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ANOTHER TWO FAMILIES OF INTEGER–VALUED POLYNOMIALS ASSOCIATED WITH FINITE TRIGONOMETRIC SUMS

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As a sequel to our recent paper, its general approach was here extended to finite alternating trigonometric sums giving rise to polynomials which were systematically examined in full detail as well as in a unified manner using simple arguments. Two new general families of integer–valued polynomials (along with four other families derived from them, also integer–valued, including two already known) were deduced. Also, these polynomials enable closed-form summation of a great deal of general families of finite sums.

1. INTRODUCTION

A polynomial $P_n(x) := a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $n \in \mathbb{N} := \{1, 2, 3, \ldots, \}$, where the coefficients a_k, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, are, in general, rational numbers, is called *integer–valued* (numerical or integral–valued) if it takes an integer value whenever x is an integer. These polynomials have been studied considerably ever since (see, for instance, [10, pp. 129–133] and [4]; see also [7] for a recent interesting result) and the most known example is the sequence of binomial coefficients $\{\binom{x}{n}\}_{n=0}^{\infty}$ with $\binom{x}{0} = 1$ and $\binom{x}{n} = x(x-1)(x-2)\cdots(x-n+1)/n!$.

Observe that, by making use of the Lagrange interpolation formula, Byrne and Smith [3, Theorem 2] derived for the first time the integer–valued polynomial sequence associated with the cotangent sum $\sum_{p=1}^{q} \cot^{2n}[(2p-1)\pi/(4\,q)]$ whose coefficients could be determined recursively from certain relations. Two additional the integer–valued polynomials associated with $\sum_{p=1}^{q} \cot^{2n}[(2p-1)\pi/(2(2q+1))]$

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and $\sum_{p=1}^{q} \csc^{2n}\left[(2p-1)\pi/(2(2q+1))\right]$ were explicitly defined by Hassan (see, respectively, Theorem 4.3 and Eq. (3.18) together with Remark 4.5 (2) in [8, pp. 822 and 817]) utilizing a sampling theorem associated with second–order discrete eigenvalue problem.

Recently, in an attempt to examine in a more detail various integral–valued polynomials arising in summation of some finite trigonometric sums [1–3, 8], Cvijović [5, Theorem 1] deduced two new very general families of integer–valued polynomials with rational coefficients, $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$. In addition, six other polynomial families were derived from them, also integer–valued, including three previously studied [3, 8]. These polynomial sequences are associated with and provide easy closed-form summations of numerous families of finite trigonometric sums (for more details about this topic see, for instance, [6, 9] and relevant references therein) in a compact and simple form [5, Corollary 2], the most important instances being [5, Theorem 2]:

$$\sum_{p=1}^{q} \cot^{2n}\left(\frac{(2p-1)\pi}{2\,q}\right) = A_{2n}(q) \quad \text{and} \quad \sum_{p=1}^{q} \csc^{2n}\left(\frac{(2p-1)\pi}{2\,q}\right) = B_{2n}(q) \qquad (n, q \in \mathbb{N}).$$

Note, however, that, instead technical and specialized (numerical analysis) methods, by making use of simpler and more familiar arguments commonly used in work with polynomials in general, Cvijović [5] established the existence and properties of $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ (and their special cases) in a systematic and unified manner as well as in a general context.

Herein, as a sequel to our recent work on integer–valued polynomials, it was aimed to extend the paper's general approach [5] to finite alternating trigonometric sums giving rise to such polynomials. In doing so, by avoiding specialized methods, examples being the Lagrange and Hermite interpolation, the main intention was to provide, in a general context as well as in systematic and unified manner, more straightforward proofs for some already known and to (possibly) generate and prove new results using simple and more familiar arguments commonly applied in characterisation of polynomial sequences. In addition, it was also intended to fully examine closed–form summation of certain associated finite sums.

2. STATEMENT OF MAIN RESULTS

Observe that, throughout the text, as usual, we set an empty sum to be zero. Our main results are as follows.

Theorem 1. *Define two sequences of real functions in x, $\{\mathscr{A}_{2n+1}(x)\}_{n=0}^{\infty}$ and $\{\mathscr{B}_{2n+1}(x)\}_{n=0}^{\infty}$, by generating relations*

$$G_{\mathscr{A}}(x,t) = \sum_{n=0}^{\infty} \mathscr{A}_{2n+1}(x)\, t^{2n+1} \qquad \textit{and} \qquad G_{\mathscr{B}}(x,t) = \sum_{n=0}^{\infty} \mathscr{B}_{2n+1}(x)\, t^{2n+1},$$

where

$$G_{\mathscr{A}}(x,t) = \frac{2\,tx}{1+t^2}\,\sec\left[2x\arctan t\right]$$

and

$$G_{\mathscr{B}}(x,t) = \frac{(2\,x+1)\,t}{\sqrt{1-t^2}}\,\sec\left[(2\,x+1)\arcsin t\right].$$

Then, we have that $\{\mathscr{A}_{2n+1}(x)\}_{n=0}^{\infty}$ *and* $\{\mathscr{B}_{2n+1}(x)\}_{n=0}^{\infty}$ *are sequences of integer–valued polynomials with rational coefficients defined explicitly by*

