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ON RAPIDLY VARYING SEQUENCES

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In this paper we investigate certain connections between the class $\mathsf{R}_{s,\infty}$ of rapidly varying sequences (in the sense of de Haan) and the rapid equivalence, selection principles and game theory.

1. INTRODUCTION

In 1930, J. Karamata [**8**] initiated investigation in asymptotic analysis of divergent processes, nowadays known as *Karamata's theory of regular variation.* In 1970, de Haan [**11**] defined and investigated rapid variation and so stimulated further development in asymptotic analysis. Two important objects in de Haan's theory of rapid variation are the class of rapidly varying functions and the class of rapidly varying sequences. The theory of regular and rapid variability has many applications in different branches of mathematics: probability theory, number theory, differential and difference equations, in particular in description of asymptotic properties of solutions of these equations, time scales theory, dynamic equations, q-calculus, and so on. The book [**1**] is a nice exposition of Karamata theory and the theory of rapid variability (see also [**12, 11**]).

We recall the definitions of rapidly varying functions and sequences.

Definition 1. ([**11, 1**]) A function $\varphi : [a,\infty) \to (0,\infty)$, $a > 0$, is said to be *rapidly varying of index of variability* ∞ if it is measurable and satisfies the asymptotic condition

$$\lim_{t\to\infty} \frac{\varphi(\lambda t)}{\varphi(t)} = \infty,\ \lambda > 1.$$

The class of rapidly varying functions of index of variability ∞ is denoted by $\mathsf{R}_{f,\infty}$.

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Definition 2. ([**2**]) A sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ of positive real numbers is *rapidly varying* (of index of variability ∞) if the following asymptotic condition is satisfied:

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = \infty,\ \ \lambda > 1,$$

where for a real number x, $[x]$ denotes the greatest integer part of x. $\mathsf{R}_{s,\infty}$ denotes the class of rapidly varying sequences (see [**2, 3, 4**]).

Throughout the paper $\mathbb{N}$ will denote the set of natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{S}$ the set of sequences of positive real numbers, and $\mathbb{S}_1$ the set of nondecreasing sequences from $\mathbb{S}$. Further, for two positive real functions g, h the symbol $\sim$ is used to denote the *strong asymptotic equivalence relation* defined by

$$g(x) \sim h(x), \quad x \to \infty, \quad \Leftrightarrow \quad \lim_{x\to\infty} \frac{g(x)}{h(x)} = 1,$$

while the symbol $\asymp$ denotes the *weak asymptotic equivalence relation* defined by

$$g(x) \asymp h(x), \quad x \to \infty, \quad \Leftrightarrow \quad 0 < \underline{\lim}_{x\to\infty} \frac{g(x)}{h(x)} \leq \overline{\lim}_{x\to\infty} \frac{g(x)}{h(x)} < \infty.$$

(see [**1**]).

In this paper we define and study a new equivalence relation in the class $\mathsf{R}_{s,\infty} \cap \mathbb{S}_1$, in particular its relations with selection principles and game theory.

2. RESULTS

We begin this section with definitions and concepts that we use in this article.

Real functions $g, h : [a, \infty) \to \mathbb{R}, \quad (a > 0)$, are *mutually asymptotically inverse*, denoted by

$$g(x) \overset{*}{\sim} h(x), \quad \text{as} \quad x \to \infty,$$

(see [**1, 6, 5**]), if for each $\lambda > 1$ there is an $x_0 = x_0(\lambda) \geq a$ such that the inequality

$$g\left(\frac{x}{\lambda}\right) \leq h(x) \leq g(\lambda x),$$

is satisfied for each $x \geq x_0$.

Especially, real functions (which are mutually asymptotically inverse) $g, h : [a, \infty) \to (0, \infty),\ (a > 0)$, are *mutually rapidly equivalent*, in denotation

$$g(x) \overset{r}{\sim} h(x), \quad \text{as} \quad x \to \infty,$$

(see [**7, 13**]) if

$$\lim_{x\to\infty} \frac{g(\lambda x)}{h(x)} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{h(\lambda x)}{g(x)} = \infty$$

hold for each $\lambda > 1$.

