Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 697–709.
https://doi.org/10.2298/AADM190221047A

A NOTE ON POLYLOGARITHMS AND INCOMPLETE GAMMA FUNCTION

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

Aykut Ahmet Aygüneş

In this paper, we firstly introduce the polylogarithms and incomplete gamma function. Then, we claim that there is a relation between polylogarithms and a generalization of incomplete gamma function. Secondly, we give a formula related to polylogarithms. Also,we obtain a relation between incomplete gamma function and the derivatives of polylogarithms. Finally, we find a generating function for the values of incomplete gamma function.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Throughout this article, we use the following standard notations:

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Also,

$$\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and the n-th derivative of any function f at z_0 is denoted by $f^{(n)}(z_0)$.

The polylogarithm (or de Jonquiére's function) $Li_s(z)$ (cf. [**9**]) is defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1)$$

2010 Mathematics Subject Classification. Primary 11G55, 11M35 ; Secondary 33B15, 33B20.
Keywords and Phrases. Polylogarithms, incomplete gamma function, a generalization of incomplete gamma function.

$$(s \in \mathbb{C} \text{ when } |z| < 1; \; Re(s) > 1 \text{ when } |z| = 1).$$

where $\Phi(z, s, w)$ is the Hurwitz-Lerch zeta function (cf. [**2**], [**5**]) defined by

$$\Phi(z, s, w) = \sum_{n=0}^{\infty} \frac{z^n}{(n+w)^s}$$

$$(s \in \mathbb{C} \text{ when } |z| < 1; \; Re(s) > 1 \text{ when } |z| = 1)$$

for $w \in \mathbb{C} \backslash \{0, -1, -2, -3, \cdots\}$.

The integral representation of Hurwitz-Lerch zeta function is as follows (cf. [**5**]):

$$\Phi(z, s, w) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 \frac{(\log t)^{s-1} t^{w-1}}{1 - zt} dt \tag{1.1}$$

or

$$\Phi(z, s, w) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-wt}}{1 - ze^{-t}} dt. \tag{1.2}$$

Let $N \in \mathbb{N}_0$. For $w = 1$ and $s = N + 1$, by using equation (1.1), we obtain

$$\begin{aligned} Li_{N+1}(z) &= z\Phi(z, N+1, 1) \\ &= z\frac{(-1)^N}{N!} \int_0^1 \frac{(\log t)^N}{1 - zt} dt. \end{aligned} \tag{1.3}$$

By choosing $z = 1$ in (1.3), the polylogarithms can be reduced to Riemann zeta function $\zeta(N+1)$ (cf. [**4**]) given by

$$\zeta(N+1) = \sum_{n=1}^{\infty} \frac{1}{n^{N+1}}$$

where $N > 0$.

Recently, many authors have investigated the poly-Bernoulli polynomials $B_n^{(k)}(x)$ (cf. [**6**], [**7**], [**8**]), by using the properties of polylogarithms. A. Bayad and Y. Hamahata gave a generating function of $B_n^{(k)}(x)$ as follows (cf. [**1**]):

$$\frac{Li_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}$$

for every integer k.

Also, Kaneko defined $B_n^{(k)}$ for every integer k by generating function (cf. [**12**])

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}$$

where $B_n^{(k)} = B_n^{(k)}(0)$. In [**12**], an explicit formula for $B_n^{(k)}$ was given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ \begin{array}{c} n \\ m \end{array} \right\}}{(m+1)^k}$$

where $\left\{ \begin{array}{c} n \\ m \end{array} \right\}$ is the Stirling number of the second kind (cf. [**13**]) defined by

$$\frac{1}{m!}(e^t-1)^m = \sum_{n=0}^{\infty} \left\{ \begin{array}{c} n \\ m \end{array} \right\} \frac{t^n}{n!}.$$

Polylogarithms also appear in generating function of generalized harmonic numbers $H_{n,r}$ (cf. [**10**]) defined by

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r}$$

where $r \in \mathbb{C}$.

From [**1**] and [**12**], we note that the polylogarithms are associated with poly-Bernoulli polynomials, specially poly-Bernoulli numbers and the Stirling number of the second kind.