(1)

$$\mathscr{A}_{2n+1}(x) = (-1)^n\,2\,x\sum_{k=0}^{2\,n}\frac{1}{2^k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{m=0}^{n}\binom{2\,(2\,l+1)\,x}{2\,m}\binom{(2\,l+1)\,x+n-m}{n-m}$$

and by

$$\mathscr{B}_{2n+1}(x) = (-1)^n\,(2\,x+1)\sum_{k=0}^{2\,n}2^{2\,n-k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{(2\,x+1)\,l+x+n}{2\,n}. \tag{2}$$

Also, $\{\mathscr{A}^{*}_{2n+1}(x)\}_{n=0}^{\infty}$, $\{\mathscr{B}^{*+}_{2n+1}(x)\}_{n=0}^{\infty}$, $\{\mathscr{B}^{*-}_{2n+1}(x)\}_{n=0}^{\infty}$ *and* $\{\mathscr{B}^{**}_{2n}(x)\}_{n=0}^{\infty}$ *with the polynomials given as* $\mathscr{A}^{*}_{2n+1}(x) = \mathscr{A}_{2n+1}(x)/2$, $\mathscr{B}^{*+}_{2n+1}(x) = (\mathscr{B}_{2n+1}(x)+1)/2$, $\mathscr{B}^{*-}_{2n+1}(x) = (\mathscr{B}_{2n+1}(x)-1)/2$ *and* $\mathscr{B}_{2n+1}(x) = (-1)^n(2\,x+1)\mathscr{B}^{**}_{2n}(x)$ *are sequences of integer–valued polynomials.*

Theorem 2. *Let* $\{\mathscr{A}_{2n+1}(x)\}_{n=0}^{\infty}$ *and* $\{\mathscr{B}_{2n+1}(x)\}_{n=0}^{\infty}$ *be the sequences of integer–valued polynomials defined as in Theorem* 1.

Then, for any non–negative integer n *and any positive integer* q*, we have that*

$$\sum_{p=1}^{2\,q}(-1)^{p-1}\cot^{2n+1}\left(\frac{(2p-1)\pi}{4\,q}\right) = \mathscr{A}_{2n+1}(q)$$

and

$$\sum_{p=1}^{2\,q+1}(-1)^{p-1}\sec^{2n+1}\left(\frac{p\,\pi}{2\,q+1}\right) = \mathscr{B}_{2n+1}(q).$$

Remark 1. Among six sequences of integer–valued polynomials given by Theorem 1, only $\{\mathscr{A}^{*}_{2n+1}(x)\}_{n=0}^{\infty}$ and $\{\mathscr{B}^{**}_{2n}(x)\}_{n=0}^{\infty}$ were studied previously.

Byrne and Smith [3, Theorem 1], evaluating the alternating cotangent sum (3) in closed form, established the polynomials $\mathscr{A}^{*}_{2n+1}(x)$ with coefficients specified by recursive relations, which was also used in the summation (up to a multiplicative constant) of the finite sum (4) [8, Eqs. 3.13 and 4.21]. Note, however, that $\mathscr{A}^{*}_{2n+1}(x)$ are here defined explicitly (divide the formula (1) by 2) and that they are special case of more general $\mathscr{A}_{2n+1}(x)$.

Hassan showed that the sums (5) [8, Eqs. 3.21 and 4.23] and (6) [8, Eq. 3.48] are integer–valued and summed them up by means of the following polynomials [8, Lemma 4.1]

$$2^n P_n(x) = \frac{2^n}{n!}\frac{\partial^n}{\partial\lambda^n}\,\frac{1}{P(x,\lambda)}\bigg|_{\lambda=1} \quad \text{with} \quad P(x) = \frac{1}{2}\Big[\cos\big((x+1)\,\theta\big) + \cos\big(x\,\theta\big)\Big],$$

where $\lambda = \cos\theta$, which, in essence, are our $\mathscr{B}^{**}_{2n}(x)$ (equal to the double sum in (2)) [*cf.* (5) and (6) with [8, Remark 4.2 and Eq. 4.23]; 2^{r-2} in [8, Eq. 4.23] should be 2^{r-1}].

Corollary 1. *In terms of the integer–valued polynomials introduced by Theorem* 1*, for non–negative integers* n *and positive integers* q*, the following summations hold*

$$\mathscr{A}^{*}_{2n+1}(q) = \frac{1}{2}\,\mathscr{A}_{2n+1}(q) = \sum_{p=1}^{q}(-1)^{p-1}\cot^{2n+1}\left(\frac{(2p-1)\pi}{4\,q}\right) \tag{3}$$

$$= (-1)^{q-1}\sum_{p=1}^{q}(-1)^{p-1}\tan^{2n+1}\left(\frac{(2p-1)\pi}{4\,q}\right), \tag{4}$$

$$\frac{1}{2}\Big(\mathscr{B}_{2n+1}(q) - (-1)^q\Big) = \sum_{p=1}^{q}(-1)^{p-1}\csc^{2n+1}\left(\frac{(2p-1)\pi}{2\,(2\,q+1)}\right) \tag{5}$$

$$= (-1)^q\sum_{p=1}^{q}\sec^{2n+1}\left(\frac{2\,p\pi}{2\,q+1}\right) = (-1)^{q-1}\sum_{p=1}^{q}\sec^{2n+1}\left(\frac{(2p-1)\pi}{2\,q+1}\right)$$

$$= (-1)^{q-1}\sum_{p=1}^{q}(-1)^{p-1}\sec^{2n+1}\left(\frac{p\,\pi}{2\,q+1}\right) = \frac{1}{2}\Big((-1)^n(2\,q+1)\,\mathscr{B}^{**}_{2n}(q) - (-1)^q\Big). \tag{6}$$

