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SINE AND COSINE TYPES OF GENERATING FUNCTIONS

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We introduce two sine and cosine types of generating functions in a general case and apply them to the generating functions of classical hypergeometric orthogonal polynomials as well as some widely investigated combinatorial numbers such as Bernoulli, Euler and Genocchi numbers. This approach can also be applied to other celebrated sequences.

1. INTRODUCTION

The generating function of a sequence of polynomials $\{P_n(x)\}$ is defined by a bivariate function, say $G(x,t)$, whose expansion in powers of t has the form

$$G(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n, \tag{1}$$

for sufficiently small $|t|$. Since $\lambda_n^* P_n^*(x) = P_n(x)$ is also a polynomial, relation (1) can be transformed to

$$G(x,t) = \sum_{n=0}^{\infty} \lambda_n^* P_n^*(x) t^n. \tag{2}$$

For $\lambda_n^* = \frac{1}{n!}$, (2) is called the exponential type of generating functions. For instance, the Sheffer polynomials $\{s_n(x)\}$ [**33, 35**] are generated by an exponential generating function as

$$f(t) \exp\big(xH(t)\big) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{3}$$

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where

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} \qquad (f_0 \neq 0),$$

and

$$H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!} \qquad (h_0 = 0).$$

The Sheffer polynomials can also be defined by means of a pair of functions, say $\big(u(t), v(t)\big)$, where $u(t)$ is an invertible series and $v(t)$ is a delta series, i.e.

$$u(t) = \sum_{n=0}^{\infty} u_n \frac{t^n}{n!} \qquad (u_0 \neq 0),$$

and

$$v(t) = \sum_{n=1}^{\infty} v_n \frac{t^n}{n!} \qquad (v_1 \neq 0).$$

If $v^{-1}(t)$ denotes the compositional inverse of $v(t)$, the exponential generating function of Sheffer polynomials is given by

$$\frac{1}{u\big(v^{-1}(t)\big)} \exp\big(xv^{-1}(t)\big) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!},$$

for

$$f(t) = \frac{1}{u\big(v^{-1}(t)\big)} \qquad \text{and} \qquad H(t) = v^{-1}(t).$$

When $u(t) \equiv 1$, the sequence corresponding to the pair $\big(1, v(t)\big)$ is called the associated Sheffer sequence for $v(t)$ denoted by $\{\sigma_n(x)\}$, and its exponential generating function is represented as

$$\exp\big(xv^{-1}(t)\big) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}.$$

Many special polynomials such as Bernoulli polynomials of the second kind, Boole polynomials, Laguerre polynomials, Meixner polynomials of the first and second kinds, Poisson-Charlier polynomials and Stirling polynomials are particular cases of Sheffer sequences. Also, Abel polynomials, Bell polynomials, central factorial, falling factorial, Mahler polynomials, Mittag-Leffler polynomials, Mott polynomials and power polynomials are some particular examples of the associated Sheffer sequences [**28**].

As an important case of Sheffer sequences, Appell polynomials $\{A_n(x)\}$ appear when $H(t) \equiv v^{-1}(t) \equiv t$ in (3). In other words

$$f(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!},$$

represents the generating function of these polynomials where f is a formal power series in t.

The Appell polynomials have found remarkable applications in different branches of mathematics, theoretical physics and chemistry [**2**]. Three special cases of them are respectively Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$ and Genocchi polynomials $G_n(x)$ which are defined by

$$\frac{t}{\mathrm{e}^t - 1}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad (|t| < 2\pi),$$
$$\frac{2}{\mathrm{e}^t + 1}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad (|t| < \pi),$$

and

$$\frac{2t}{\mathrm{e}^t + 1}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \qquad (|t| < \pi).$$

In this sense, Bernoulli numbers $B_n := B_n(0)$, Euler numbers $E_n := 2^n E_n(\frac{1}{2})$ and Genocchi numbers $G_n := G_n(0)$ have found various applications in number theory, combinatorics and numerical analysis [**8**].

Another important case of Appell polynomials is the Apostol type of Bernoulli, Euler and Genocchi polynomials which are respectively generated by

$$\frac{t}{\lambda\mathrm{e}^t - 1}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} B_n(x;\lambda)\frac{t^n}{n!} \qquad (|t + \ln\lambda| < 2\pi), \tag{4}$$

$$\frac{2}{\lambda\mathrm{e}^t + 1}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} E_n(x;\lambda)\frac{t^n}{n!} \qquad (|t + \ln\lambda| < \pi), \tag{5}$$

and

$$\frac{2t}{\lambda\mathrm{e}^t + 1}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} G_n(x;\lambda)\frac{t^n}{n!} \qquad (|t + \ln\lambda| < \pi), \tag{6}$$

where

$$B_{n,\lambda} := B_n(0;\lambda), \;\; E_{n,\lambda} := E_n(0;\lambda) \;\;\text{ and }\;\; G_{n,\lambda} := G_n(0;\lambda), \tag{7}$$

denote the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi numbers [**14, 16**].

Up to now, many authors have studied these numbers and polynomials in the literature in detail, for example see [**4, 13, 26**]. In [**14, 15**], the author computes the Fourier expansions and integral representations of Apostol-Bernoulli and Apostol-Euler polynomials while in [**27**], the author investigates the Padé approximation of these polynomials. Also in [**9**] and [**12**], a q-extension of Apostol-Euler polynomials is given. For Genocchi polynomials and their various extensions see e.g. [**5, 10, 11**].

Besides the aforesaid references, some new types of Bernoulli, Euler and Genocchi polynomials and their Apostol type have been recently introduced in [**17, 18, 19**] and [**31, 32**]. In [**31, 32**] the following generating functions

$$f(t;\lambda)\,\mathrm{e}^{pt}\cos qt=\sum_{n=0}^{\infty}A_n^{(c)}(p,q;\lambda)\frac{t^n}{n!},\tag{8}$$

and

$$f(t;\lambda)\,\mathrm{e}^{pt}\sin qt=\sum_{n=0}^{\infty}A_n^{(s)}(p,q;\lambda)\frac{t^n}{n!},\tag{9}$$

are introduced for $p,q\in\mathbb{R}$, and the functions

$$f(t;\lambda)=\frac{t}{\lambda\mathrm{e}^t-1},\quad f(t;\lambda)=\frac{2}{\lambda\mathrm{e}^t+1}\quad\text{and}\quad f(t;\lambda)=\frac{2t}{\lambda\mathrm{e}^t+1},$$

are then replaced in (8) and (9) to derive new families of polynomials $A_n^{(c)}(p,q;\lambda)$ and $A_n^{(s)}(p,q;\lambda)$, where $A_n^{(c)}(p,q;\lambda)$ denotes the cosine type of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and $A_n^{(s)}(p,q;\lambda)$ denotes the sine type of them. Note that $\lambda=1$ gives the cosine and sine types of usual Bernoulli, Euler and Genocchi polynomials which are separately studied in [**17, 18, 19**].

The main aim of this paper is to introduce sine and cosine types of generating functions in a general case containing all above-mentioned examples as particular cases.

Research in the field of generating functions is not limited only to their role in combinatorics. Another subject which they are invloved in concerns to the classical orthogonal polynomials, because generating functions of such polynomials help us investigate their basic properties and applications. For instance, the generating function of Legendre polynomials appears in electromagnetism [**29**]. Also, the polynomial value at some specific points such as 1 or -1 can be computed by its generating function. Since classical orthogonal polynomials have found interesting applications in physics and engineering, extending them may somehow lead to new applications that logically develop the previous known applications, see e.g. [**7**].