Also, we introduce the incomplete gamma function $\Gamma(s,x)$ (cf. [**10**]):

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t} dt \tag{1.4}$$

where $Re(s) > 0$ and $x \in \mathbb{R}$.

By choosing $x = 0$ in (1.4), we obtain the classical Euler gamma function (cf. [**10**]) given by

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt.$$

Polylogarithms and incomplete gamma function are useful functions in Analytic Number Theory and Mathematical Physics.

2. A GENERALIZATION OF INCOMPLETE GAMMA FUNCTION

In this section we introduce a generalization of incomplete gamma function $\Gamma_{\mu,z}(s,x)$. Then, by using the integral representation of polylogarithms $Li_{N+1}(z)$, we obtain a relation between polylogarithms and a general case of incomplete gamma function.

Recently, some authors have studied on generalizations of incomplete gamma function $\Gamma(\alpha, x; \beta)$ for $Re(\alpha) > 0$ and $\beta \in \mathbb{C}$.

In [**11**], Chaudhry and Zubair studied on the generalized incomplete gamma function given by the following integral:

$$\Gamma(\alpha, x; \beta) = \int_x^\infty t^{\alpha-1} e^{-t-\beta t^{-1}} dt. \tag{1.5}$$

In [**3**], Miller derived several reduction formulas for specializations of a certain generalized incomplete gamma function $\Gamma(\alpha, x; \beta)$ and its associated Kampé De Fériet function.

It is possible to define a generalization of different type from equation (1.5). Let $\mu \in \mathbb{R}$. Then, we define

$$\Gamma_{\mu,z}(N+1, x) = \int_x^\infty \frac{t^N e^{-\mu t}}{1 - z e^{-\mu t}} dt.$$

In this section, we claim that $\Gamma_{\mu,z}(s,x)$ is associated with polylogarithms. Firstly, we give the following key theorem for our claim:

Theorem 1. *Let $b > a > 0$ and $N \in \mathbb{N}_0$. Then, we have*

$$\int_{\log a}^{\log b} \frac{t^N e^t}{1 - z e^t} dt = \frac{1}{z} \sum_{k=0}^{N} \binom{N}{k} (-1)^k k! \left\{ (\log b)^{N-k} Li_{k+1}(bz) - (\log a)^{N-k} Li_{k+1}(az) \right\}.$$

Proof. From equation (1.3), we get

$$\int_0^1 \frac{(\log t)^N}{1 - zt} dt = N!(-1)^N \frac{Li_{N+1}(z)}{z}.$$

We have

$$\int_a^b \frac{(\log t)^N}{1 - zt} dt = \int_0^{b-a} \frac{(\log(u+a))^N}{1 - za - zu} du, \tag{1.6}$$

by substituting $t = u + a$.

Then we have

$$\int_0^{b-a} \frac{(\log(u+a))^N}{1-za-zu}dt = \int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v}(b-a)dv, \tag{1.7}$$

by substituting $u = (b-a)v$ into the right side of equation (1.7).

From (1.6) and (1.7), we arrive

$$\int_a^b \frac{(\log t)^N}{1-zt}dt = \int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v}(b-a)dv. \tag{1.8}$$

On the other hand,

$$\int_0^b \frac{(\log t)^N}{1-zt}dt = \int_0^1 \frac{(\log b + \log u)^N}{1-bzu}bdu,$$

by substituting $t = bu$.

By using binomial expansion,

$$\begin{aligned}
\int_0^b \frac{(\log t)^N}{1-zt}dt &= \int_0^1 \frac{(\log b + \log u)^N}{1-bzu}bdu \\
&= b\int_0^1 \frac{1}{1-bzu}\left\{\sum_{k=0}^{N}\binom{N}{k}(\log b)^{N-k}(\log u)^k\right\}du \\
&= b\sum_{k=0}^{N}\binom{N}{k}(\log b)^{N-k}\left\{\int_0^1 \frac{(\log u)^k}{1-bzu}du\right\} \\
&= b\sum_{k=0}^{N}\binom{N}{k}(\log b)^{N-k}k!(-1)^k\frac{Li_{k+1}(bz)}{bz} \\
&= \frac{1}{z}\sum_{k=0}^{N}\binom{N}{k}(\log b)^{N-k}k!(-1)^k Li_{k+1}(bz). \qquad (1.9)
\end{aligned}$$