3. PROOF OF THE RESULTS

To establish the main results, for a better clarity of the proofs, we need several auxiliary (mainly known) results collected as two lemmas and proved in detail for the sake of a self-contained presentation. Observe that the summation in Lemma 1 (b) could not be found in the literature, and it may be of some independent interest.

Lemma 1. *Let* θ *and* δ *be real numbers and let* n *be a positive integer. Then, the following summations holds true:*

a) $$\sum_{k=0}^{n-1}(-1)^k \frac{\sin\left(\delta+\frac{(2\,k+1)\pi}{2n}\right)\cos\left(\delta+\frac{(2\,k+1)\pi}{2n}\right)}{\cos^2\theta-\cos^2\left(\delta+\frac{(2\,k+1)\pi}{2n}\right)} = \frac{2\,n\cos n\theta\cos n\delta}{\cos(2\,n\theta)+\cos(2\,n\delta)} \quad (n \text{ is even});$$

b) $$\sum_{k=0}^{n-1}(-1)^k \frac{\cos\left(\delta+\frac{k\pi}{n}\right)}{\sin^2\theta-\cos^2\left(\delta+\frac{k\pi}{n}\right)} = \frac{2\,n\sin\left(n\frac{\pi}{2}\right)\cos n\theta\cos n\delta}{\cos\theta\big(\cos(2\,n\theta)+\cos(2\,n\delta)\big)} \quad (n \text{ is odd}).$$

Proof of Lemma 1. It is necessary to begin by deriving the next two summation formulae (see, for instance, [13, Eqs. (1.0.2a) and (1.0.2b)],

$$\frac{n(1-x^{2n})}{x^{2n}-2\,x^n\cos n\delta+1} = \sum_{k=0}^{n-1}\frac{1-x^2}{x^2-2x\cos\left(\delta+\frac{2\,k\,\pi}{n}\right)+1} \tag{7}$$

and

$$\frac{nx^n\sin n\delta}{x^{2n}-2\,x^n\cos n\delta+1} = \sum_{k=0}^{n-1}\frac{x\,\sin\left(\delta+\frac{2\,k\,\pi}{n}\right)}{x^2-2x\cos\left(\delta+\frac{2\,k\,\pi}{n}\right)+1}. \tag{8}$$

Let $w_n = e^{\frac{2\,\pi\,\imath}{n}}$, $n\in\mathbb{N}$, be a primitive root of unity. Then, the factorisation $1-z^n = \prod_{k=0}^{n-1}\left(1-z\,w_n^k\right)$ of a polynomial $P_n(z) = 1-z^n$ leads to the partial fraction decomposition of $n/(1-z^n)$

$$\frac{n}{1-z^n} = \sum_{k=0}^{n-1}\frac{1}{1-z\,w_n^k}. \tag{9}$$

Upon setting $z = x\,e^{\imath\delta}$ in (9) followed by taking real and imaginary parts, one obtains respectively

$$\frac{n\big(1-x^n\cos n\delta\big)}{x^{2n}-2\,x^n\cos n\delta+1} = \sum_{k=0}^{n-1}\frac{1-x\,\cos\left(\delta+\frac{2\,k\,\pi}{n}\right)}{x^2-2x\cos\left(\delta+\frac{2\,k\,\pi}{n}\right)+1} \tag{10}$$

and the summation (8). Replace x by $1/x$ in (10), so that (10) yields

$$\frac{nx^n\big(x^n-\cos n\delta\big)}{x^{2n}-2\,x^n\cos n\delta+1} = \sum_{k=0}^{n-1}\frac{x^2-x\cos\left(\delta+\frac{2\,k\,\pi}{n}\right)}{x^2-2x\cos\left(\delta+\frac{2\,k\,\pi}{n}\right)+1}. \tag{11}$$

At last, the desired formula (7) results upon subtracting (11) from (10).