In this paper, we introduce sine and cosine types of generating functions in a general case and apply them for two main classes, i.e. for the generating functions of classical hypergeometric orthogonal polynomials and for widely-investigated sequences of numbers appeared in number theory. For this purpose, we first introduce two trigonometric types of generating functions in Section 2. Then in Section 3 we introduce two families of generating functions for Jacobi polynomials and its subcases (i.e. Chebyshev, Legendre and ultraspherical polynomials) as well as Laguerre and Hermite polynomials. Finally, Section 4 is devoted to sine and cosine types of generating functions of the well known sequences of numbers such as Bernoulli, Euler, Genocchi and Stirling numbers, though our approach can be applied for other celebrated sequences, too.

2. SINE AND COSINE TYPES OF GENERATING FUNCTIONS

Let us start with a sequence of bivariate functions, as defined in [**24**], as

$$F_m\big(p,q;\{\lambda_r,a_r,b_r\}_{r=1}^m\big)=\sum_{r=1}^m\lambda_r f(a_r p+b_r q)\,, \tag{10}$$

in which $\{\lambda_r,a_r,b_r\}_{r=1}^m$ are real or complex numbers and f is an arbitrary differentiable function. By noting the bivariate Taylor expansion

$$F(p,q)=\sum_{k=0}^\infty\frac{1}{k!}\Big(\sum_{j=0}^k\binom{k}{j}\left.\frac{\partial^k F}{\partial p^j\partial q^{k-j}}\right|_{(0,0)}p^j q^{k-j}\Big),$$

we find that

$$F_m\big(p,q;\{\lambda_r,a_r,b_r\}_{r=1}^m\big)=\sum_{k=0}^\infty\frac{f^{(k)}(0)}{k!}\Big(\sum_{j=0}^k\binom{k}{j}\Big(\sum_{r=1}^m\lambda_r\,a_r^j\,b_r^{k-j}\Big)\,p^j q^{k-j}\Big). \tag{11}$$

For simplicity, if in (11)

$$A_k\big(p,q;\{\lambda_r,a_r,b_r\}_{r=1}^m\big)=\sum_{j=0}^k\binom{k}{j}\Big(\sum_{r=1}^m\lambda_r\,a_r^j\,b_r^{k-j}\Big)\,p^j q^{k-j}=\sum_{r=1}^m\lambda_r(a_r p+b_r q)^k, \tag{12}$$

then (10) is simplified as

$$\sum_{r=1}^m\lambda_r f(a_r p+b_r q)=\sum_{k=0}^\infty\frac{f^{(k)}(0)}{k!}A_k\big(p,q;\{\lambda_r,a_r,b_r\}_{r=1}^m\big). \tag{13}$$

Moreover, by noting (12), the relation

$$A_k\big(p,px;\{\lambda_r,a_r,b_r\}_{r=1}^m\big)=p^k\sum_{j=0}^k\binom{k}{j}\Big(\sum_{r=1}^m\lambda_r\,a_r^j\,b_r^{k-j}\Big)\,x^{k-j}=p^k\sum_{r=1}^m\lambda_r(a_r+b_r x)^k,$$

implies that (13) is a generating function for the polynomials

$$P_k\big(x;\{\lambda_r,a_r,b_r\}_{r=1}^m\big)=p^{-k}A_k\big(p,px;\{\lambda_r,a_r,b_r\}_{r=1}^m\big)=\sum_{r=1}^m\lambda_r(a_r+b_r x)^k,$$

as follows

$$\sum_{r=1}^m\lambda_r f\big(p(a_r+b_r x)\big)=\sum_{k=0}^\infty P_k\big(x;\{\lambda_r,a_r,b_r\}_{r=1}^m\big)f^{(k)}(0)\,\frac{p^k}{k!}.$$

Now, suppose in (13) that

$$m=2,\ \ \lambda_1=-\lambda_2=\frac{1}{2\mathrm{i}},\ \ a_1=a_2=1\qquad\text{and}\qquad b_1=-b_2=\mathrm{i}=\sqrt{-1},$$

to reach the identity

$$\frac{1}{2\mathrm{i}}\Big(f(p+\mathrm{i}q)-f(p-\mathrm{i}q)\Big)=\sum_{k=1}^{\infty}\frac{f^{(k)}(0)}{k!}\,S_k(p,q), \tag{14}$$

where, according to (12),

$$S_k(p,q)=\frac{(p+\mathrm{i}q)^k-(p-\mathrm{i}q)^k}{2\mathrm{i}}=\sum_{j=0}^{[(k-1)/2]}(-1)^j\binom{k}{2j+1}p^{k-2j-1}q^{2j+1}. \tag{15}$$

Once again, if in (13) we assume that

$$m=2,\ \ \lambda_1=\lambda_2=\frac{1}{2},\ \ a_1=a_2=1 \qquad \text{and} \qquad b_1=-b_2=\mathrm{i}=\sqrt{-1},$$

then we get

$$\frac{1}{2}\Big(f(p+\mathrm{i}q)+f(p-\mathrm{i}q)\Big)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\,C_k(p,q), \tag{16}$$

where

$$C_k(p,q)=\frac{(p+\mathrm{i}q)^k+(p-\mathrm{i}q)^k}{2}=\sum_{j=0}^{[k/2]}(-1)^j\binom{k}{2j}p^{k-2j}q^{2j}. \tag{17}$$

The two sequences $C_k(p,q)$ and $S_k(p,q)$ which are respectively the real and imaginary parts of the complex function z^k for $z=p+\mathrm{i}q$, will play a key role in introducing our generating functions.

The two real sequences (15) and (17) show that the left-hand side of the expansions (14) and (16) are real-valued functions. The following lemma which can be proved directly and independantly in standard textbooks of complex analysis such as [**3**] has found interesting applications, see e.g. [**21, 24**].

Lemma 1. If f is a complex function such that

$$f(x+\mathrm{i}y)=u(x,y)+\mathrm{i}\,v(x,y),$$

then

$$u(x,y)=\frac{f(x+\mathrm{i}y)+f(x-\mathrm{i}y)}{2}\ \in\mathbb{R},$$

and

$$v(x,y)=\frac{f(x+\mathrm{i}y)-f(x-\mathrm{i}y)}{2\mathrm{i}}\ \in\mathbb{R}.$$

For example, if $f(z)=z^{\alpha}$ for any $\alpha\in\mathbb{C}$, then substituting it into Lemma 1 leads to the identity

$$z^{\alpha}=C_{\alpha}(x,y)+\mathrm{i}\,S_{\alpha}(x,y),$$

where

$$C_\alpha(x,y)=\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{2j}x^{\alpha-2j}y^{2j},$$

and

$$S_\alpha(x,y)=\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{2j+1}x^{\alpha-2j-1}y^{2j+1}.$$

Moreover, the equality $(z^\alpha)^\beta=z^{\alpha\beta}$ implies the following interesting identity

$$\Big(C_\alpha(x,y)+\mathrm{i}\,S_\alpha(x,y)\Big)^\beta=C_{\alpha\beta}(x,y)+\mathrm{i}\,S_{\alpha\beta}(x,y),$$

for any $\alpha,\beta\in\mathbb{C}$.