Similarly,

$$\int_0^a \frac{(\log t)^N}{1-zt}dt = \frac{1}{z}\sum_{k=0}^{N}\binom{N}{k}(\log a)^{N-k}k!(-1)^k Li_{k+1}(az). \tag{1.10}$$

Suppose that $b > a > 0$. Then, we set

$$\int_a^b \frac{(\log t)^N}{1-zt}dt = \int_0^b \frac{(\log t)^N}{1-zt}dt - \int_0^a \frac{(\log t)^N}{1-zt}dt.$$

From (1.9) and (1.10), we have

(1.11)
$$\int_a^b \frac{(\log t)^N}{1-zt}dt = \frac{1}{z}\sum_{k=0}^{N}\binom{N}{k} k!(-1)^k \left\{(\log b)^{N-k}Li_{k+1}(bz) - (\log a)^{N-k}Li_{k+1}(az)\right\}.$$

By using (1.8) and (1.11), we obtain

$$\int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v}dv$$
$$= \frac{1}{(b-a)z}\sum_{k=0}^{N}\binom{N}{k} k!(-1)^k \left\{(\log b)^{N-k}Li_{k+1}(bz) - (\log a)^{N-k}Li_{k+1}(az)\right\}.$$

By substituting $\log((b-a)v+a) = t$ into the left side of the above equation, we get

$$\int_0^1 \frac{(\log((b-a)v+a))^N}{1-za-z(b-a)v}dv = \int_{\log a}^{\log b} \frac{t^N}{1-za-z(e^t-a)}\frac{e^t}{(b-a)}dt. \tag{1.12}$$

By using the equation (1.12), we arrive at the desired result. □

Remark 1. *By choosing $a = 1$ in Theorem 1, we have*

(1.13)
$$\int_0^{\log b} \frac{t^N e^t}{1-ze^t}dt = \frac{1}{z}\left\{-Li_{N+1}(z)(-1)^N N! + \sum_{k=0}^{N}\binom{N}{k}(-1)^k k!(\log b)^{N-k}Li_{k+1}(bz)\right\}.$$

By substituting $t = -u$ into the left side of equation (1.13), we have

$$\int_0^{\log b} \frac{t^N e^t}{1-ze^t}dt = (-1)^{N+1}\int_0^{-\log b} \frac{u^N e^{-u}}{1-ze^{-u}}du = (-1)^N \int_{-\log b}^{0} \frac{u^N e^{-u}}{1-ze^{-u}}du. \tag{1.14}$$

For $b > 1$,

$$\int_{-\log b}^{\infty} \frac{u^N e^{-u}}{1-ze^{-u}}du = \int_{-\log b}^{0} \frac{u^N e^{-u}}{1-ze^{-u}}du + \int_0^{\infty} \frac{u^N e^{-u}}{1-ze^{-u}}du \tag{1.15}$$

where

$$\int_0^{\infty} \frac{u^N e^{-u}}{1-ze^{-u}}du = N!\Phi(z, N+1, 1) = N!\frac{Li_{N+1}(z)}{z}. \tag{1.16}$$

for $s = N+1$ and $w = 1$ in equation (1.2).

From (1.14), (1.15) and (1.16), we have

$$\int_{-\log b}^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \left\{ (-1)^N \int_{0}^{\log b} \frac{t^N e^{t}}{1 - ze^{t}} dt \right\} + N! \frac{Li_{N+1}(z)}{z}$$

or

$$\int_{-\log b}^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \frac{(-1)^N}{z} \sum_{k=0}^{N} \left(\begin{array}{c} N \\ k \end{array} \right) (-1)^k k! (\log b)^{N-k} Li_{k+1}(bz)$$

by using equation (1.13).