In order to establish Part (a) (*cf.* [13, Eq. (3.0.1a)]), first note that the summation

$$\sum_{k=0}^{n-1}\frac{\sin\left(\delta+\frac{k\pi}{n}\right)\cos\left(\delta+\frac{k\pi}{n}\right)}{\cos^2\theta-\cos^2\left(\delta+\frac{k\pi}{n}\right)} = \frac{n\sin(2\,n\delta)}{\cos(2\,n\theta)-\cos(2\,n\delta)} \tag{12}$$

follows at once from (8) on putting $x = e^{2\imath\theta}$ and $\cos^2\theta = (1+x)^2/(4\,x)$ in the expression LHS = RHS with

$$\text{LHS} := \frac{n x^n \sin(2\,n\delta)}{x^{2\,n} - 2\,x^n\ \cos(2\,n\delta) + 1} = \frac{n\sin(2\,n\delta)}{x^n + x^{-n} - 2\ \cos(2\,n\delta)}$$

and

$$\begin{aligned}\text{RHS} :=& \sum_{k=0}^{n-1} \frac{x\ \sin\left(2\,\delta + \frac{2\,k\,\pi}{n}\right)}{x^2 - 2x\cos\left(2\,\delta + \frac{2\,k\,\pi}{n}\right) + 1} = \sum_{k=0}^{n-1} \frac{x\,2\ \sin\left(\delta + \frac{k\,\pi}{n}\right)\cos\left(\delta + \frac{k\,\pi}{n}\right)}{(1+x)^2 - 2x\left[1 + \cos\left(2\,(\delta + \frac{k\,\pi}{n})\right)\right]}\\ =&\frac{1}{2}\sum_{k=0}^{n-1} \frac{\sin\left(\delta + \frac{k\,\pi}{n}\right)\cos\left(\delta + \frac{k\,\pi}{n}\right)}{(1+x)^2/(4\,x) - \cos^2\left(\delta + \frac{k\,\pi}{n}\right)}.\end{aligned}$$

Next, after replacing δ with $\delta + \pi/(2\,n)$, the summation formula (12) becomes

$$\sum_{k=0}^{n-1} \frac{\sin\left(\delta + \frac{(2\,k+1)\pi}{2n}\right)\cos\left(\delta + \frac{(2\,k+1)\pi}{2n}\right)}{\cos^2\theta - \cos^2\left(\delta + \frac{(2\,k+1)\pi}{2n}\right)} = -\frac{n\sin(2\,n\delta)}{\cos(2\,n\theta) + \cos(2\,n\delta)}, \tag{13}$$

while (13) could be further rearranged to give

$$\sum_{\substack{n=0\\ k\ is\ even}}^{n-1} \frac{2\ \sin\left(\delta + \frac{(2\,k+1)\pi}{2n}\right)\cos\left(\delta + \frac{(2\,k+1)\pi}{2n}\right)}{\cos^2\theta - \cos^2\left(\delta + \frac{(2\,k+1)\pi}{2n}\right)} = \frac{n\cos n\delta}{\sin n\delta + \cos n\theta}. \tag{14}$$

Subtracting (13) from (14) results finally in the proposed summation in Part (a).

To prove Part (b), we need the following easily derivable summation

$$\sum_{k=0}^{n-1} \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos^2\left(\delta + \frac{k\pi}{n}\right)} = \frac{n\sin(2\,n\theta)}{\cos(2\,n\theta) - \cos(2\,n\delta)} \qquad (n \in \mathbb{N}). \tag{15}$$

Indeed, the deduction of (15) is enabled by (7) on putting $x = e^{2\imath\theta}$ and $\cos^2\theta = (1+x)^2/(4\,x)$ in LHS = RHS, with

$$\text{LHS} := \frac{n(1 - x^{2\,n})(1+x)^2}{\left(x^{2\,n} - 2\,x^n\ \cos(2\,n\delta) + 1\right)\left(1 - x^2\right)} = -\frac{n(x^n - x^{-n})}{x^n + x^{-n} - 2\ \cos(2\,n\delta)}\frac{1+x}{1-x}$$

and

$$\text{RHS} := \sum_{k=0}^{n-1} \frac{(1+x)^2}{1 + x^2 - 2\,x\cos\left(2\,\delta + \frac{2\,k\pi}{n}\right)} = \sum_{k=0}^{n-1} \frac{(1+x)^2/(4x)}{(1+x)^2/(4x) - \cos^2\left(\delta + \frac{k\,\pi}{n}\right)}.$$

On the other hand, (7) can be also rewritten as

$$\frac{n(1-x^{2n})(1+x^2)}{(1+x^{2n}-2\,x^n\cos n\delta)(1-x^2)}=\sum_{k=0}^{n-1}\frac{(1+x^2)/(2\,x)}{(1+x^2)/(2\,x)-\cos\left(\delta+\frac{2\,k\,\pi}{n}\right)},$$

which, through substitutions $x=e^{\imath\theta}$ and $(1+x^2)/(2\,x)=\cos\theta$, results in

$$\frac{n\sin n\theta\cot\theta}{\cos n\theta-\cos n\delta}=\sum_{k=0}^{n-1}\frac{\cos\theta}{\cos\theta-\cos\left(\delta+\frac{2k\pi}{n}\right)}=\sum_{k=0}^{n-1}\frac{\cos\theta}{\cos\theta-(-1)^k\cos\left(\delta+\frac{k\pi}{n}\right)} \tag{16}$$

when n is an odd natural number. Now, the difference between (16) and (15) times $\cos\theta$ gives

$$\sum_{k=0}^{n-1}(-1)^k\frac{\cos\left(\delta+\frac{k\pi}{n}\right)}{\cos^2\theta-\cos^2\left(\delta+\frac{k\pi}{n}\right)}=\frac{2\,n\sin n\theta\cos n\delta}{\sin\theta\big(\cos(2\,n\theta)-\cos(2\,n\delta)\big)}\qquad (n\,\text{is}\,\text{odd}).$$

Finally, the sought formula given in Part (b) follows from the last equation after replacing θ with $\theta+\pi/2$.