One of the advantages of the sequences $C_\alpha(p,q)$ and $S_\alpha(p,q)$ is that they can be represented in polar forms. In other words, if

$$r=\sqrt{p^2+q^2}\qquad\text{and}\qquad\theta=\arg(p+\mathrm{i}q),$$

for $p,q\in\mathbb{R}$ such that $p^2+q^2\neq 0$, then we have

$$S_\alpha(p,q)=r^\alpha\sin(\alpha\theta)\qquad\text{and}\qquad C_\alpha(p,q)=r^\alpha\cos(\alpha\theta), \tag{18}$$

or equivalently

$$S_\alpha(p,q)=(p^2+q^2)^{\frac{\alpha}{2}}\sin\big(\alpha\arctan\frac{q}{p}\big), \tag{19}$$

and

$$C_\alpha(p,q)=(p^2+q^2)^{\frac{\alpha}{2}}\cos\big(\alpha\arctan\frac{q}{p}\big). \tag{20}$$

A remarkable case in relations (19) and (20) is when $q=\lambda p$, so that we have

$$S_\alpha(p,\lambda p)=\big(1+\lambda^2\big)^{\frac{\alpha}{2}}\sin\big(\alpha\arctan\lambda\big)\,p^\alpha,$$

and

$$C_\alpha(p,\lambda p)=\big(1+\lambda^2\big)^{\frac{\alpha}{2}}\cos\big(\alpha\arctan\lambda\big)\,p^\alpha.$$

Moreover, it is not difficult to verify from (18) that

$$|C_\alpha(p,q)|\leq(p^2+q^2)^{\frac{\alpha}{2}}\qquad\text{and}\qquad|S_\alpha(p,q)|\leq(p^2+q^2)^{\frac{\alpha}{2}}. \tag{21}$$

By using (21), we can directly conclude that if the limit

$$R=\lim_{k\to\infty}\left|\frac{f^{(k+1)}(0)}{(k+1)f^{(k)}(0)}\right|,$$

exists, then the expansions (14) and (16) are absolutely convergent provided that

$$\sqrt{p^2+q^2} < \frac{1}{R}.$$

Throughout the paper, we assume that p and q are always chosen to be small enough such that any appeared series converges absolutely.

We are now in a good position to define sine and cosine types of generating functions in two different cases.

2.1 First Kind of Trigonometric-Type Generating Functions

Suppose that the generating function of a sequence of polynomials is given by

$$G(z,t) = \sum_{n=0}^{\infty} P_n(z) t^n,$$

with

$$P_n(z) = \sum_{k=0}^{n} c^*_{n,k} (a^* z + b^*)^k, \tag{22}$$

in which $\{c^*_{n,k}\}_{k=0}^{n}$ are pre-determined coefficients and $a^*, b^* \in \mathbb{R}$.

By referring to the real-valued functions (14) and (16) and taking $z = p \pm \mathrm{i}q$, we respectively obtain

$$\begin{aligned}
\mathrm{G}_{\mathrm{I}}^{(s)}(p,q;\,t) &= \frac{1}{2\mathrm{i}}\big(G(p+\mathrm{i}q,t) - G(p-\mathrm{i}q,t)\big) = \sum_{n=0}^{\infty} \frac{1}{2\mathrm{i}}\big(P_n(p+\mathrm{i}q) - P_n(p-\mathrm{i}q)\big)t^n \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} c^*_{n,k}\frac{1}{2\mathrm{i}}\big((a^*p+b^*+\mathrm{i}a^*q)^k - (a^*p+b^*-\mathrm{i}a^*q)^k\big)\Big)t^n \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} c^*_{n,k}\, S_k(a^*p+b^*, a^*q)\Big)t^n,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{G}_{\mathrm{I}}^{(c)}(p,q;\,t) &= \frac{1}{2}\big(G(p+\mathrm{i}q,t) + G(p-\mathrm{i}q,t)\big) = \sum_{n=0}^{\infty} \frac{1}{2}\big(P_n(p+\mathrm{i}q) + P_n(p-\mathrm{i}q)\big)t^n \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} c^*_{n,k}\frac{1}{2}\big((a^*p+b^*+\mathrm{i}a^*q)^k + (a^*p+b^*-\mathrm{i}a^*q)^k\big)t^n \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} c^*_{n,k}\, C_k(a^*p+b^*, a^*q)\Big)t^n,
\end{aligned}$$

as the first kind of the sine and cosine types of the generating function $G(z,t)$. In fact, they are respectively the imaginary and real parts of the generating function $G(z,t)$ with respect to the variable z.

2.2 Second Kind of Trigonometric-Type Generating Functions

This turn, we consider $t = p \pm \mathrm{i}q$ to respectively obtain

$$\begin{aligned}\mathrm{G}_{\mathrm{II}}^{(s)}(z;\, p, q) &= \frac{1}{2\mathrm{i}}\big(G(z, p+\mathrm{i}q) - G(z, p-\mathrm{i}q)\big)\\ &= \sum_{n=0}^{\infty}\frac{1}{2\mathrm{i}}\big((p+\mathrm{i}q)^n - (p-\mathrm{i}q)^n\big)P_n(z) = \sum_{n=0}^{\infty} S_n(p,q)P_n(z),\end{aligned}$$

and

$$\begin{aligned}\mathrm{G}_{\mathrm{II}}^{(c)}(z;\, p, q) &= \frac{1}{2}\big(G(z, p+\mathrm{i}q) + G(z, p-\mathrm{i}q)\big)\\ &= \sum_{n=0}^{\infty}\frac{1}{2}\big((p+\mathrm{i}q)^n + (p-\mathrm{i}q)^n\big)P_n(z) = \sum_{n=0}^{\infty} C_n(p,q)P_n(z),\end{aligned}$$

as the imaginary and real parts of the generating function $G(z, t)$ with respect to the variable t. We call them the second kind of the sine and cosine types of the function $G(z, t)$, respectively.

3. SINE AND COSINE TYPES OF THE GENERATING FUNCTIONS OF CLASSICAL ORTHOGONAL POLYNOMIALS

All classical orthogonal polynomials [**6, 34**] can be defined in terms of hypergeometric series

$$ {}_pF_q\left(\begin{array}{cccc} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{array}\,\middle|\, z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k}\,\frac{z^k}{k!}, \tag{23}$$

in which

$$(r)_k = \prod_{j=0}^{k-1}(r+j),$$

denotes the Pochhammer symbol [**1**] and z may be a complex variable.

According to the ratio test, the series (23) is convergent for any $p \le q+1$. In fact, it converges in $|z| < 1$ for $p = q+1$, converges everywhere for $p < q+1$ and converges nowhere ($z \neq 0$) for $p > q+1$. Moreover, for $p = q+1$ it absolutely converges for $|z| = 1$ if the condition

$$A^* = \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) > 0,$$

holds and is conditionally convergent for $|z| = 1$ and $z \neq 1$ if $-1 < A^* \le 0$ and is finally divergent for $|z| = 1$ and $z \neq 1$ if $A^* \le -1$.