For $b \to e^{-b}$, we get the following equation:

$$\int_{b}^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \frac{N!}{z} \sum_{k=0}^{N} \frac{b^{N-k}}{(N-k)!} Li_{k+1}(e^{-b} z). \tag{1.17}$$

where $b \in \mathbb{R}$.

Remark 2. *While $z \to 0$ in equation (1.17), we arrive*

$$\Gamma(N+1, b) = \int_{b}^{\infty} t^N e^{-t} dt = e^{-b} N! \sum_{k=0}^{N} \frac{b^k}{k!}$$

where

$$\lim_{z \to 0} \frac{Li_{k+1}(e^{-b} z)}{z} = e^{-b}.$$

By substituting $b = \mu c$ into equation (1.17), we get

$$\int_{\mu c}^{\infty} \frac{t^N e^{-t}}{1 - ze^{-t}} dt = \frac{N!}{z} \sum_{k=0}^{N} \frac{(\mu c)^{N-k}}{(N-k)!} Li_{k+1}(e^{-\mu c} z). \tag{1.18}$$

By substituting $t = \mu v$ into the left side of equation (1.18), we arrive at the following corollary:

Corollary 1. *Let $c, \mu \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Then, we have*

$$\Gamma_{\mu,z}(N+1, c) = \int_{c}^{\infty} \frac{v^N e^{-\mu v}}{1 - ze^{-\mu v}} dv = \frac{N!}{\mu^{N+1} z} \sum_{k=0}^{N} \frac{(\mu c)^{N-k}}{(N-k)!} Li_{k+1}(e^{-\mu c} z).$$

Remark 3. *By choosing $\mu = 1$ in Corollary 1, we have*

$$\lim_{z \to 0} \Gamma_{1,z}(N+1, c) = \Gamma(N+1, c).$$

Therefore, we note that $\Gamma_{\mu,z}(N+1, c)$ is a generalization of the incomplete gamma function $\Gamma(N+1, c)$.

3. MAIN RESULTS

In this section, we firstly give a formula for the polylogarithms. Secondly, we obtain a relation between incomplete gamma function and the derivatives of polylogarithms. Finally, we find the generating function for the values of incomplete gamma function.

From equation (1.3), we get

$$\int_0^1 \frac{(\log t)^N}{1-Mzt}dt = \frac{Li_{N+1}(Mz)}{Mz}(-1)^N N! \tag{1.19}$$

By substituting $u = Mt$ into the left side of equation (1.19), we get

$$\int_0^1 \frac{(\log t)^N}{1-Mzt}dt = \frac{1}{M}\int_0^M \frac{(\log u - \log M)^N}{1-zu}du$$

Denote such a partition P, that is $P = \{0, 1, 2, \cdots, M\}$. For $k \in \mathbb{N}_0$, we integrate on $[k, k+1]$:

$$\int_0^M \frac{(\log u - \log M)^N}{1-zu}du = \sum_{k=0}^{M-1}\int_k^{k+1} \frac{(\log u - \log M)^N}{1-zu}du. \tag{1.20}$$

By substituting $u = v+k$ into the left side of equation (1.20) for each integral, we have

$$\int_k^{k+1} \frac{(\log u - \log M)^N}{1-zu}du = \int_0^1 \frac{(\log(v+k) - \log M)^N}{1-kz-zv}dv.$$

Then, we arrive

$$\int_0^1 \frac{(\log t)^N}{1-Mzt}dt = \frac{1}{M}\sum_{k=0}^{M-1}\int_0^1 \frac{(\log(v+k) - \log M)^N}{1-kz-zv}dv.$$

By using binomial expansion,

$$\int_0^1 \frac{(\log(v+k) - \log M)^N}{1-kz-zv}dv \tag{1.21}$$

$$= \int_0^1 \frac{1}{1-kz-zv}\left\{\sum_{j=0}^{N}\binom{N}{j}(-\log M)^{N-j}(\log(v+k))^j\right\}dv$$

$$= \sum_{j=0}^{N}\binom{N}{j}(-\log M)^{N-j}\left\{\int_0^1 \frac{(\log(v+k))^j}{1-kz-zv}dv\right\} \tag{1.22}$$