Lemma 2. *We have that:*

a) *The formal power series expansion of the secant function is given by*

$$\sec\theta=\sum_{k=0}^{\infty}2^{-k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}e^{\imath(2\,l+1)\theta};$$

b) *If $t=\tan\theta$, then*

$$\operatorname{Re}\left(e^{\alpha\imath\theta}\right)=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{2\,k}\frac{t^{2\,k}}{(\sqrt{1+t^2})^{\alpha}};$$

c) *If $t=\sin\theta$, then*

$$\operatorname{Re}\left(e^{\alpha\imath\theta}\right)=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{2\,k}t^{2\,k}(\sqrt{1-t^2})^{\alpha-2\,k}.$$

Proof of Lemma 2. The required formal power series follows without difficulty

$$\sec\theta=\frac{e^{\imath\theta}}{1-(1-e^{2\,\imath\theta})/2}=e^{\imath\theta}\sum_{k=0}^{\infty}2^{-k}\big(1-e^{2\,\imath\theta}\big)^k=e^{\imath\theta}\sum_{k=0}^{\infty}2^{-k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}e^{\imath 2\,l\theta}.$$

To obtain $\operatorname{Re}\left(e^{\alpha\imath\theta}\right)$ with $t=\tan\theta$, recall that $\tan\theta=\imath(e^{2\imath\theta}-1)/(e^{2\imath\theta}+1)$, then

$$e^{2\,\imath\theta}=\frac{1-\imath t}{1+\imath t}=\frac{(1-\imath t)^2}{1+t^2},\qquad\text{hence}\qquad e^{\imath\theta}=\frac{1-\imath t}{\sqrt{1+t^2}},$$

and the sought follows by the expansion of $1 - \imath t$. Likewise, $e^{\imath\theta} = \imath t + \sqrt{1-t^2}$ comes from $t = \sin\theta$ and $2\,\imath\sin\theta = e^{\imath\theta} - e^{-\imath\theta}$, thus the series of $\imath t + \sqrt{1-t^2}$ gives $\mathrm{Re}\left(e^{\alpha\imath\theta}\right)$.

Proof of Corollary 1. Knowing that Theorem 2 holds (see its proof below), the proposed summations are readily derivable by simple arguments and by making use of elementary series identities. For instance, to show that

$$S_1 := \sum_{p=1}^{q} (-1)^{p-1} \cot^{2\,n-1}\left[(2\,p-1)\pi/(4\,q)\right] = \mathscr{A}_{2n-1}(q)/2,$$

it suffices to note that $\mathscr{A}_{2n-1}(q) = S_1 + S_2$ where

$$S_2 := \sum_{p=q+1}^{2\,q} (-1)^{p-1} \cot^{2\,n-1}\left[(2\,p-1)\pi/(4\,q)\right]$$

amounts to

$$S_2 = \sum_{p=1}^{q} (-1)^{2\,q-p} \cot^{2\,n-1}\left(\frac{\left[2\,(2\,q+1-p)-1\right]\pi}{4\,q}\right) = S_1.$$

Proof of Theorem 1. This theorem gives explicit definitions of the polynomials $\mathscr{A}_{2n+1}(x)$ and $\mathscr{B}_{2n+1}(x)$ and to deduce them we need Lemma 2.

To derive the proposed formula for $\mathscr{A}_{2n+1}(x)$ in (1), set $t = \tan\theta$ and combine Parts (a) and (b) of Lemma 2. Then we have

$$\begin{aligned} G_{\mathscr{A}}(x,t) &= \frac{2\,tx}{1+t^2} \sec\left[2\,x\arctan t\right] = \frac{2\,tx}{1+t^2} \sum_{k=0}^{\infty} 2^{-k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \mathrm{Re}\left(e^{\imath 2\,(2\,l+1)\,x\theta}\right) \\ &= x \sum_{k=0}^{\infty} 2^{1-k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} X_{m,\,l}(x,t) \qquad (17) \end{aligned}$$

where $X_{m,\,l}(x,t)$ can be further rearranged as follows

$$\begin{aligned} X_{m,\,l}(x,t) &= \sum_{m=0}^{\infty} (-1)^m \binom{2(2\,l+1)x}{2\,m} \frac{t^{2\,m+1}}{\left(1+t^2\right)^{(2\,l+1)x+1}} \\ &= \sum_{m=0}^{\infty} (-1)^m t^{2\,m+1} \binom{2(2\,l+1)x}{2\,m} \sum_{n=0}^{\infty} (-1)^n t^{2\,n} \binom{(2\,l+1)x+n}{n} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{m+n} t^{2\,m+1+2\,n} \binom{2(2\,l+1)x}{2\,m} \binom{(2\,l+1)x+n}{n} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{(n-m)+m} t^{2\,m+1+2\,(n-m)} \binom{2(2\,l+1)x}{2\,m} \binom{(2\,l+1)x+n-m}{n-m} \\ &= \sum_{n=0}^{\infty} (-1)^n t^{2\,n+1} \sum_{m=0}^{n} \binom{2(2\,l+1)x}{2\,m} \binom{(2\,l+1)x+n-m}{n-m} \qquad (18) \end{aligned}$$