Sequences of positive real numbers $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are *mutually rapidly equivalent* in denotation

$$c_n \overset{r}{\sim} d_n, \text{ as } \; n \to \infty,$$

if

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{d_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{d_{[\lambda n]}}{c_n} = \infty$$

hold for each $\lambda > 1$.

Proposition 3. *Let sequences* $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *and* $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ *be elements from* $\mathbb{S}_1$. *If* $c_n \overset{r}{\sim} d_n$, *as* $n \to \infty$, *then* $\mathbf{c} \in \mathsf{R}_{s,\infty}$ *and* $\mathbf{d} \in \mathsf{R}_{s,\infty}$.

Proof. From $c_n \overset{r}{\sim} d_n$ as $n \to \infty$ it follows that for each $\lambda > 1$ we have

$$\begin{aligned}
\underline{\lim}_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} &\geq \underline{\lim}_{n\to\infty} \frac{c_{\left[\frac{\lambda n}{[\sqrt{\lambda} n]}\cdot\, [\sqrt{\lambda} n]\right]}}{d_{[\sqrt{\lambda} n]}} \cdot \underline{\lim}_{n\to\infty} \frac{d_{[\sqrt{\lambda} n]}}{c_n} \\
&\geq \underline{\lim}_{n\to+\infty} \frac{c_{\left[\frac{\lambda n}{\sqrt{\lambda} n}\cdot [\sqrt{\lambda} n]\right]}}{d_{[\sqrt{\lambda} n]}} \cdot (\infty) \\
&= \underline{\lim}_{n\to\infty} \frac{c_{[\sqrt{\lambda}[\sqrt{\lambda} n]]}}{d_{[\sqrt{\lambda} n]}} \cdot (\infty) = (\infty)\cdot(\infty) = \infty.
\end{aligned}$$

Therefore $\mathbf{c} \in \mathsf{R}_{s,\infty}$. Analogously we prove $\mathbf{d} \in \mathsf{R}_{s,\infty}$. □

Proposition 4. *Relation* $\overset{r}{\sim}$ *is an equivalence relation in* $\mathsf{R}_{s,\infty} \cap \mathbb{S}_1$.

Proof. 1. (Reflexivity) Let $\mathbf{c} \in \mathsf{R}_{s,\infty}$, then $\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = \infty$ holds, for each $\lambda > 1$, hence $c_n \overset{r}{\sim} c_n$, as $n \to \infty$, so that reflexivity holds.

2.(Symmetry) According to the definition of relation $\overset{r}{\sim}$, symmetry holds.

3. (Transitivity) Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$, $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ and $\mathbf{e} = (e_n)_{n\in\mathbb{N}}$ be elements from $\mathsf{R}_{s,\infty} \cap \mathbb{S}_1$ such that $c_n \overset{r}{\sim} d_n$, as $n \to \infty$, and $d_n \overset{r}{\sim} e_n$, as $n \to \infty$. Then for each $\lambda > 1$ we have

$$\begin{aligned}
\underline{\lim}_{n\to\infty} \frac{c_{[\lambda n]}}{e_n} &\geq \underline{\lim}_{n\to\infty} \frac{c_{[\lambda n]}}{d_{[\sqrt{\lambda} n]}} \cdot \underline{\lim}_{n\to\infty} \frac{d_{[\sqrt{\lambda} n]}}{e_n} \\
&\geq \underline{\lim}_{n\to\infty} \frac{c_{[\sqrt{\lambda}[\sqrt{\lambda} n]]}}{d_{[\sqrt{\lambda} n]}} \cdot (\infty) = (\infty)\cdot(\infty) = \infty.
\end{aligned}$$

Analogously we prove that $\underline{\lim}_{n\to\infty} \frac{e_{[\lambda n]}}{c_n} = \infty$ holds, hence $c_n \overset{r}{\sim} e_n$ holds, as $n \to \infty$. □

