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A NEW FAMILY OF COMBINATORIAL NUMBERS AND POLYNOMIALS ASSOCIATED WITH PETERS NUMBERS AND POLYNOMIALS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

Yilmaz Simsek

The aim of this paper is to define new families of combinatorial numbers and polynomials associated with Peters polynomials. These families are also a modification of the special numbers and polynomials in [**11**]. Some fundamental properties of these polynomials and numbers are given. Moreover, a combinatorial identity, which calculates the Fibonacci numbers with the aid of binomial coefficients and which was proved by Lucas in 1876, is proved by different method with the help of these combinatorial numbers. Consequently, by using the same method, we give a new recurrence formula for the Fibonacci numbers and Lucas numbers. Finally, relations between these combinatorial numbers and polynomials with their generating functions and other well-known special polynomials and numbers are given.

1. INTRODUCTION

The motivation of this paper is given as follows: using generating functions and their functional equation of new families of combinatorial numbers and polynomials, we give a new approach in order to prove and evaluate combinatorial sums involving Stirling numbers, Apostol type numbers and also some new formulas including Lucas formula for the Fibonacci numbers. By using this approach,

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we give some formulas and relations for the family of combinatorial numbers and polynomials including Apostol-Bernoulli polynomials and numbers, Apostol-Euler polynomials and numbers and Apostol-Genocchi polynomials and numbers, Peters numbers and polynomials and Stirling numbers.

In this paper, the following definitions, notations, and other well-known preliminaries are needed. Throughout this paper, we use $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and also $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. The principal value $\log z$ is the logarithm whose imaginary part lies in the interval $(-\pi, \pi]$ and

$$\binom{\lambda}{v} = \frac{\lambda(\lambda-1)\cdots(\lambda-v+1)}{v!} = \frac{(\lambda)_v}{v!},$$

where $v \in \mathbb{N}$, $\lambda \in \mathbb{C}$ (*cf.* [**1**]-[**16**]).

1.1. Generating functions for some special numbers and polynomials

The Fibonacci numbers f_n and Lucas numbers L_n are defined by means of the following generating functions, respectively:

$$F_f(t) = \frac{t}{1-t-t^2} = \sum_{n=0}^{\infty} f_n t^n \tag{1}$$

where $f_0 = 0$ and $f_1 = f_2 = 1$ (*cf.* [**2**], [**4**], [**7**, p. 229]), and

$$F_L(t) = \frac{2-t}{1-t-t^2} = \sum_{n=0}^{\infty} L_n t^n \tag{2}$$

where $L_0 = 2$ and $L_1 = 1$ (*cf.* [**2**], [**4**], [**7**, p. 229]).

The Apostol-Bernoulli polynomials, The Apostol-Euler polynomials, and the Apostol-Genocchi polynomials are given by means of the following generating functions, respectively:

$$F_B(t, x; \lambda) = \frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!}, \tag{3}$$

$$F_E(t, x; \lambda) = \frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x; \lambda) \frac{t^n}{n!}, \tag{4}$$

and

$$F_G(t, x; \lambda) = tF_E(t, x; \lambda) = \sum_{n=0}^{\infty} \mathcal{G}_n(x; \lambda) \frac{t^n}{n!}, \tag{5}$$

(*cf.* [**2**], [**11**], [**12**], [**13**], [**14**]). Substituting $x = 0$ into (3), (4) and (5), the following well-known families of numbers including the Apostol-Bernoulli numbers, Apostol-Euler numbers and Apostol-Genocchi numbers are obtained, respectively: $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$, $\mathcal{E}_n(\lambda) = \mathcal{E}_n(0;\lambda)$, and $\mathcal{G}_n(\lambda) = \mathcal{G}_n(0;\lambda)$. On the other hand, setting $\lambda = 1$ in (3), (4) and (5), the following well-known families of polynomials including the Bernoulli polynomials, the Euler polynomials and the Genocchi polynomials are obtained, respectively: $B_n(x) = \mathcal{B}_n(x;1)$, $E_n(x) = \mathcal{E}_n(x;1)$, and $G_n(x) = \mathcal{G}_n(x;1)$. Substituting $x = 0$ into the above relations, the Bernoulli numbers, the Euler numbers and the Genocchi numbers are obtained, respectively: $B_n = B_n(0)$, $E_n = E_n(0)$, and $G_n = G_n(0)$ (*cf.* [**2**], [**11**], [**12**], [**13**], [**14**]).

The Stirling numbers of the first kind and the second kind are given by means of the following generating functions, respectively:

$$F_{S_1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} s(n,k)\frac{t^n}{n!} \tag{6}$$

and

$$F_{S_2}(t,k) = \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!}, \tag{7}$$

so that the Stirling numbers of the second kind satisfy the following explicit formula:

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n, \tag{8}$$

(*cf.* [**2**], [**10**], [**13**], [**14**], [**15**]).