by making use of the familiar summation $1/(1-t)^{\alpha+1} = \sum_{n=0}^{\infty}\binom{\alpha+n}{n}t^n$ [11, Eq. 5.2.11.16, p. 710] as well as the identity $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,\,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,\,n-k}$ [12, Eq. (1), p. 56]. Next, in view of both equations, (17) and (18), coefficients with odd powers of t in the expansion of $G_{\mathscr{A}}(x,t)$ with respect to t are:

$$(-1)^n\, x \sum_{k=0}^{\infty} 2^{1-k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \sum_{m=0}^{n} \binom{2(2\,l+1)x}{2\,m}\binom{(2\,l+1)x+n-m}{n-m}.$$

The inner double sum (with respect to l and m) in the last expression is the kth difference operation on a polynomial with degree $2\,n$, therefore it will vanish if $k > 2\,n$. Hence, the upper limit for k is $2\,n$, which leads to the explicit formula for the polynomial sequence $\{\mathscr{A}_{2n+1}(x)\}_{n=0}^{\infty}$ given in (1).

To deduce the formula for the polynomials $\mathscr{B}_{2n+1}(x)$ in (2) we apply Parts (a) and (c) of Lemma 2 and set $t = \sin\theta$.

$$\begin{aligned} G_{\mathscr{B}}(x,t) &= \frac{(2\,x+1)\,t}{\sqrt{1-t^2}} \sec\big[(2\,x+1)\arcsin t\big] \\ &= \frac{(2\,x+1)\,t}{\sqrt{1-t^2}} \sum_{k=0}^{\infty} 2^{-k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \operatorname{Re}\big(e^{\imath(2\,l+1)(2\,x+1)\theta}\big) \\ &= (2\,x+1) \sum_{k=0}^{\infty} 2^{-k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} Y_{m,\,l}(x,t) \qquad (19) \end{aligned}$$

with $Y_{m,\,l}(x,t)$ given by

$$\begin{aligned} Y_{m,\,l}(x,t) &= \sum_{m=0}^{\infty} (-1)^m t^{2\,m+1} \binom{(2\,l+1)(2\,x+1)}{2\,m} \big(\sqrt{1-t^2}\big)^{2(2\,l+1)x+2(l-m)} \\ &= \sum_{m=0}^{\infty} (-1)^m t^{2\,m+1} \binom{(2\,l+1)(2\,x+1)}{2\,m} \sum_{n=0}^{\infty} (-1)^n t^{2\,n} \binom{(2\,x+1)\,l+x-m}{n} \\ &= \sum_{n=0}^{\infty} (-1)^n t^{2\,n+1} \sum_{m=0}^{n} \binom{(2\,l+1)(2\,x+1)}{2\,m}\binom{(2\,x+1)\,l+x-m}{n-m} \\ &= \sum_{n=0}^{\infty} (-1)^n t^{2\,n+1} 2^{2\,n} \binom{(2\,x+1)\,l+x+n}{2\,n} \qquad (20) \end{aligned}$$

where, to obtain the second, third and last line, we utilized respectively the binomial summation $(1+x)^{\alpha} = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$, the above–listed double series identity and the following identity involving binomial coefficients $\sum_{m=0}^{n}\binom{2\,a+1}{2\,m}\binom{a-m}{n-m} = 2^{2\,n}\binom{a+n}{2\,n}$ [11, Eq. 4.2.5.81, p. 621]. Finally, the explicit formula for the polynomial sequence $\{\mathscr{B}_{2n+1}(x)\}_{n=0}^{\infty}$ given by (2) follows by extracting coefficients with odd powers of t in the expansion of $G_{\mathscr{B}}(x,t)$ (*i.e.,* (19) and (20) combined together) with respect to t and using the above–detailed "difference operation" argument.

To conclude the proof, it is known that $\mathscr{A}^{*}_{2n+1}(x)$ [3, Theorem 1] and $\mathscr{B}^{**}_{2n}(x)$ [8, Eq. 4.23] are integer–valued (see Remark 1). What remains is to demonstrate that $\mathscr{A}_{2n+1}(x)$ and $\mathscr{B}_{2n+1}(x)$ as well as their two other special cases introduced by Theorem 1 are all integer–valued. For $\mathscr{A}_{2n+1}(x)$ and $\mathscr{B}_{2n+1}(x)$ this is straightforward: recall the relationships $\mathscr{A}_{2n+1}(x) = 2\,\mathscr{A}^{*}_{2n+1}(x)$ and $\mathscr{B}_{2n+1}(x) = (-1)^n(2\,x+1)\,\mathscr{B}^{**}_{2n}(x)$. Moreover, since $\mathscr{B}^{**}_{2n}(x)$ is always an odd number for any integer x [8, Eq. 4.23], it follows then that $\mathscr{B}_{2n+1}(x)$ is such, hence $\mathscr{B}^{*+}_{2n+1}(x)$ and $\mathscr{B}^{*-}_{2n+1}(x)$ are integer–valued too on account of their definitions.

