486}\,t^{12} - \frac{13}{162}\,t^{10} + \frac{314}{360}\,t^8 - \frac{1031}{270}\,t^6 + \frac{151}{36}\,t^4 - \frac{1}{3}\,t^2 + \frac{1}{72}.
\end{aligned}
$$

Thus, in the limit as $p \to \infty$, we obtain

$$
\begin{aligned}
f_p(t) \sim \varphi(t)\Bigg\{ & 1 + \sqrt{2}\left(\frac{1}{3}\,t^3 - t\right)\frac{1}{p^{1/2}} + \left(\frac{1}{9}\,t^6 - \frac{7}{6}\,t^4 + 2t^2 - \frac{1}{6}\right)\frac{1}{p}\\
& + \sqrt{2}\left(\frac{1}{81}\,t^9 - \frac{5}{18}\,t^7 + \frac{47}{30}\,t^5 - \frac{37}{18}\,t^3 + \frac{1}{6}\,t\right)\frac{1}{p^{3/2}}\\
& + \left(\frac{1}{486}\,t^{12} - \frac{13}{162}\,t^{10} + \frac{314}{360}\,t^8 - \frac{1031}{270}\,t^6 + \frac{151}{36}\,t^4 - \frac{1}{3}\,t^2 + \frac{1}{72}\right)\frac{1}{p^2} + \cdots\Bigg\},
\end{aligned} \tag{17}
$$

which obviously develops the Chen-Wang result (2) in order to produce a complete asymptotic expansion.

Remark 1. We can calculate the coefficients b_j in (13) by using the formulas (16) and (11), namely,

$$
b_n = \frac{1}{n!}\begin{vmatrix}
1!\,a_1 & \binom{n-1}{1}2!\,a_2 & \binom{n-1}{2}3!\,a_3 & \binom{n-1}{3}4!\,a_4 & \cdots & \cdots & n!\,a_n\\
-1 & 1!\,a_1 & \binom{n-2}{1}2!\,a_2 & \binom{n-2}{2}3!\,a_3 & \cdots & \cdots & (n-1)!\,a_{n-1}\\
0 & -1 & 1!\,a_1 & \binom{n-3}{1}2!\,a_2 & \cdots & \cdots & (n-2)!\,a_{n-2}\\
0 & 0 & -1 & 1!\,a_1 & \cdots & \cdots & (n-3)!\,a_{n-3}\\
0 & 0 & 0 & -1 & \cdots & \cdots & (n-4)!\,a_{n-4}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & -1 & 1!\,a_1
\end{vmatrix}. \tag{18}
$$

3. AN ALTERNATIVE REPRESENTATION FOR THE COEFFICIENTS b_j

By mainly using the partition function, we can provide an alternative representation formula for calculating the coefficients $b_j \equiv b_j(x)$ $(j \in \mathbb{N})$ in (13) as in Theorem 2. In order to do this, we introduce the following set of partitions of an integer $n \in \mathbb{N}$:

$$\mathcal{A}_n := \{(k_1, k_2, \cdots, k_n) \in \mathbb{N}_0^n : k_1 + 2k_2 + \cdots + nk_n = n\}. \tag{19}$$

In Number Theory, the partition function $p(n)$ represents the number of possible partitions of $n \in \mathbb{N}$, that is, the number of distinct ways of representing n as a sum of natural numbers (with their order being irrelevant). By convention, $p(0) = 1$ and $p(-n) = 0$ $(n \in \mathbb{N})$. Beginning with $p(0) = 1$, the first several values of the partition function $p(n)$ given by $p(0) = 1$):

$$1,\ 1,\ 2,\ 3,\ 5,\ 7,\ 11,\ 15,\ 22,\ 30,\ 42,\ \cdots.$$

It is easy to see that the cardinality of the set $\mathcal{A}_n$ is equal to the partition function $p(n)$.

Theorem 3. *The coefficients $b_j \equiv b_j(t)$ $(j \in \mathbb{N})$ in* (13) *can be calculated by using the following formula*:

$$b_j = \sum_{(k_1,k_2,\cdots,k_j)\in\mathcal{A}_j} \frac{a_1^{k_1}\, a_2^{k_2}\, \cdots\, a_j^{k_j}}{k_1!\, k_2!\, \cdots\, k_j!}, \tag{20}$$

where $\mathcal{A}_j$ are given in (19) *and a_j $(j \in \mathbb{N})$ are given in* (6).