The Peters polynomials, which are a member of the family of the Sheffer polynomials, are defined by means of the following generating function:

$$F_P(t,x;\lambda,\mu) = \frac{(1+t)^x}{\left(1+(1+t)^{\lambda}\right)^{\mu}} = \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!} \tag{9}$$

(*cf.* [**5**], [**6**], [**8**], [**10**]). Substituting $x = 0$ into (9), these polynomials are reduced to the Peters numbers: $s_n(\lambda,\mu) = s_n(0;\lambda,\mu)$. Substituting $\mu = 1$ into (9), these polynomials are reduced to the Boole polynomials: $\xi_n(x;\lambda) = s_n(x;\lambda,1)$ (*cf.* [**5**], [**10**]) and also substituting $x = 0$ and $\mu = 1$ into (9), these polynomials are reduced to the Boole numbers: $\xi_n(\lambda) = s_n(0;\lambda,1)$ (*cf.* [**5**]) and also $Ch_n = \xi_n(1) = s_n(0;1,1)$ denotes the Changhee numbers (*cf.* [**6**]).

In [**11**], by using p-adic q-integral on p-adic integers, we constructed the following special polynomials $Y_n(x;\lambda)$, which are a member of the family of the Peters polynomials, as follows:

$$F_Y(t,x;\lambda) = \frac{(1+\lambda t)^x}{\lambda^2 t+\lambda-1} = \sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!} \tag{10}$$

in which if we take $x=0$, then we have the numbers $Y_n(\lambda)=Y_n(0;\lambda)$ (*cf.* [**11**]).

In [**12**], we defined the following generating function for the combinatorial numbers, $y_1(n,k;\lambda)$:

$$F_{y_1}(t,k;\lambda)=\frac{1}{k!}\left(\lambda e^t+1\right)^k=\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!}, \tag{11}$$

where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$ (*cf.* [**12**, p. 10, Eq-(10)]). When $\lambda=1$, we get the combinatorial numbers

$$B(n,k)=y_1(n,k;1)$$

(*cf.* [**12**, p. 10, Eq-(10)], [**3**]).

2. New family of combinatorial numbers and polynomials

In this section, we construct the following generating function for combinatorial polynomials $Y_{n,2}(x,\lambda)$, which are a member of the family of the Peters polynomials, as follows:

$$F_{Y_2}(t,x;\lambda)=\frac{2(1+\lambda t)^x}{\lambda^2 t+2(\lambda-1)}=\sum_{n=0}^{\infty}Y_{n,2}(x;\lambda)\frac{t^n}{n!} \tag{12}$$

in which if we set $x=0$, then we have combinatorial numbers $Y_{n,2}(\lambda)=Y_{n,2}(0;\lambda)$.

We set

$$F_{Y_2}(t,x;\lambda)=(1+\lambda t)^x G_{Y_2}(t;\lambda), \tag{13}$$

where

$$G_{Y_2}(t;\lambda)=\frac{2}{\lambda^2 t+2(\lambda-1)}=\sum_{n=0}^{\infty}Y_{n,2}(\lambda)\frac{t^n}{n!} \tag{14}$$

Combining equation (13) with (12) and (14) yields

$$\sum_{n=0}^{\infty}Y_{n,2}(x;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}Y_{j,2}(\lambda)(x)_{n-j}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the following theorem:

Theorem 1.

$$Y_{n,2}(x;\lambda)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}Y_{j,2}(\lambda)(x)_{n-j}. \tag{15}$$

By (13), we have

$$\sum_{n=0}^{\infty} Y_{n,2}\left(x;\lambda\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty} x^n F_{S_1}\left(\lambda t,n\right)\sum_{n=0}^{\infty} Y_{n,2}\left(\lambda\right)\frac{t^n}{n!}.$$

Combining the above equation with (6), we obtain

$$\sum_{m=0}^{\infty} Y_{m,2}\left(x;\lambda\right)\frac{t^m}{m!}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{n=0}^{k}\left(\begin{array}{c} m \\ k \end{array}\right) x^n\lambda^k s(k,n) Y_{m-k,2}\left(\lambda\right)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the following theorem:

Theorem 2.

$$Y_{m,2}\left(x;\lambda\right)=\sum_{k=0}^{m}\sum_{n=0}^{k}\left(\begin{array}{c} m \\ k \end{array}\right)\lambda^k s(k,n) Y_{m-k,2}\left(\lambda\right)x^n. \tag{16}$$

Theorem 3.

$$\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\lambda^{n-j} Y_{j,2}\left(\lambda\right)\left(x+y\right)_{n-j}=\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\lambda^{n-j} Y_{j,2}\left(x;\lambda\right)\left(y\right)_{n-j}. \tag{17}$$

Proof. We set the following functional equation:

$$F_{Y_2}\left(t,x+y;\lambda\right)=\left(1+\lambda t\right)^y F_{Y_2}\left(t,x;\lambda\right) \tag{18}$$