Proof of Theorem 2. To prove this theorem the main ingredient in the proof is Lemma 1.

The summation formula in Lemma 1 (a) (which is valid for even positive integers) enables verification of the proposed summation involving the integer–valued polynomials $\mathscr{A}_{2n+1}(x)$. Indeed, bearing in mind that q is a positive integer and that $t=\tan\theta$, we have

$$
\begin{aligned}
G_{\mathscr{A}}(x,t)\big|_{x=q} &= \frac{2\,xt}{1+t^2}\sec\left(2\,x\arctan t\right)\bigg|_{x=q} = \frac{\sin\theta\sin\theta\,2\,x\cos x\theta\cos x\delta}{\cos(2\,x\delta)+\cos(2\,x\theta)}\bigg|_{x=2\,q,\delta=0}\\
&= \sum_{p=1}^{2\,q}(-1)^{p-1}\frac{\sin\theta\cos\theta\sin\left(\frac{(2\,p-1)\pi}{4\,q}\right)\cos\left(\frac{(2\,p-1)\pi}{4\,q}\right)}{\cos^2\theta-\cos^2\left(\frac{(2\,p-1)\pi}{4\,q}\right)}\\
&= \sum_{p=1}^{2\,q}(-1)^{p-1}\frac{\sin\theta\cos\theta\sin\left(\frac{(2\,p-1)\pi}{4\,q}\right)\cos\left(\frac{(2\,p-1)\pi}{4\,q}\right)}{\sin^2\left(\frac{(2\,p-1)\pi}{4\,q}\right)\cos^2\theta-\cos^2\left(\frac{(2\,p-1)\pi}{4\,q}\right)\sin^2\theta}\\
&= \sum_{p=1}^{2\,q}(-1)^{p-1}\frac{\tan\theta}{\tan\left(\frac{(2\,p-1)\pi}{4\,q}\right)}\,\frac{\tan^2\left(\frac{(2\,p-1)\pi}{4\,q}\right)}{\tan^2\left(\frac{(2\,p-1)\pi}{4\,q}\right)-\tan^2\theta}\\
&= \sum_{p=1}^{2\,q}(-1)^{p-1}\frac{\tan\theta}{\tan\left(\frac{(2\,p-1)\pi}{4\,q}\right)}\sum_{n=0}^{\infty}\left(\frac{\tan\theta}{\tan\left(\frac{(2\,p-1)\pi}{4\,q}\right)}\right)^{2\,n}\\
&= \sum_{n=0}^{\infty}\left(\sum_{p=1}^{2\,q}(-1)^{p-1}\cot^{2\,n+1}\left(\frac{(2\,p-1)\pi}{4\,q}\right)\right)\tan^{2\,n+1}\theta\\
&= \sum_{n=0}^{\infty}\mathscr{A}_{2\,n+1}(x)\,t^{2\,n+1}.
\end{aligned}
$$

Analogously, set $t=\sin\theta$ and use Lemma 1 (b) (which is valid for odd positive integers) to obtain summation involving the integer–valued polynomials $\mathscr{B}_{2n+1}(x)$:

$$\begin{aligned}
G_{\mathscr{B}}(x,t)\big|_{x=q} &= \frac{(2\,xt+1)\,t}{\sqrt{1-t^2}}\sec\big((2\,x+1)\arcsin t\big)\bigg|_{x=q} \\
&= \sin\theta\;\frac{2\,x\sin(x\frac{\pi}{2})\cos x\theta\cos x\delta}{\cos\theta\Big(\cos(2\,x\delta)+\cos(2\,x\theta)\Big)}\Bigg|_{\delta=\frac{\pi}{x},\,x=2\,q+1} \\
&= \sum_{p=1}^{2\,q+1}(-1)^{p-1}\frac{\sin\theta\cos\left(\frac{p\,\pi}{2\,q+1}\right)}{\sin^2\theta-\cos^2\left(\frac{p\,\pi}{2\,q+1}\right)} \\
&= \sum_{p=1}^{2\,q+1}(-1)^{p-1}\frac{\sin\theta}{\cos\left(\frac{p\,\pi}{2\,q+1}\right)}\sum_{n=0}^{\infty}\left(\frac{\sin\theta}{\cos\left(\frac{p\,\pi}{2\,q+1}\right)}\right)^{2\,n} \\
&= \sum_{n=0}^{\infty}\left(\sum_{p=1}^{2\,q+1}(-1)^{p-1}\sec^{2\,n+1}\left(\frac{p\,\pi}{2\,q+1}\right)\right)\sin^{2\,n+1}\theta \\
&= \sum_{n=0}^{\infty}\mathscr{B}_{2\,n+1}(x)\,t^{2\,n+1}.
\end{aligned}$$