Proof. In view of (13), we can let

$$\ln\left(\frac{f_{p^2}(t)}{\varphi(t)}\right) = \ln\left(1 + \sum_{j=1}^{m} \frac{b_j}{p^j}\right) + O\left(\frac{1}{p^{m+1}}\right) \qquad (p \to \infty),$$

where $b_1, \cdots, b_m$ are real numbers to be determined. Then, by using the Fundamental Theorem of Algebra, we see that there exist unique complex numbers $\lambda_1, \cdots, \lambda_m$ such that

$$1 + \frac{b_1}{p} + \cdots + \frac{b_m}{p^m} = \left(1 + \frac{\lambda_1}{p}\right)\cdots\left(1 + \frac{\lambda_m}{p}\right). \tag{21}$$

Also, by applying the following series expansion:

$$\ln\left(1 + \frac{z}{p}\right) = \sum_{j=1}^{m} \frac{(-1)^{j-1} z^j}{j p^j} + O\left(\frac{1}{p^{m+1}}\right)$$

for $|z| < |p|$ and $p \to \infty$, we obtain

$$\text{(22)} \qquad \ln\left(1 + \frac{b_1}{p} + \cdots + \frac{b_m}{p^m}\right) = \sum_{j=1}^{m} \frac{(-1)^{j-1} S_j}{j p^j} + O\left(\frac{1}{p^{m+1}}\right) \qquad (p \to \infty),$$

where

$$S_j = \lambda_1^j + \cdots + \lambda_m^j \qquad (j = 1, 2, \cdots, m).$$

It follows from (15) that

$$\text{(23)} \qquad \ln\left(\frac{f_{p^2}(t)}{\varphi(t)}\right) = \sum_{j=1}^{m} \frac{a_j}{p^j} + O\left(\frac{1}{p^{m+1}}\right) \qquad (p \to \infty).$$

From (22) and (23), we obtain

$$\text{(24)} \qquad S_j = (-1)^{j-1} j a_j \qquad (j = 1, 2, \cdots, m),$$

that is,

$$\text{(25)} \qquad \begin{cases} \lambda_1 + \cdots + \lambda_m = S_1, \\ \lambda_1^2 + \cdots + \lambda_m^2 = S_2, \\ \qquad\quad \vdots \\ \qquad\quad \vdots \\ \lambda_1^m + \cdots + \lambda_m^m = S_m. \end{cases}$$

We now let

$$P_m(p) = p^m + c_1 p^{m-1} + \cdots + c_{m-1} p + c_m$$

be a polynomial with zeros $\lambda_1, \cdots, \lambda_m$, which satisfy the system of equations (25). So we have

$$\text{(26)} \qquad P_m(p) = (p - \lambda_1) \cdots (p - \lambda_m).$$

Then the Newton formulas (see, for example, [**7**]) give the following connection between the coefficients c_j and the power sums S_j:

$$S_j + S_{j-1} c_1 + S_{j-2} c_2 + \cdots + S_1 c_{j-1} + j c_j = 0 \quad \text{for} \quad j = 1, \cdots, m.$$

It is known (see [**7**]) that c_j can be expressed in terms of S_j as follows:

$$c_j = \sum_{(k_1, k_2, \cdots, k_j) \in \mathcal{A}_j} \frac{(-1)^{k_1 + k_2 + \cdots + k_j}}{k_1! k_2! \cdots k_j!} \left(\frac{S_1}{1}\right)^{k_1} \left(\frac{S_2}{2}\right)^{k_2} \cdots \left(\frac{S_j}{j}\right)^{k_j},$$

where the $\mathcal{A}_j$ ($j \in \mathbb{N}$) are given in (19).

From (26), we obtain

$$\frac{(-1)^m}{p^m} P_m(-p) = \left(1 + \frac{\lambda_1}{p}\right) \cdots \left(1 + \frac{\lambda_m}{p}\right).$$

We thus have

$$1 + \frac{(-1)c_1}{p} + \frac{(-1)^2 c_2}{p^2} + \cdots + \frac{(-1)^m c_m}{p^m} = \left(1 + \frac{\lambda_1}{p}\right) \cdots \left(1 + \frac{\lambda_m}{p}\right). \tag{27}$$

Thus, in light of (21) and (27), the coefficients b_j are given by

$$\begin{aligned} b_j &= (-1)^j c_j \\ &= (-1)^j \sum_{(k_1,\, k_2,\, \cdots,\, k_j) \in \mathcal{A}_j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1! k_2! \cdots k_j!} \left(\frac{S_1}{1}\right)^{k_1} \left(\frac{S_2}{2}\right)^{k_2} \cdots \left(\frac{S_j}{j}\right)^{k_j} \\ &= \sum_{(k_1, k_2, \cdots, k_j) \in \mathcal{A}_j} \frac{a_1^{k_1}\, a_2^{k_2}\, \cdots\, a_j^{k_j}}{k_1!\, k_2!\, \cdots\, k_j!}, \end{aligned}$$

where the S_j are specified in (24). The proof of Theorem 3 is now completed. □

We next give explicit numerical values of the first few b_j by using the partition set (19) and the formula (20). Noting that (10) holds true, we find that

$$b_1 = \sum_{k_1=1} \frac{a_1^{k_1}}{k_1!} = a_1 = \sqrt{2}\left(\frac{1}{3}\, t^3 - t\right).$$