Also, by substituting $\log(v+k)=t$ into the right side of equation (1.22), we have

$$\int_0^1 \frac{(\log(v+k))^j}{1-kz-zv}dv = \int_{\log k}^{\log(k+1)} \frac{t^j e^t}{1-ze^t}dt.$$

Hence,

$$\begin{aligned}
\frac{Li_{N+1}(Mz)}{Mz}(-1)^N N! &= \int_0^1 \frac{(\log t)^N}{1-Mzt}dt \\
&= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{j=0}^{N}\binom{N}{j}(-\log M)^{N-j}\left\{\int_{\log k}^{\log(k+1)} \frac{t^j e^t}{1-ze^t}dt\right\} \\
&= \frac{1}{M}\sum_{j=0}^{N}\binom{N}{j}(-\log M)^{N-j}\left\{\int_{-\infty}^{0} \frac{t^j e^t}{1-ze^t}dt\right\} \\
&\quad +\frac{1}{M}\sum_{k=1}^{M-1}\sum_{j=0}^{N}\binom{N}{j}(-\log M)^{N-j}\left\{\int_{\log k}^{\log(k+1)} \frac{t^j e^t}{1-ze^t}dt\right\}. \qquad (1.23)
\end{aligned}$$

where

$$\begin{aligned}
\int_{-\infty}^{0} \frac{t^j e^t}{1-ze^t}dt &= (-1)^j \int_0^\infty \frac{t^j e^{-t}}{1-ze^{-t}}dt \\
&= (-1)^j j!\Phi(z,j+1,1) \\
&= (-1)^j j!\frac{Li_{j+1}(z)}{z}.
\end{aligned}$$

By using Theorem 1 for $b=k+1$ and $a=k$, the equation (1.23) can be written as

$$\frac{Li_{N+1}(Mz)}{Mz}(-1)^N N! - \frac{1}{Mz}\sum_{j=0}^{N}\binom{N}{j}(-\log M)^{N-j}(-1)^j j! Li_{j+1}(z) =$$

$$\frac{1}{Mz}\sum_{k=1}^{M-1}\sum_{j=0}^{N}\binom{N}{j}(-\log M)^{N-j}\sum_{r=0}^{j}\binom{j}{r}(-1)^r r!\left\{\begin{array}{c}(\log(k+1))^{j-r}Li_{r+1}(kz+z)\\ -(\log k)^{j-r}Li_{r+1}(kz)\end{array}\right\}.$$

Consequently, we arrive at the following theorem:

Theorem 2. *Let $M, N \in \mathbb{N}_0$. Then, we have*

$$(-1)^N N! Li_{N+1}((M+2)z) - \sum_{j=0}^{N}\binom{N}{j}(-\log(M+2))^{N-j}(-1)^j j! Li_{j+1}(z) =$$

$$\sum_{k=0}^{M}\sum_{j=0}^{N}\sum_{r=0}^{j}\binom{N}{j}\binom{j}{r}(-\log(M+2))^{N-j}(-1)^r r!\left\{\begin{array}{c}(\log(k+2))^{j-r}Li_{r+1}((k+2)z)\\ -(\log(k+1))^{j-r}Li_{r+1}((k+1)z)\end{array}\right\}.$$

For our second main result, we use

$$\frac{e^{-\mu t}}{1-ze^{-\mu t}}=\frac{1}{z}\sum_{n=1}^{\infty}z^n e^{-n\mu t}. \tag{1.24}$$

From equation (1.24), we have

$$\begin{aligned}\int_c^{\infty}\frac{t^N z e^{-\mu t}}{1-ze^{-\mu t}}dt &= \int_c^{\infty}t^N\left(\sum_{n=1}^{\infty}z^n e^{-n\mu t}\right)dt\\ &= \sum_{n=1}^{\infty}z^n\left(\int_c^{\infty}t^N e^{-n\mu t}dt\right).\end{aligned} \tag{1.25}$$