There are two important cases of the series (23) respectively as follows

$$
\begin{aligned}
{}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle|\, z\right) &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \\
&= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt
\end{aligned} \tag{24}
$$

$$(\operatorname{Re} c > \operatorname{Re} b > 0),$$

which converges in $|z| \leq 1$ and the second case, which converges everywhere as

$$
{}_1F_1\left(\begin{array}{c} a \\ c \end{array} \middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt}\, dt \tag{25}
$$

$$(\operatorname{Re} c > \operatorname{Re} a > 0).$$

Since the same approach as used in Sections 2.1 and 2.2 can be applied for two hypergeometric series (24) and (25), we can directly find that

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{S_k(p,q)}{k!} \\
&= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\Big((1-pt)^2+q^2t^2\Big)^{-\frac{a}{2}} \sin\Big(-a \arctan\frac{qt}{1-pt}\Big)\, dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{C_k(p,q)}{k!} \\
&= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\Big((1-pt)^2+q^2t^2\Big)^{-\frac{a}{2}} \cos\Big(a \arctan\frac{qt}{1-pt}\Big)\, dt,
\end{aligned}
$$

and similarly, corresponding to (25), we obtain

$$
\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{S_k(p,q)}{k!} = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{pt} \sin qt\, dt,
$$

and

$$
\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{C_k(p,q)}{k!} = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{pt} \cos qt\, dt.
$$

Note that the aforesaid approach can be applied for other special functions [**25**] and well-known inequalities [**20**]. For example, recently in [**22, 23**], we have applied this approach for the integral representation of gamma function

$$
\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx,
$$

to respectively define

$$\Gamma_s(p,q) = \frac{1}{2\mathrm{i}}\Big(\Gamma(p+\mathrm{i}q) - \Gamma(p-\mathrm{i}q)\Big) = \int_0^\infty x^{p-1}e^{-x}\sin\big(q\ln x\big)dx,$$

and

$$\Gamma_c(p,q) = \frac{1}{2}\Big(\Gamma(p+\mathrm{i}q) + \Gamma(p-\mathrm{i}q)\Big) = \int_0^\infty x^{p-1}e^{-x}\cos\big(q\ln x\big)dx,$$

for any $p > 0$ and $q \in \mathbb{R}$. Also, by using the limit definition of the gamma function, we obtain

$$\Gamma_s(p,q) = \lim_{n\to\infty} \frac{n!\,n^p \sin\Big(q\ln n - \sum\limits_{k=0}^{n}\arctan\dfrac{q}{p+k}\Big)}{\prod\limits_{k=0}^{n}\Big((p+k)^2+q^2\Big)^{\frac{1}{2}}},$$

and

$$\Gamma_c(p,q) = \lim_{n\to\infty} \frac{n!\,n^p \cos\Big(q\ln n - \sum\limits_{k=0}^{n}\arctan\dfrac{q}{p+k}\Big)}{\prod\limits_{k=0}^{n}\Big((p+k)^2+q^2\Big)^{\frac{1}{2}}}.$$

Since the beta function has a close relationship with the gamma function as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \qquad (\mathrm{Re}\,x > 0,\ \mathrm{Re}\,y > 0),$$

the equality

$$\frac{1}{2}\Big(t^{\mathrm{i}r}(1-t)^{\mathrm{i}s} + t^{-\mathrm{i}r}(1-t)^{-\mathrm{i}s}\Big) = \cos\Big(r\ln t + s\ln(1-t)\Big),$$

implies to define the sine and cosine types of the beta function as follows

$$\begin{aligned} B_s(p,q,r,s) &= \frac{1}{2\mathrm{i}}\Big(B(p+\mathrm{i}r, q+\mathrm{i}s) - B(p-\mathrm{i}r, q-\mathrm{i}s)\Big) \\ &= \int_0^1 t^{p-1}(1-t)^{q-1}\sin\Big(r\ln t + s\ln(1-t)\Big)\,dt, \end{aligned}$$

and

$$\begin{aligned} B_c(p,q,r,s) &= \frac{1}{2}\Big(B(p+\mathrm{i}r, q+\mathrm{i}s) + B(p-\mathrm{i}r, q-\mathrm{i}s)\Big) \\ &= \int_0^1 t^{p-1}(1-t)^{q-1}\cos\Big(r\ln t + s\ln(1-t)\Big)\,dt. \end{aligned}$$

In what follows, we now define the sine and cosine types of generating functions of classical orthogonal polynomials.

3.1 Trigonometric-Type Generating Functions of Jacobi Polynomials

The Jacobi polynomials

$$(26)\qquad P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n, & 1+\alpha+\beta+n \\ & \alpha+1 \end{matrix}\;\middle|\;\frac{1-z}{2}\right) = \sum_{k=0}^{n} c_{n,k}^{(\alpha,\beta)}\left(\frac{1-z}{2}\right)^k,$$

for

$$c_{n,k}^{(\alpha,\beta)} = \frac{(\alpha+1)_n}{n!}\,\frac{(-n)_k(1+\alpha+\beta+n)_k}{(\alpha+1)_k\,k!},$$

satisfy the relation

$$\frac{2^{\alpha+\beta}}{\rho(1+\rho-t)^{\alpha}(1+\rho+t)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(z)\,t^n \qquad |t|<1,$$

where

$$\rho = \rho(z,t) = \sqrt{1-2tz+t^2} \qquad \text{and} \qquad \alpha,\beta > -1.$$

By noting that

$$c_{n,k}^{*} = c_{n,k}^{(\alpha,\beta)}, \quad a^* = -\frac{1}{2} \quad \text{and} \quad b^* = \frac{1}{2},$$

in (22), there are two different types of generating functions to extend the Jacobi polynomials (26). To reach this goal, we first simplify the following relations

$$(27)\quad \mathrm{G}_{\mathrm{I}}^{(s)}(p,q;\,t;\,\alpha,\beta) = \frac{2^{\alpha+\beta-1}}{\mathrm{i}}\times$$

$$\left(\frac{1}{\sqrt{1+t^2-2t(p+\mathrm{i}q)}\left(1-t+\sqrt{1+t^2-2t(p+\mathrm{i}q)}\right)^{\alpha}\left(1+t+\sqrt{1+t^2-2t(p+\mathrm{i}q)}\right)^{\beta}}\right.$$

$$\left.-\frac{1}{\sqrt{1+t^2-2t(p-\mathrm{i}q)}\left(1-t+\sqrt{1+t^2-2t(p-\mathrm{i}q)}\right)^{\alpha}\left(1+t+\sqrt{1+t^2-2t(p-\mathrm{i}q)}\right)^{\beta}}\right),$$

and

$$(28)\quad \mathrm{G}_{\mathrm{I}}^{(c)}(p,q;\,t\,;\alpha,\beta) = 2^{\alpha+\beta-1}\times$$

$$\left(\frac{1}{\sqrt{1+t^2-2t(p+\mathrm{i}q)}\left(1-t+\sqrt{1+t^2-2t(p+\mathrm{i}q)}\right)^{\alpha}\left(1+t+\sqrt{1+t^2-2t(p+\mathrm{i}q)}\right)^{\beta}}\right.$$

$$\left.+\frac{1}{\sqrt{1+t^2-2t(p-\mathrm{i}q)}\left(1-t+\sqrt{1+t^2-2t(p-\mathrm{i}q)}\right)^{\alpha}\left(1+t+\sqrt{1+t^2-2t(p-\mathrm{i}q)}\right)^{\beta}}\right).$$