For $k_1 + 2k_2 = 2$, since $p(2) = 2$, the partition set $\mathcal{A}_2$ in (19) is seen to have 2 elements given by

$$\mathcal{A}_2 = \{(0, 1),\, (2, 0)\}.$$

From (20), we find that

$$b_2 = \sum_{(k_1, k_2) \in \mathcal{A}_2} \frac{a_1^{k_1}\, a_2^{k_2}}{k_1!\, k_2!} = \frac{a_1^0\, a_2^1}{0!\, 1!} + \frac{a_1^2\, a_2^0}{2!\, 0!} = \frac{1}{9}\, t^6 - \frac{7}{6}\, t^4 + 2t^2 - \frac{1}{6}.$$

For $k_1 + 2k_2 + 3k_3 = 3$, since $p(3) = 3$, as above, the partition set $\mathcal{A}_3$ in (19) contains 3 elements given by

$$\mathcal{A}_3 = \{(0, 0, 1),\, (1, 1, 0),\, (3, 0, 0)\}.$$

We then find from (20) that

$$\begin{aligned} b_3 &= \sum_{(k_1, k_2, k_3) \in \mathcal{A}_3} \frac{a_1^{k_1}\, a_2^{k_2}\, a_3^{k_3}}{k_1!\, k_2!\, k_3!} = \frac{a_1^0\, a_2^0\, a_3^1}{0!\, 0!\, 1!} + \frac{a_1^1\, a_2^1\, a_3^0}{1!\, 1!\, 0!} + \frac{a_1^3\, a_2^0\, a_3^0}{3!\, 0!\, 0!} \\ &= \sqrt{2}\left(\frac{1}{81}\, t^9 - \frac{5}{18}\, t^7 + \frac{47}{30}\, t^5 - \frac{37}{18}\, t^3 + \frac{1}{6}\, t\right). \end{aligned}$$

In a similar manner, the partition set $\mathcal{A}_4$ can be found to have $5 = p(4)$ elements given by

$$\mathcal{A}_4 = \{(0,0,0,1),\ (1,0,1,0),\ (0,2,0,0),\ (2,1,0,0),\ (4,0,0,0)\},$$

which yields

$$b_4 = \frac{1}{486}\,t^{12} - \frac{13}{162}\,t^{10} + \frac{314}{360}\,t^{8} - \frac{1031}{270}\,t^{6} + \frac{151}{36}\,t^{4} - \frac{1}{3}\,t^{2} + \frac{1}{72}.$$

We note that the values of b_j (for $j = 1, 2, 3, 4$) above are equal to the coefficients appearing in (17). The representation using a recursive algorithm for the coefficients b_j in (14) is more practical for numerical evaluation than the expressions in (18) and (20).

The standard normal distribution and χ^2-distribution are clearly among the most common distributions in statistics. The formula (3) is a textbook fact. By Scheffé's theorem [**11**], this implies the convergence of the total variation distance:

$$d_{TV}(p) = \frac{1}{2}\int_{-\infty}^{\infty} |f_p(x) - \varphi(x)|\mathrm{d}x \tag{28}$$

tends to 0 as $p \to \infty$ (see also the work by Pinelis [**9**]).

Remark 2. As an application of (17), we obtain

$$\begin{aligned} d_{TV}(p) &= \frac{1}{2}\int_{-\infty}^{\infty} |f_p(x) - \varphi(x)|\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \varphi(x)\left|\frac{\sqrt{2}\left(\frac{1}{3}x^3 - x\right)}{\sqrt{p}} + O\left(\frac{1}{p}\right)\right|\mathrm{d}x \longrightarrow 0 \end{aligned} \tag{29}$$

in the limit as $p \to \infty$.

4. CONCLUSION

For a χ^2-distribution $\chi^2(x; n)$, the probability density function $\chi^2(x; n)$ with n degrees of freedom is known to be asymmetrical. In our present investigation, by successfully applying several interesting recent developments by (for example) Chen and Wang [**4**], we have derived a potentially useful asymptotic formula in terms of $\frac{1}{n}$ as $n \to \infty$ as well as a complete asymptotic expansion for the corresponding probability density function. Our main results in this article are stated and proved as theorems (see Theorems 1, 2 and 3 of the preceding section). One of our results has been shown to lead to a known result for the probability density function of the standard normal distribution.

It is believed that our main results and their corollaries and consequences will prove to be worthy of motivating further developments in the study and analysis of probability distributions and probability generating functions.

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