By substituting $n\mu t=u$, we get

$$\int_c^{\infty}t^N e^{-n\mu t}dt=\frac{1}{(n\mu)^{N+1}}\int_{n\mu c}^{\infty}u^N e^{-u}du. \tag{1.26}$$

From equation (1.26), equation (1.25) becomes

$$\begin{aligned}\int_c^{\infty}\frac{t^N z e^{-\mu t}}{1-ze^{-\mu t}}dt &= \frac{1}{\mu^{N+1}}\sum_{n=1}^{\infty}\frac{z^n}{n^{N+1}}\left(\int_{n\mu c}^{\infty}u^N e^{-u}du\right)\\ &= \frac{1}{\mu^{N+1}}\sum_{n=1}^{\infty}\frac{\Gamma(N+1,n\mu c)}{n^{N+1}}z^n\end{aligned}$$

Then, from Corollary 1, we deduce

$$\sum_{n=1}^{\infty}\frac{\Gamma(N+1,n\mu c)}{n^{N+1}}z^n=N!\sum_{k=0}^{N}\frac{(\mu c)^{N-k}}{(N-k)!}Li_{k+1}(e^{-\mu c}z)$$

or

$$\sum_{n=1}^{\infty}\frac{\Gamma(N+1,n\xi)}{n^{N+1}}z^n=N!\sum_{k=0}^{N}\frac{\xi^{N-k}}{(N-k)!}Li_{k+1}(e^{-\xi}z) \tag{1.27}$$

for $\xi=\mu c$.

We define

$$f_{N,\xi}(z)=N!\sum_{k=0}^{N}\frac{\xi^{N-k}}{(N-k)!}Li_{k+1}(e^{-\xi}z).$$

The n-th derivative of function $f_{N,\xi}$:

$$\frac{d^n}{dz^n}f_{N,\xi}(z)=N!e^{-n\xi}\sum_{k=0}^{N}\frac{\xi^{N-k}}{(N-k)!}Li_{k+1}^{(n)}(e^{-\xi}z).$$

For $z = 0$,

$$f_{N,\xi}^{(n)}(0) = N! e^{-n\xi} \sum_{k=0}^{N} \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}^{(n)}(0).$$

In the left side of equation (1.27), by using the property of Taylor series, we obtain

$$\begin{aligned} \frac{\Gamma(N+1, n\xi)}{n^{N+1}} &= \frac{f_{N,\xi}^{(n)}(0)}{n!} \\ &= \frac{N!}{n!} e^{-n\xi} \sum_{k=0}^{N} \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}^{(n)}(0). \end{aligned}$$

Consequently, we arrive at the following theorem:

Theorem 3. *Let $\xi \in \mathbb{R}$, $N \in \mathbb{N}_0$ and $1 \leq n \in \mathbb{N}$. Then, we have*

$$\frac{e^{n\xi} n! \Gamma(N+1, n\xi)}{n^{N+1} N!} = \sum_{k=0}^{N} \frac{\xi^{N-k}}{(N-k)!} Li_{k+1}^{(n)}(0). \tag{1.28}$$

Remark 4. *The n-th derivative of function $Li_{k+1}(z)$ is given as follows:*

$$\begin{aligned} \frac{d^n}{dz^n} Li_{k+1}(z) &= \sum_{m=n}^{\infty} \frac{(m-1)(m-2)(m-3)\cdots(m-n+1)}{m^k} z^{m-n} \\ &= \sum_{m=0}^{\infty} \frac{(m+n-1)(m+n-2)(m+n-3)\cdots(m+1)}{(m+n)^k} z^m \\ &= \sum_{m=0}^{\infty} \frac{(m+n-1)!}{(m+n)^k m!} z^m. \end{aligned}$$

For $z = 0$, we arrive

$$Li_{k+1}^{(n)}(0) = \frac{(n-1)!}{n^k}. \tag{1.29}$$

From (1.28) and (1.29), we get

$$\frac{e^{n\xi} \Gamma(N+1, n\xi)}{n^N N!} = \sum_{k=0}^{N} \frac{\xi^{N-k}}{(N-k)! n^k}.$$