In order to simplify (27) and (28), we directly use Lemma 1 and the well-known Euler identity $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ to respectively obtain

$$\sqrt{1+t^2-2pt-\mathrm{i}\,2qt} = \sqrt{\rho^*}\, e^{-\mathrm{i}\theta^*},$$

$$\Big(1-t+\sqrt{1+t^2-2pt-\mathrm{i}\,2qt}\Big)^{-\alpha} = \\ \Big(\rho^*+(1-t)^2+2(1-t)\sqrt{\rho^*}\cos\theta^*\Big)^{-\frac{\alpha}{2}}\, e^{\alpha\mathrm{i}\arctan\frac{\sqrt{\rho^*}\sin\theta^*}{1-t+\sqrt{\rho^*}\cos\theta^*}},$$

and

$$\Big(1+t+\sqrt{1+t^2-2pt-\mathrm{i}\,2qt}\Big)^{-\beta} = \\ \Big(\rho^*+(1+t)^2+2(1+t)\sqrt{\rho^*}\cos\theta^*\Big)^{-\frac{\beta}{2}}\, e^{\beta\mathrm{i}\arctan\frac{\sqrt{\rho^*}\sin\theta^*}{1+t+\sqrt{\rho^*}\cos\theta^*}},$$

in which

$$\rho^* = \sqrt{(1+t^2-2pt)^2+4q^2t^2} \qquad \text{and} \qquad \theta^* = \frac{1}{2}\arctan\frac{2qt}{1+t^2-2pt}.$$

On the other hand, since

$$\cos^2\big(\frac{1}{2}\arctan z\big) = \frac{1+\sqrt{1+z^2}}{2\sqrt{1+z^2}},$$

we have

$$\sqrt{\rho^*}\,\cos\theta^* = \frac{1}{\sqrt{2}}\sqrt{\rho^*+|1+t^2-2pt|},$$

and

$$\sqrt{\rho^*}\,\sin\theta^* = \frac{1}{\sqrt{2}}\sqrt{\rho^*-|1+t^2-2pt|}.$$

The above results finally give the first kind of the sine and cosine generating functions of Jacobi polynomials as follows

$$\frac{\Big(\rho^*+(1-t)^2+\sqrt{2}(1-t)\sqrt{\rho^*+|1+t^2-2pt|}\Big)^{-\frac{\alpha}{2}}\Big(\rho^*+(1+t)^2+\sqrt{2}(1+t)\sqrt{\rho^*+|1+t^2-2pt|}\Big)^{-\frac{\beta}{2}}}{2^{-(\alpha+\beta)}\sqrt{\rho^*}} \\ \times\sin\Big(\theta^*+\alpha\arctan\frac{\sqrt{\rho^*-|1+t^2-2pt|}}{\sqrt{2}(1-t)+\sqrt{\rho^*+|1+t^2-2pt|}}+\beta\arctan\frac{\sqrt{\rho^*-|1+t^2-2pt|}}{\sqrt{2}(1+t)+\sqrt{\rho^*+|1+t^2-2pt|}}\Big) \\ = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_{n,k}^{(\alpha,\beta)}\, S_k\Big(\frac{1-p}{2},-\frac{q}{2}\Big)\right)t^n,$$

and

(29)

$$\frac{\left(\rho^*+(1-t)^2+\sqrt{2}(1-t)\sqrt{\rho^*+|1+t^2-2pt|}\right)^{-\frac{\alpha}{2}}\left(\rho^*+(1+t)^2+\sqrt{2}(1+t)\sqrt{\rho^*+|1+t^2-2pt|}\right)^{-\frac{\beta}{2}}}{2^{-(\alpha+\beta)}\sqrt{\rho^*}}$$

$$\times\cos\left(\theta^*+\alpha\arctan\frac{\sqrt{\rho^*-|1+t^2-2pt|}}{\sqrt{2}(1-t)+\sqrt{\rho^*+|1+t^2-2pt|}}+\beta\arctan\frac{\sqrt{\rho^*-|1+t^2-2pt|}}{\sqrt{2}(1+t)+\sqrt{\rho^*+|1+t^2-2pt|}}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}c_{n,k}^{(\alpha,\beta)}\,C_k\Big(\frac{1-p}{2},-\frac{q}{2}\Big)\right)t^n.$$

An interesting case in (29) is when $q=\lambda(p-1)$ and $p=x$ leading to a generalization of the Jacobi polynomials as

$$P_n^{(\alpha,\beta)}(x;\lambda)=\sum_{k=0}^{n}c_{n,k}^{(\alpha,\beta)}\left(1+\lambda^2\right)^{\frac{k}{2}}\cos\left(k\arctan\lambda\right)\Big(\frac{1-x}{2}\Big)^k.$$

Similarly, to compute the second kind of the sine and cosine generating functions of the Jacobi polynomials, we should first compute the following relations

(30)

$$\begin{aligned}&\mathrm{G}_{\mathrm{II}}^{(s)}(z;\,p,q\,;\alpha,\beta)=\frac{2^{\alpha+\beta-1}}{\mathrm{i}}\times\\&\quad\Bigg(\Big(1+(p+\mathrm{i}q)^2-2(p+\mathrm{i}q)z\Big)^{-\frac{1}{2}}\times\\&\qquad\Big(1-(p+\mathrm{i}q)+\sqrt{1+(p+\mathrm{i}q)^2-2(p+\mathrm{i}q)z}\Big)^{-\alpha}\Big(1+p+\mathrm{i}q+\sqrt{1+(p+\mathrm{i}q)^2-2(p+\mathrm{i}q)z}\Big)^{-\beta}\\&\qquad-\Big(1+(p-\mathrm{i}q)^2-2(p-\mathrm{i}q)z\Big)^{-\frac{1}{2}}\times\\&\qquad\Big(1-(p-\mathrm{i}q)+\sqrt{1+(p-\mathrm{i}q)^2-2(p-\mathrm{i}q)z}\Big)^{-\alpha}\Big(1+p-\mathrm{i}q+\sqrt{1+(p-\mathrm{i}q)^2-2(p-\mathrm{i}q)z}\Big)^{-\beta}\Bigg),\end{aligned}$$

and

(31)

$$\begin{aligned}&\mathrm{G}_{\mathrm{II}}^{(c)}(z;\,p,q\,;\alpha,\beta)=2^{\alpha+\beta-1}\times\\&\quad\Bigg(\Big(1+(p+\mathrm{i}q)^2-2(p+\mathrm{i}q)z\Big)^{-\frac{1}{2}}\times\\&\qquad\Big(1-(p+\mathrm{i}q)+\sqrt{1+(p+\mathrm{i}q)^2-2(p+\mathrm{i}q)z}\Big)^{-\alpha}\Big(1+p+\mathrm{i}q+\sqrt{1+(p+\mathrm{i}q)^2-2(p+\mathrm{i}q)z}\Big)^{-\beta}\\&\qquad+\Big(1+(p-\mathrm{i}q)^2-2(p-\mathrm{i}q)z\Big)^{-\frac{1}{2}}\times\\&\qquad\Big(1-(p-\mathrm{i}q)+\sqrt{1+(p-\mathrm{i}q)^2-2(p-\mathrm{i}q)z}\Big)^{-\alpha}\Big(1+p-\mathrm{i}q+\sqrt{1+(p-\mathrm{i}q)^2-2(p-\mathrm{i}q)z}\Big)^{-\beta}\Bigg).\end{aligned}$$