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ be element from $\mathbb{S}$. Then the sequences

(1) $\quad \overline{\mathbf{c}} = (\overline{c}_n)_{n\in\mathbb{N}}, \;\; \overline{c}_n = \max\{c_p | 1 \leq p \leq n\} \quad$ (*cumulative minimum*)

and

(2) $\underline{\mathbf{c}} = (\underline{c}_n)_{n\in\mathbb{N}},\ \ \underline{c}_n = \min\{c_p | p \geq n\}$ (*cumulative maximum*)

are called *upper* and *lower associate of the sequence* $\mathbf{c}$, respectively (see [**1**]).

Proposition 5. *Let* $\mathbf{c} \in \mathsf{R}_{s,\infty}$ *and* $c_n \asymp d_n$*, as* $n \to \infty$*. Then* $c_n \overset{r}{\sim} d_n$*, as* $n \to \infty$*.*

Proof. For each $\lambda > 1$ it holds

$$\underline{\lim}_{n\to\infty} \frac{c_{[\lambda n]}}{d_n} \geq \underline{\lim}_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} \cdot \underline{\lim}_{n\to\infty} \frac{c_n}{d_n} = \infty,$$

and

$$\underline{\lim}_{n\to\infty} \frac{d_{[\lambda n]}}{c_n} \geq \underline{\lim}_{n\to\infty} \frac{d_{[\lambda n]}}{c_{[\lambda n]}} \cdot \underline{\lim}_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = \infty.$$

This means that $c_n \overset{r}{\sim} d_n$, as $n \to \infty$. □

Proposition 6. *Let* $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$*. Then the sequence* $\mathbf{c} \in \mathsf{R}_{s,\infty}$ *if and only if* $\underline{c}_n \overset{r}{\sim} \overline{c}_n$ *holds, as* $n \to \infty$*.*

Proof. ($\Rightarrow$) If $\mathbf{c} \in \mathsf{R}_{s,\infty}$, then according to [**2**, Theorem 2.1], the function $\varphi(x) = c_{[x]}$, $x \geq 1$, belongs to the class $R_{f,\infty}$. It means that (according to [**7**, Theorem 1.1]) $\lim_{n\to\infty} \frac{\underline{\varphi}(\lambda x)}{\overline{\varphi}(x)} = \infty$, for each $\lambda > 1$, where the cumulative minimum $\underline{\varphi}$ and cumulative maximum $\overline{\varphi}$ are defined analogously with (1) and (2): $\underline{\varphi}(x) = \inf\{\varphi(t) : t \geq x\}$ and $\overline{\varphi}(x) = \sup\{\varphi(t) : t \leq x\}$. Thus, $\lim_{n\to\infty} \frac{\underline{c}_{[\lambda n]}}{\overline{c}_n} = \infty$ for each $\lambda > 1$. The inequality $\underline{c}_n \leq c_n \leq \overline{c}_n$, for $n \in \mathbb{N}$, implies that for each $\lambda > 1$ it holds

$$\underline{\lim}_{n\to\infty} \frac{\overline{c}_{[\lambda n]}}{\underline{c}_n} \geq \lim_{n\to\infty} \frac{\underline{c}_{[\lambda n]}}{\overline{c}_n} = \infty.$$

This means $\overline{c}_n \overset{r}{\sim} \underline{c}_n$, as $n \to \infty$.