We expand the series both side of the above equation for $N \in \mathbb{N}_0$:

$$\begin{aligned} e^{n\xi} \sum_{N=0}^{\infty} \Gamma(N+1, n\xi) \frac{(t/n)^N}{N!} &= \sum_{N=0}^{\infty} \sum_{k=0}^{N} \frac{\xi^{N-k}}{(N-k)! n^k} t^N \\ &= \sum_{N=0}^{\infty} \sum_{k=0}^{N} \frac{(t\xi)^{N-k}}{(N-k)!} \left(\frac{t}{n}\right)^k. \end{aligned}$$

By using Cauchy product for the right side of the above equation, we have

$$\begin{aligned} e^{n\xi}\sum_{N=0}^{\infty}\Gamma(N+1,n\xi)\frac{(t/n)^N}{N!} &= \sum_{N=0}^{\infty}\sum_{k=0}^{N}\frac{(t\xi)^{N-k}}{(N-k)!}\left(\frac{t}{n}\right)^k \\ &= \left(\sum_{N=0}^{\infty}\frac{(t\xi)^N}{N!}\right)\left(\sum_{N=0}^{\infty}(t/n)^N\right) \\ &= e^{t\xi}\frac{1}{1-\frac{t}{n}} \end{aligned}$$

where $|t/n| < 1$.

For $t \to nt$,

$$\sum_{N=0}^{\infty}\Gamma(N+1,n\xi)\frac{t^N}{N!} = \frac{e^{-n\xi(1-t)}}{1-t}$$

where $|t| < 1$.

Hence, we obtain the following corollary for $n\xi = x$:

Corollary 2. *Let $x \in \mathbb{R}$ and $|t| < 1$. Then, the values of incomplete gamma function is given by the following generating function:*

$$\frac{e^{-x(1-t)}}{1-t} = \sum_{N=0}^{\infty}\Gamma(N+1,x)\frac{t^N}{N!}.$$

Acknowledgements. The author is supported by the research fund of Antalya Akev University.

REFERENCES

1. A. Bayad, Y. Hamahata, Polylogarithms and Poly-Bernoulli Polynomials, Kyushu J. Math. 65, 15-24 (2011).
2. A. Erdélyi, *Higher Transcendental Functions*, vol. 1, New York: McGrow-Hill, 1955.
3. A. R. Miller, Reductions of a generalized incomplete gamma function, related Kampé De Fériet functions, and incomplete Weber integrals, Rocky Mt. J. Math. 30 (2), 703-714 (2000).
4. E. C. Titchmarsh, *The Theory of the Riemann-zeta function*, Oxford University (Clarendon) Press, Oxford and London, 1951; Second Edition (Revised by D. R. Heath-Brown),1986.
5. H. M. Srivastava, J. Choi, *Series Associated with the Zeta and Related Functions,* Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
6. D. S. Kim, T. Kim, A note on polyexponential and unipoly functions, Russ. J. Math. Phys. 26 (1), 40-49 (2019).

7. D. S. Kim, T. Kim, A note on degenarate poly-Bernoulli numbers and polynomials, Adv. Difference Equ. 2015, 2015:258, 8 pp.

8. D. S. Kim, T. Kim, Poly-Cauchy and Peters mixed-type polynomials, Adv. Difference Equ. 2014 (4), 18 pp.

9. L. Lewin, *Polylogarithms and Associated Functions*, New York: North-Holland, 1981.

10. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bareau of Standards, Applied Mathematics Series 55, 4th printing, Washington, 1965.

11. M. A. Chaudhry, S. M. Zubair, Generalized incomplete gamma functions with applications, J. Comput. Appl. Math. 55, 99-124 (1994).

12. M. Kaneko, Poly-Bernoulli numbers, Jour. Th. Nombre Bordeaux 9, 199–206 (1997).

13. Y. Simsek, Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications, Fixed Point Theory 87, 1-28 (2013).

Aykut Ahmet Aygüneş
Department of Software Engineering,
Faculty of Engineering and Architecture,
Antalya Akev University,
Turkey
E-mail: *aykutahmet1981@hotmail.com*

(Received 21.02.2019)
(Revised 18.06.2019)