In order to simplify (30) and (31), we can use Lemma 1 once again to respectively get

$$\sqrt{1+(p+\mathrm{i}q)^2-2(p+\mathrm{i}q)z}=\sqrt{R^*}\,e^{\mathrm{i}\phi^*},$$
$$\Big(1-p-\mathrm{i}q+\sqrt{R^*}\,\cos\phi^*+\mathrm{i}\sqrt{R^*}\,\sin\phi^*\Big)^{-\alpha}$$
$$=\Big(q^2+(1-p)^2+R^*+2\sqrt{R^*}\big((1-p)\cos\phi^*-q\sin\phi^*\big)\Big)^{-\frac{\alpha}{2}}e^{-\alpha\mathrm{i}\arctan\frac{\sqrt{R^*}\sin\phi^*-q}{\sqrt{R^*}\cos\phi^*+1-p}},$$

and

$$\Big(1+p+\mathrm{i}q+\sqrt{R^*}\,\cos\phi^*+\mathrm{i}\sqrt{R^*}\,\sin\phi^*\Big)^{-\beta}$$
$$=\Big(q^2+(1+p)^2+R^*+2\sqrt{R^*}\big((1+p)\cos\phi^*+q\sin\phi^*\big)\Big)^{-\frac{\beta}{2}}e^{-\beta\mathrm{i}\arctan\frac{\sqrt{R^*}\sin\phi^*+q}{\sqrt{R^*}\cos\phi^*+1+p}},$$

in which

$$R^*=\sqrt{\big(1+p^2-2pz-q^2\big)^2+4q^2(p-z)^2}\text{ and }\phi^*=\frac{1}{2}\arctan\frac{2q(p-z)}{1+p^2-2pz-q^2}.$$

On the other side, since

$$\sqrt{R^*}\,\cos\phi^*=\frac{1}{\sqrt{2}}\sqrt{R^*+|1+p^2-2pz-q^2|},$$

and

$$\sqrt{R^*}\,\sin\phi^*=\frac{1}{\sqrt{2}}\sqrt{R^*-|1+p^2-2pz-q^2|},$$

the above results eventually give the second kind of the sine and cosine generating functions of Jacobi polynomials as follows:

$$\begin{aligned}(32)\qquad &\frac{1}{2^{-(\alpha+\beta)}\sqrt{R^*}}\Big(q^2+(1-p)^2+R^*+2\sqrt{R^*}\big((1-p)\sin\phi^*-q\sin\phi^*\big)\Big)^{-\frac{\alpha}{2}}\\
&\times\Big(q^2+(1+p)^2+R^*+2\sqrt{R^*}\big((1+p)\cos\phi^*+q\sin\phi^*\big)\Big)^{-\frac{\beta}{2}}\\
&\times\sin\Big(\phi^*-\alpha\arctan\frac{\sqrt{R^*-|1+p^2-2pz-q^2|}-q\sqrt{z}}{\sqrt{R^*+|1+p^2-2pz-q^2|}+(1-p)\sqrt{z}}\\
&\qquad-\beta\arctan\frac{\sqrt{R^*-|1+p^2-2pz-q^2|}+q\sqrt{z}}{\sqrt{R^*+|1+p^2-2pz-q^2|}+(1+p)\sqrt{z}}\Big)\\
&=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(z)\,S_n(p,q)\qquad\qquad(p^2+q^2<1),\end{aligned}$$

and

$$\begin{aligned}(33)\qquad &\frac{1}{2^{-(\alpha+\beta)}\sqrt{R^*}}\Big(q^2+(1-p)^2+R^*+2\sqrt{R^*}\big((1-p)\cos\phi^*-q\sin\phi^*\big)\Big)^{-\frac{\alpha}{2}}\\
&\times\Big(q^2+(1+p)^2+R^*+2\sqrt{R^*}\big((1+p)\cos\phi^*+q\sin\phi^*\big)\Big)^{-\frac{\beta}{2}}\\
&\times\cos\Big(\phi^*-\alpha\arctan\frac{\sqrt{R^*-|1+p^2-2pz-q^2|}\;-q\sqrt{z}}{\sqrt{R^*+|1+p^2-2pz-q^2|}\;+(1-p)\sqrt{z}}\\
&\qquad-\beta\arctan\frac{\sqrt{R^*-|1+p^2-2pz-q^2|}\;+q\sqrt{z}}{\sqrt{R^*+|1+p^2-2pz-q^2|}\;+(1+p)\sqrt{z}}\Big)\\
&=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(z)\,C_n(p,q)\qquad (p^2+q^2<1).\end{aligned}$$

For instance, replacing $p=0$ in (32) and (33) respectively give the generating functions of odd and even degrees of Jacobi polynomials as follows

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(z)\,S_n(0,q)=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(z)\,\sin\frac{n\pi}{2}\,q^n=\sum_{j=0}^{\infty}(-1)^jP_{2j+1}^{(\alpha,\beta)}(z)\,q^{2j+1},$$

and

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(z)\,C_n(0,q)=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(z)\,\cos\frac{n\pi}{2}\,q^n=\sum_{j=0}^{\infty}(-1)^jP_{2j}^{(\alpha,\beta)}(z)\,q^{2j}.$$

There are some particular cases of Jacobi polynomials which can be defined by a different type of their generating functions [**6**]. For instance, the Ultraspherical (or Gegenbauer) polynomials

$$\begin{aligned}C_n^{(\lambda)}(z)&=\frac{(2\lambda)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\quad n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix}\;\middle|\;\frac{1-z}{2}\right)\\
&=\frac{(2\lambda)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+2\lambda)_k}{(\lambda+\frac{1}{2})_k\,k!}\Big(\frac{1-z}{2}\Big)^k,\end{aligned}$$

have a generating function for $\lambda>-\frac{1}{2}$ and $\lambda\neq 0$ as

$$(34)\qquad (1-2tz+t^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(z)\,t^n\qquad |t|<1.$$

Hence, the first kind of the sine and cosine types of (34) are computed as

$$\begin{aligned}&\Big((1+t^2-2pt)^2+4q^2t^2\Big)^{-\frac{\lambda}{2}}\sin\Big(\lambda\arctan\frac{2qt}{1+t^2-2pt}\Big)\\
&\qquad=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{(-n)_k(n+2\lambda)_k}{(\lambda+\frac{1}{2})_k\,k!}\,S_k\big(\tfrac{1-p}{2},-\tfrac{q}{2}\big)\Big)(2\lambda)_n\,\frac{t^n}{n!},\end{aligned}$$

and

$$\left((1+t^2-2pt)^2+4q^2t^2\right)^{-\frac{\lambda}{2}}\cos\left(\lambda\arctan\frac{2qt}{1+t^2-2pt}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-n)_k(n+2\lambda)_k}{(\lambda+\frac{1}{2})_k\,k!}\,C_k\left(\tfrac{1-p}{2},-\tfrac{q}{2}\right)\right)(2\lambda)_n\,\frac{t^n}{n!}.$$

As well, the second kind of the sine and cosine generating functions of ultraspherical polynomials are eventually computed as

$$\left((1+p^2-q^2-2pz)^2+4q^2(p-z)^2\right)^{-\frac{\lambda}{2}}\sin\left(\lambda\arctan\frac{2q(z-p)}{1+p^2-q^2-2pz}\right)$$
$$=\sum_{n=0}^{\infty}C_n^{(\lambda)}(z)\,S_n(p,q), \tag{35}$$

and

$$\left((1+p^2-q^2-2pz)^2+4q^2(p-z)^2\right)^{-\frac{\lambda}{2}}\cos\left(\lambda\arctan\frac{2q(p-z)}{1+p^2-q^2-2pz}\right)$$
$$=\sum_{n=0}^{\infty}C_n^{(\lambda)}(z)\,C_n(p,q). \tag{36}$$