4. CONCLUDING REMARKS

Two new very general families of integer–valued polynomials with rational coefficients which are associated with finite alternating trigonometric summation, $\mathscr{A}_{2n+1}(x)$ and $\mathscr{B}_{2n+1}(x)$, were introduced. As illustrative examples for reference, we list a few of them, respectively generated by

$$\mathscr{A}_{2n+1}(x) = \frac{1}{(2n+1)!}\,\frac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}}\,G_{\mathscr{A}}(x,t)\bigg|_{t=0} \qquad (n\in\mathbb{N}_0)$$

and

$$\mathscr{B}_{2n+1}(x) = \frac{1}{(2n+1)!}\,\frac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}}\,G_{\mathscr{B}}(x,t)\bigg|_{t=0} \qquad (n\in\mathbb{N}_0)$$

or by means of the corresponding formula given in Theorem 1. In general, $\mathscr{A}_{2n+1}(x)$ is of degree $2\,n+1$ in x, $\mathscr{A}_{2n+1}(0)=0$ and these first several polynomials are

$$\begin{aligned}
&\mathscr{A}_1(x) = 2\,x, \\
&\mathscr{A}_3(x) = 4\,x^3 - 2\,x, \\
&\mathscr{A}_5(x) = \tfrac{20}{3}\,x^5 - \tfrac{20}{3}\,x^3 + 2\,x, \\
&\mathscr{A}_7(x) = \tfrac{488}{45}\,x^7 - \tfrac{140}{9}\,x^5 + \tfrac{392}{45}x^3 - 2\,x \\
&\mathscr{A}_9(x) = \tfrac{1108}{63}\,x^9 - \tfrac{488}{15}\,x^7 + \tfrac{76}{3}\,x^5 - \tfrac{3272}{315}\,x^3 + 2\,x \\
&\mathscr{A}_{11}(x) = \tfrac{404168}{14175}\,x^{11} - \tfrac{12188}{189}\,x^9 + \tfrac{42944}{675}\,x^7 - \tfrac{20108}{567}\,x^5 + \tfrac{2068}{175}\,x^3 - 2\,x.
\end{aligned}$$

Likewise, $\mathscr{B}_{2\,n+1}(x)$ are of degree $2\,n+1$, $\mathscr{B}_{2\,n+1}(0)=1$ and first examples are

$$
\begin{aligned}
\mathscr{B}_1(x) &= 2\,x+1,\\
\mathscr{B}_3(x) &= 4x^3+6x^2+4x+1,\\
\mathscr{B}_5(x) &= \tfrac{20}{3}\,x^5+\tfrac{50}{3}\,x^4+20x^3+\tfrac{40}{3}\,x^2+\tfrac{16}{3}\,x+1,\\
\mathscr{B}_7(x) &= \tfrac{488}{45}\,x^7+\tfrac{1708}{45}\,x^6+\tfrac{2912}{45}x^5+\tfrac{602}{9}x^4+\tfrac{2072}{45}x^3+\tfrac{952}{45}x^2+\tfrac{32}{5}x+1\\
\mathscr{B}_9(x) &= \tfrac{1108}{63}\,x^9+\tfrac{554}{7}\,x^8+\tfrac{18328}{105}\,x^7+\tfrac{1208}{5}\,x^6+\tfrac{3476}{15}\,x^5+160x^4\\
&\quad+\tfrac{25456}{315}\,x^3+\tfrac{1024}{35}\,x^2+\tfrac{256}{35}\,x+1\\
\mathscr{B}_{11}(x) &= \tfrac{404168}{14175}\,x^{11}+\tfrac{2222924}{14175}\,x^{10}+\tfrac{1202872}{2835}\,x^9+\tfrac{98978}{135}\,x^8+\tfrac{1413016}{1575}\,x^7+\tfrac{551012}{675}\,x^6\\
&\quad+\tfrac{1606088}{2835}\,x^5+\tfrac{856592}{2835}\,x^4+\tfrac{1749088}{14175}\,x^3+\tfrac{2816}{75}\,x^2+\tfrac{512}{63}\,x+1.
\end{aligned}
$$

In addition, four more example sets for $\mathscr{A}^{*}_{2\,n+1}(x)$, $\mathscr{B}^{*+}_{2\,n+1}(x)$, $\mathscr{B}^{*-}_{2\,n+1}(x)$ and $B^{**}_{2n}(x)$ are readily available, since $\mathscr{A}_{2\,n+1}(x)$ and $\mathscr{B}_{2\,n+1}(x)$ include these integer–valued polynomials as special cases. Observe that four of these six polynomial sequences were not previously studied, $\mathscr{A}_{2\,n+1}(x)$, $\mathscr{B}_{2\,n+1}(x)$, $\mathscr{B}^{*+}_{2\,n+1}(x)$ and $\mathscr{B}^{*-}_{2\,n+1}(x)$. The considered polynomials enable closed-form summation of a great deal of general families of finite sums involving odd-powered trigonometric functions.

In conclusion, it is noteworthy that, thanks to considering integral–valued polynomials arising in finite summation of various trigonometric sums by making use of simple and familiar arguments commonly used in work with polynomials in general, instead of applying rather specialized methods of previous works in this topic, more straightforward proofs for some already known and a number of new results were provided in a general context as well as in a systematic and unified manner.

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