($\Leftarrow$) It holds $\lim_{n\to\infty} \frac{\underline{c}_{[\lambda n]}}{\overline{c}_n} = \infty$, for each $\lambda > 1$, and thus

$$\underline{\lim}_{x\to\infty} \frac{\underline{c}_{[\lambda x]}}{\overline{c}_{[x]}} = \underline{\lim}_{x\to\infty} \frac{\underline{c}_{\left[\frac{\lambda x}{[x]}\cdot[x]\right]}}{\overline{c}_{[x]}} \geq \underline{\lim}_{x\to\infty} \frac{\underline{c}_{[\lambda[x]]}}{\overline{c}_{[x]}} = \infty$$

for each $\lambda > 1$, because the cumulative minimum is a nondecreasing function. Again, according to [**7**, Theorem 1.1] the function $\varphi(x) = c_{[x]}$, $x > 1$, belongs to the class $\mathsf{R}_{f,\infty}$ and hence the sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$, as its restriction on $\mathbb{N}$, is an element of the class $\mathsf{R}_{s,\infty}$. □

Proposition 7. *Let* $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$ *and let the sequence* $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ *be an element of the class* $\mathsf{R}_{s,\infty}$*. If* $\underline{d}_n \leq c_n \leq \overline{d}_n$*, for* $n \geq n_0 \geq 1$*, then* $\mathbf{c} \in \mathsf{R}_{s,\infty}$*.*

Proof. Let $c = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. As $\mathbf{d} = (d_n)_{n\in\mathbb{N}}$ is in the class $\mathsf{R}_{s,\infty}$, according to [**2**, Theorem 2.1], the function $\varphi(x) = d_{[x]}$, $x \geq 1$, belongs to the class $\mathsf{R}_{f,\infty}$. Therefore, according to [**7**, Theorem 1.1], for each $\lambda > 1$ we have

$$\underline{\lim}_{n\to\infty}\frac{c_{[\lambda n]}}{c_n} \geq \underline{\lim}_{n\to\infty}\frac{d_{[\lambda n]}}{d_n} \geq \lim_{x\to\infty}\frac{d_{[\lambda x]}}{d_{[x]}} = \infty.$$

It means that $\mathbf{c} \in \mathsf{R}_{s,\infty}$. □

Let $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. Then the sequence

$$\widetilde{\mathbf{c}} = (\widetilde{c}_n)_{n\in\mathbb{N}}, \;\; \widetilde{c}_n = \frac{1}{n}\sum_{k=1}^{n-1} c_k \tag{3}$$

is called the sequence of *additive midpoint of the sequence* $\mathbf{c}$ (see [**1**]).

Proposition 8. *Let* $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathsf{R}_{s,\infty}$*. Then* $\widetilde{\mathbf{c}} \in \mathsf{R}_{s,\infty}$ *and* $c_n \overset{r}{\sim} \widetilde{c}_n$*, as* $n \to \infty$*.*

Proof. Since $\mathbf{c} = (c_n)_{n\in\mathbb{N}} \in \mathsf{R}_{s,\infty}$, then, according to [**2**, Theorem 2.1], the function $\varphi(x) = c_{[x]}$, $x \geq 1$, belongs to the class $\mathsf{R}_{f,\infty}$. It follows that (see [**7**, p. 890])

$$\widetilde{\varphi}(x) = \frac{1}{x}\int_1^x \varphi(t)dt, \;\; x \geq 2,$$

belongs to the class $\mathsf{R}_{f,\infty}$. Consequently,

$$\widetilde{\varphi}(n) = \frac{1}{n}\int_1^n \varphi(t)dt = \frac{1}{n}\sum_{k=1}^{n-1} c_k, \;\; n \geq 2,$$

is an element of the class $\mathsf{R}_{s,\infty}$, and it holds that $\widetilde{\varphi}(n) = \widetilde{c}_n$. By [**7**, Theorem 1.3],

$$\widetilde{\varphi}(x) \overset{r}{\sim} \varphi(x), \;\text{ as } x \to \infty,$$

and thus

$$c_n = \varphi(n) \overset{r}{\sim} \widetilde{\varphi}(n) = \widetilde{c}_n, \;\text{ as } n \to \infty,$$

also holds. □

Let us state the definition of well known selection principles, which we call α_i selection principles (see [**9**]).