For example, since

$$C_n^{(\lambda)}(1)=\frac{(2\lambda)_n}{n!}=\binom{n+2\lambda-1}{n},$$

replacing $z=1$ in equations (35) and (36) yields

$$\sum_{n=0}^{\infty}\binom{n+2\lambda-1}{n}S_n(p,q)=\left((1-p)^2+q^2\right)^{-\lambda}\sin\left(\lambda\arctan\frac{2q(1-p)}{(1-p)^2-q^2}\right),$$

and

$$\sum_{n=0}^{\infty}\binom{n+2\lambda-1}{n}C_n(p,q)=\left((1-p)^2+q^2\right)^{-\lambda}\cos\left(\lambda\arctan\frac{2q(p-1)}{(1-p)^2-q^2}\right),$$

where $\lambda>-\frac{1}{2}$ and $p^2+q^2<1$.

3.2 Trigonometric-Type Generating Functions of Laguerre Polynomials

For $\alpha>-1$, the Laguerre polynomials

$$L_n^{(\alpha)}(z)=\binom{n+\alpha}{n}{}_1F_1\left(\begin{array}{c}-n\\ \alpha+1\end{array}\,\middle|\,z\right)=\sum_{k=0}^{n}(-1)^k\frac{1}{k!}\binom{n+\alpha}{n-k}z^k,$$

satisfy the relation

$$(1-t)^{-\alpha-1}\exp\big(\frac{tz}{t-1}\big)=\sum_{n=0}^{\infty}L_n^{(\alpha)}(z)\,t^n \qquad |t|<1. \tag{37}$$

Hence, they can be extended through their sine and cosine generating functions as

$$(1-t)^{-\alpha-1}\exp\big(\frac{pt}{t-1}\big)\sin\big(\frac{qt}{t-1}\big)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(-1)^k\frac{1}{k!}\binom{n+\alpha}{n-k}S_k(p,q)\Big)t^n,$$

and

$$(1-t)^{-\alpha-1}\exp\big(\frac{pt}{t-1}\big)\cos\big(\frac{qt}{t-1}\big)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(-1)^k\frac{1}{k!}\binom{n+\alpha}{n-k}C_k(p,q)\Big)t^n. \tag{38}$$

An interesting case in (38) is when $q=\lambda p$ and $p=x$ leading to a generalization of the Laguerre polynomials as

$$L_n^{(\alpha)}(x;\lambda)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n+\alpha}{n-k}\big(1+\lambda^2\big)^{\frac{k}{2}}\cos\big(k\arctan\lambda\big)\,x^k.$$

Similarly, replacing $t=p\pm \mathrm{i}q$ in (37) respectively yields

(39)
$$-\Big((1-p)^2+q^2\Big)^{-\frac{\alpha+1}{2}}\exp\Big(\frac{(p(p-1)+q^2)z}{(1-p)^2+q^2}\Big)\sin\Big(\frac{qz}{(1-p)^2+q^2}+(\alpha+1)\arctan\frac{q}{p-1}\Big)$$
$$=\sum_{n=0}^{\infty}L_n^{(\alpha)}(z)\,S_n(p,q),$$

and

(40)
$$\Big((1-p)^2+q^2\Big)^{-\frac{\alpha+1}{2}}\exp\Big(\frac{(p(p-1)+q^2)z}{(1-p)^2+q^2}\Big)\cos\Big(\frac{qz}{(1-p)^2+q^2}+(\alpha+1)\arctan\frac{q}{p-1}\Big)$$
$$=\sum_{n=0}^{\infty}L_n^{(\alpha)}(z)\,C_n(p,q),$$

for $p^2+q^2<1$. For instance, if $q=2p$ in (39) and (40) then

$$\sum_{n=0}^{\infty}(\sqrt{5})^n\sin(n\arctan 2)\,L_n^{(\alpha)}(z)\,p^n=$$
$$-\Big(1-2p+5p^2\Big)^{-\frac{\alpha+1}{2}}\exp\Big(\frac{(5p-1)pz}{1-2p+5p^2}\Big)\sin\Big(\frac{2pz}{1-2p+5p^2}+(\alpha+1)\arctan\frac{2p}{p-1}\Big),$$

and

$$\sum_{n=0}^{\infty}(\sqrt{5})^n \cos(n \arctan 2)\, L_n^{(\alpha)}(z)\, p^n =$$

$$\Big(1-2p+5p^2\Big)^{-\frac{\alpha+1}{2}} \exp\Big(\frac{(5p-1)pz}{1-2p+5p^2}\Big) \cos\Big(\frac{2pz}{1-2p+5p^2}+(\alpha+1)\arctan\frac{2p}{p-1}\Big),$$

which are valid for any $p \in (-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5})$.

3.3 Generating Functions of Hermite Polynomials

The Hermite polynomials

$$H_n(z) = (2z)^n\, {}_2F_0\left(\begin{array}{cc} -\frac{n}{2}, & -\frac{n-1}{2} \\ & - \end{array}\Bigg| -z^{-2}\right) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k\, n!\, 2^{n-2k}}{k!(n-2k)!}\, z^{n-2k},$$

satisfy the relation

$$\exp\big(2tz - t^2\big) = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}. \tag{41}$$

Hence

$$\exp\big(2pt - t^2\big)\sin(2qt) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k\, n!\, 2^{n-2k}}{k!\,(n-2k)!}\, S_{n-2k}(p,q)\Big)\frac{t^n}{n!},$$

and

$$\exp\big(2pt - t^2\big)\cos(2qt) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k\, n!\, 2^{n-2k}}{k!\,(n-2k)!}\, C_{n-2k}(p,q)\Big)\frac{t^n}{n!},$$

are known as the first kind of the sine and cosine types of Hermite polynomials generating functions. Moreover, taking $t = p \pm \mathrm{i}q$ in (41) gives the second kind of the sine and cosine generating functions as follows

$$\exp\big(2pz + q^2 - p^2\big)\sin\big(2q(z-p)\big) = \sum_{n=0}^{\infty}\frac{H_n(z)}{n!}\, S_n(p,q),$$

and

$$\exp\big(2pz + q^2 - p^2\big)\cos\big(2q(z-p)\big) = \sum_{n=0}^{\infty}\frac{H_n(z)}{n!}\, C_n(p,q).$$

For instance, for $q = p$ we have

$$\exp(2pz)\sin\big(2p(z-p)\big) = \sum_{n=0}^{\infty}(\sqrt{2})^n \sin(\frac{n\pi}{4})\, H_n(z)\, \frac{p^n}{n!},$$

and

$$\exp(2pz)\cos\big(2p(z-p)\big)=\sum_{n=0}^{\infty}(\sqrt{2})^n\cos(\frac{n\pi}{4})\,H_n(z)\,\frac{p^n}{n!},$$

where $\lambda_n^*=\frac{(\sqrt{2})^n}{n!}\sin(\frac{n\pi}{4})$ and $\lambda_n^*=\frac{(\sqrt{2})^n}{n!}\cos(\frac{n\pi}{4})$ in (2) respectively.