Definition 9. Let $\mathcal{A}$ and $\mathcal{B}$ be subfamilies of the set $\mathbb{S}$. The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i \in \{2, 3, 4\}$, denotes the following selection hypotheses: for each sequence $(A_n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

1. $\alpha_2(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n)\cap \mathrm{Im}(B)$ is infinite for each $n\in\mathbb{N}$;
2. $\alpha_3(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n)\cap \mathrm{Im}(B)$ is infinite for infinitely many $n\in\mathbb{N}$;
3. $\alpha_4(\mathcal{A},\mathcal{B})$: the set $\mathrm{Im}(A_n)\cap \mathrm{Im}(B)$ is nonempty for infinitely many $n\in\mathbb{N}$,

where Im denotes the image of the corresponding set.

Let us recall the definition of an infinitely long game related to α_2 (see [**9, 10**]).

Definition 10. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of the set $\mathbb{S}$. The symbol $\mathsf{G}_{\alpha_2}(\mathcal{A},\mathcal{B})$ denotes the following infinitely long game for two players, I and II, who play a round for each natural number n. In the first round I chooses an arbitrary element $(A_{1,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$, and II chooses a subsequence $y_{r_1}=(A_{1,r_1(j)})_{j\in\mathbb{N}}$ of the sequence A_1. At the k^{th} round, $k\geq 2$, I chooses an arbitrary element $A_k=(A_{k,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$ and II chooses a subsequence $y_{r_k}=(A_{k,r_k(j)})_{j\in\mathbb{N}}$ of the sequence A_k, such that $\mathrm{Im}(r_{k(j)})\cap \mathrm{Im}(r_{p(j)})=\emptyset$ is satisfied, for each $p\leq k-1$. II wins a play

$$A_1, y_{r_1};\ldots;A_k, y_{r_k};\ldots$$

if and only if all elements from $Y=\bigcup_{k\in\mathbb{N}}\bigcup_{j\in\mathbb{N}}A_k, r_{k(j)}$, with respect to second index, form a subsequence $\mathbf{y}=(y_m)_{m\in\mathbb{N}}\in\mathcal{B}$.

A strategy σ for the player II is a *coding strategy* if II remembers only the most recent move by I and by II before deciding how to play the next move.

Let $\mathbf{c}=(c_n)_{n\in\mathbb{N}}\in\mathbb{S}$. Then we define

$$[\mathbf{c}]_r=\{\mathbf{d}=(d_n)_{n\in\mathbb{N}}\in\mathbb{S}|\ c_n\overset{r}{\sim}d_n, n\to\infty\}\tag{4}$$

in $\mathsf{R}_{s,\infty}$, and for $\mathbf{c}=(c_n)_{n\in\mathbb{N}}\in\mathbb{S}_1$ we define

$$[\mathbf{c}]_r^{'}=\{\mathbf{d}=(d_n)_{n\in\mathbb{N}}\in\mathbb{S}_1|\ c_n\overset{r}{\sim}d_n, n\to\infty\}\tag{5}$$

as the equivalence class in $\mathsf{R}_{\infty,s}\cap\mathbb{S}_1$, with regard to Propositions 3 and 4.

Proposition 11. *The player II has a winning coding strategy in the game* $G_{\alpha_2}([c]'_r,$ $[c]_r)$*, for each fixed element* $\mathbf{c}\in\mathsf{R}_{s,\infty}\cap\mathbb{S}_1$*.*

Proof. (1^{st} round): Let the sequence $\mathbf{c}=(c_n)_{n\in\mathbb{N}}\in R_{s,\infty}\cap\mathbb{S}_1$ generating the class $[\mathbf{c}]_r^{'}\in\mathbb{S}_1$ be given and let σ be the strategy of the player II. The player I chooses a sequence $\mathbf{x_1}=(x_{1,n})_{n\in\mathbb{N}}\in[\mathbf{c}]_r^{'}$ arbitrary. Then the player II chooses the subsequence $\sigma(\mathbf{x_1})=(x_{1,k_1(n)})_{n\in\mathbb{N}}$ of the sequence $\mathbf{x_1}$, where $\mathrm{Im}(k_1)$ is the set of natural numbers greater then or equal to n_1 which are divisible by 2 and not divisible by 2^2, and n_1 is a natural number such that $\frac{c_{[\lambda n]}}{x_{1,n}}\geq 2$ and $\frac{x_{1,[\lambda n]}}{c_n}\geq 2$ hold for each $\lambda>1+\varepsilon$, $\varepsilon>0$, and each $n\geq n_1$. The last inequalities are possible because of monotonicity of sequences $\mathbf{c}$ and $\mathbf{x_1}$.

(m^{th} round, $m \geq 2$): The player I chooses the sequence $\mathbf{x_m} = (x_{m,n})_{m\in\mathbb{N}} \in [\mathbf{c}]_r^{'}$ arbitrary. Then the player II chooses the subsequence

$$\sigma(\mathbf{x_m}, (x_{m-1,k_{m-1}(n)})_{n\in\mathbb{N}}) = (x_{m,k_m(n)})_{n\in\mathbb{N}}$$

of the sequence $\mathbf{x_m}$, so that $\mathrm{Im}(k_m)$ is the set of natural numbers greater then or equal to n_m, which are divisible by 2^m and not divisible by 2^{m+1}, $n_m \in \mathbb{N}$, and $\frac{c_{[\lambda n]}}{x_{m,n}} \geq 2^m$ and $\frac{x_{m,[\lambda n]}}{c_n} \geq 2^m$ hold for each $\lambda > 1+\varepsilon$, $\varepsilon > 0$, and each $n \geq n_m$. The last inequalities hold because the sequences $(c_n)_{n\in\mathbb{N}}$ and $(x_{m,n})_{n\in\mathbb{N}}$ are nondecreasing.

Now, we will look at the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} x_{m,k_m(n)}$ of positive real numbers indexed by two indexes. This set we can consider as the subsequence of the sequence $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ given by:

$$y_i = \begin{cases} x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m,n \in \mathbb{N}; \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction of the sequence $\mathbf{y}$, we have that $\mathbf{y} \in \mathbb{S}$. Also, the intersection of $\mathbf{y}$ and $\mathbf{x_m}$, $m \in \mathbb{N}$, is an infinite set.

Let us prove that $y_m \overset{r}{\sim} c_m$, as $m \to \infty$. Let $M > 0$. Choose the smallest $m \in \mathbb{N}$ such that $2^m > M$. For each $k \in \{1,2,...,m-1\}$ there is $n_k^* \in \mathbb{N}$, so that $\frac{c_{[\lambda n]}}{x_{k,n}} \geq M$ and $\frac{x_{k,[\lambda n]}}{c_n} \geq M$ hold for each $n \geq n_k^*$. Let $n^* = \max\{n_1^*, \ldots, n_{m-1}^*\}$. Therefore, the inequalities $\frac{c_{[\lambda i]}}{y_i} \geq M$ and $\frac{y_{[\lambda i]}}{c_i} \geq M$ hold for each $\lambda > 1+\varepsilon$, $\varepsilon > 0$, and each $i \geq n^*$. Therefore, $y_i \overset{r}{\sim} c_i$, as $i \to \infty$, because M was arbitrary. One concludes that $\mathbf{y} \in [\mathbf{c}]_r$. □

Corollary 12. *The selection principle* $\alpha_2([\mathbf{c}]_r^{'}, [\mathbf{c}]_r)$ *holds for each fixed element* $\mathbf{c}$ *from the class* $\mathsf{R}_{s,\infty} \cap \mathbb{S}_1$.

From Corollary 12 and [**10**, p. 109] we have

Corollary 13. *The selection principles* $\alpha_i([\mathbf{c}]_r^{'}, [\mathbf{c}]_r)$ *hold for* $i \in \{3,4\}$, *where* $\mathbf{c}$ *is an arbitrary and fixed element from* $\mathsf{R}_{s,\infty} \cap \mathbb{S}_1$.

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