4. SINE AND COSINE TYPES OF GENERATING FUNCTIONS OF SOME WIDELY-INVESTIGATED NUMBERS

Following the same approach, we can now define the sine and cosine types of generating functions of Sheffer polynomials and as special cases, introduce the trigonometric types of generating functions of some well-known sequences of numbers in the literature such as Bernoulli, Euler, Genocchi and in general Apostol type of them.

By noting the identity (3), the first kind of the sine and cosine types of Sheffer polynomials generating functions are given by

$$f(t)\exp\big(pH(t)\big)\sin\big(qH(t)\big)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{s_n^{(k)}(0)}{k!}\,S_k(p,q)\Big)\frac{t^n}{n!},$$

and

$$f(t)\exp\big(pH(t)\big)\cos\big(qH(t)\big)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{s_n^{(k)}(0)}{k!}\,C_k(p,q)\Big)\frac{t^n}{n!}.$$

Also, the second kinds respectively read as

$$\frac{1}{2\mathrm{i}}\Big(f(p+\mathrm{i}q)\exp\big(zH(p+\mathrm{i}q)\big)-f(p-\mathrm{i}q)\exp\big(zH(p-\mathrm{i}q)\big)\Big)=\sum_{n=0}^{\infty}\frac{s_n(z)}{n!}S_n(p,q),$$

and

$$\frac{1}{2}\Big(f(p+\mathrm{i}q)\exp\big(zH(p+\mathrm{i}q)\big)+f(p-\mathrm{i}q)\exp\big(zH(p-\mathrm{i}q)\big)\Big)=\sum_{n=0}^{\infty}\frac{s_n(z)}{n!}C_n(p,q).$$

4.1 Trigonometric Types of Generating Functions of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi Numbers

As we pointed out, the first kind of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and their associated numbers have been introduced in [**31**] by defining six specific generating functions. For instance, the first kind of the sine and cosine types of the generating functions for Apostol-Bernoulli polynomials are given by

$$\frac{t}{\lambda\mathrm{e}^t-1}\,\mathrm{e}^{pt}\sin qt=\sum_{n=0}^{\infty}B_n^{(s)}(p,q;\lambda)\frac{t^n}{n!},$$

and

$$\frac{t}{\lambda \mathrm{e}^{t}-1}\,\mathrm{e}^{pt}\cos qt=\sum_{n=0}^{\infty}B_{n}^{(c)}(p,q;\lambda)\frac{t^{n}}{n!},$$

in which

$$B_{n}^{(s)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda}S_{k}(p,q),$$
$$B_{n}^{(c)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda}C_{k}(p,q),$$

and $B_{n-k,\lambda}$ is defined according to (7).

Now to comput the second kind of the sine and cosine types of the generating functions for Apostol-Bernoulli polynomials, it is enough to refer to the generating function of these polynomials in (4) to respectively obtain

$$\frac{e^{pz}\Big(\big(\lambda e^{p}(p\cos q+q\sin q)-p\big)\sin(qz)-\big(\lambda e^{p}(p\sin q-q\cos q)-q\big)\cos(qz)\Big)}{\lambda e^{p}(\lambda e^{p}-2\cos q)+1}$$
$$=\sum_{n=0}^{\infty}B_{n}(z;\lambda)\frac{S_{n}(p,q)}{n!},$$

and

$$\frac{e^{pz}\Big(\big(\lambda e^{p}(p\cos q+q\sin q)-p\big)\cos(qz)+\big(\lambda e^{p}(p\sin q-q\cos q)-q\big)\sin(qz)\Big)}{\lambda e^{p}(\lambda e^{p}-2\cos q)+1}$$
$$=\sum_{n=0}^{\infty}B_{n}(z;\lambda)\frac{C_{n}(p,q)}{n!}.$$

Hence, the Apostol-Bernoulli numbers $B_{n,\lambda}=B_{n}(0;\lambda)$ satisfy the relations

$$\frac{q-\lambda e^{p}(p\sin q-q\cos q)}{\lambda e^{p}(\lambda e^{p}-2\cos q)+1}=\sum_{n=0}^{\infty}B_{n,\lambda}\frac{S_{n}(p,q)}{n!},$$

and

$$\frac{\lambda e^{p}(p\cos q+q\sin q)-p}{\lambda e^{p}(\lambda e^{p}-2\cos q)+1}=\sum_{n=0}^{\infty}B_{n,\lambda}\frac{C_{n}(p,q)}{n!}.$$

Similarly, by considering the generating function of Apostol-Euler polynomials in (5), it can be shown that the corresponding sine type is given by

$$\frac{2e^{pz}\Big(\sin(qz)+\lambda e^{p}\big(\cos q\sin(qz)-\sin q\cos(qz)\big)\Big)}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}=\sum_{n=0}^{\infty}E_{n}(z;\lambda)\frac{S_{n}(p,q)}{n!},$$

and the cosine type by

$$\frac{2e^{pz}\Big(\cos(qz)+\lambda e^{p}\big(\cos q\cos(qz)+\sin q\sin(qz)\big)\Big)}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}=\sum_{n=0}^{\infty}E_n(z;\lambda)\frac{C_n(p,q)}{n!}.$$

This means that the Apostol-Euler numbers $E_{n,\lambda}=E_n(0;\lambda)$ satisfy the relations

$$\frac{-2\,\lambda e^{p}\sin q}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}=\sum_{n=0}^{\infty}E_{n,\lambda}\frac{S_n(p,q)}{n!},$$

and

$$\frac{2\left(1+\lambda e^{p}\cos q\right)}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}=\sum_{n=0}^{\infty}E_{n,\lambda}\frac{C_n(p,q)}{n!}.$$

Finally, by considering the generating function of Apostol-Genocchi polynomials in (6), we respectively obtain

$$\frac{2e^{pz}\Big(\big(p+\lambda e^{p}(p\cos q+q\sin q)\big)\sin qz+\big(q+\lambda e^{p}(q\cos q-p\sin q)\big)\cos qz\Big)}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}$$
$$=\sum_{n=0}^{\infty}G_n(z;\lambda)\frac{S_n(p,q)}{n!},$$

and

$$\frac{2e^{pz}\Big(\big(p+\lambda e^{p}(p\cos q+q\sin q)\big)\cos qz-\big(q+\lambda e^{p}(q\cos q-p\sin q)\big)\sin qz\Big)}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}$$
$$=\sum_{n=0}^{\infty}G_n(z;\lambda)\frac{C_n(p,q)}{n!}.$$

Therefore, the Apostol-Genocchi numbers $G_{n,\lambda}=G_n(0;\lambda)$ satisfy the relations

$$\frac{2\left(q+\lambda e^{p}(q\cos q-p\sin q)\right)}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}=\sum_{n=0}^{\infty}G_{n,\lambda}\frac{S_n(p,q)}{n!},$$

and

$$\frac{2\left(p+\lambda e^{p}(p\cos q+q\sin q)\right)}{\lambda e^{p}(\lambda e^{p}+2\cos q)+1}=\sum_{n=0}^{\infty}G_{n,\lambda}\frac{C_n(p,q)}{n!}.$$

Note that sine and cosine types of generating functions of other well known sequences of numbers such as combinatorial numbers and polynomials associated with Peters polynomials [**30**] can be defined through our approach.

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