Combining the above equation with (12), after some elementary calculations, we get

$$\sum_{n=0}^{\infty} Y_{n,2}\left(x+y;\lambda\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\left(y\right)_{n-j}\lambda^{n-j} Y_{j,2}\left(x;\lambda\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the following relation:

$$Y_{n,2}\left(x+y;\lambda\right)=\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\left(y\right)_{n-j}\lambda^{n-j} Y_{j,2}\left(x;\lambda\right). \tag{19}$$

Similarly, by using (15), we obtain

$$Y_{n,2}\left(x+y;\lambda\right)=\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\left(x+y\right)_{n-j}\lambda^{n-j} Y_{j,2}\left(\lambda\right). \tag{20}$$

Combining (19) with (20), we get the desired result. $\square$

By applying geometric series representation of the function $G_{Y_2}(t;\lambda)$ in (14), we obtain

$$\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!} = \frac{1}{\lambda-1}\sum_{n=0}^{\infty}(-1)^n \frac{\lambda^{2n}}{(2\lambda-2)^n}t^n.$$

From the above equation, we deduce the following theorem:

Lemma 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_{n,2}(\lambda) = 2(-1)^n n! \frac{\lambda^{2n}}{(2\lambda-2)^{n+1}} \tag{21}$$

Substituting (21) into (15), we get a explicit formula for the polynomials $Y_{n,2}(x;\lambda)$ by the following theorem:

Theorem 4. *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_{n,2}(x;\lambda) = 2\sum_{j=0}^{n}(-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda-2)^{j+1}}(x)_{n-j}. \tag{22}$$

2.1. Recurrence relations

Here, recurrence relations for the numbers $Y_{n,2}(\lambda)$ and the polynomials $Y_{n,2}(x,\lambda)$ are given.

By using (14), we obtain

$$2 = \lambda^2\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^{n+1}}{n!} + (2\lambda-2)\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{t^n}{n!}$$

Making some elementary calculations in the above equation, we deduce a recurrence relation for the numbers $Y_{n,2}(\lambda)$ by the following theorem:

Theorem 5. *Let $Y_{0,2}(\lambda) = \frac{1}{\lambda-1}$. For $n \in \mathbb{N}$, we have*

$$Y_{n,2}(\lambda) = \frac{n\lambda^2}{2-2\lambda}Y_{n-1,2}(\lambda) \tag{23}$$

Theorem 6. *Let $n \in \mathbb{N}$. Then we have*

$$2\lambda^n (x)_n = \lambda^2 n Y_{n-1,2}(x,2) + (2\lambda-2)Y_{n,2}(x,2).$$

Proof. Using (12), we have

$$2(1+\lambda t)^x = \lambda^2\sum_{n=0}^{\infty} Y_{n,2}(x;\lambda)\frac{t^{n+1}}{n!} + (2\lambda-2)\sum_{n=0}^{\infty} Y_{n,2}(x;\lambda)\frac{t^n}{n!}.$$

Thus, we get

$$\sum_{n=0}^{\infty} 2\,(x)_n\,\lambda^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\lambda^2 n Y_{n-1,2}(x;\lambda) + (2\lambda-2)\,Y_{n,2}(x;\lambda)\right) \frac{t^n}{n!}. \tag{24}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

2.2. Relation between the numbers $Y_{n,2}(\lambda)$ and others well-known special numbers

Here, relations between the numbers $Y_{n,2}(\lambda)$ and other well-known numbers are given. By using the polynomials $Y_{n,2}(x;\lambda)$, the well-known explicit formulas for Apostol type numbers including Apostol-Bernoulli numbers, Apostol-Euler numbers and Apostol-Genocchi numbers are given.

Note that, by (10) and (14), the relation between the numbers $Y_n(\lambda)$ and the numbers $Y_{n,2}(\lambda)$ is given as follows:

$$Y_{n,2}(\lambda) = \frac{1}{2^{n+1}} Y_n(\lambda).$$

Substituting $x=0$, $\lambda=\mu=1$ and $t=\frac{\theta^2 u}{\theta-1}$ into (9), we obtain

$$s_n(0;1,1) = \frac{(\theta-1)^{n+1}}{2\theta^{2n}} Y_{n,2}(\theta).$$

Theorem 7. *Let $n \in \mathbb{N}$. Then we have*

$$\mathcal{B}_n\left(x;\frac{\lambda}{2-\lambda}\right) = \frac{2-\lambda}{2} n \sum_{m=0}^{n-1} \lambda^{-m} S(n-1,m)\, Y_{m,2}(x;\lambda). \tag{25}$$

Proof. Substituting $\lambda t = e^u - 1$ into (12), we get the following functional equation:

$$F_{Y_2}\left(\frac{e^u-1}{\lambda}, x; \lambda\right) = \frac{2}{u(2-\lambda)} F_B\left(u, x; \frac{\lambda}{2-\lambda}\right)$$

Combining the above equation with (3), (12) and (7), after some elementary calculations, we get

$$\frac{2}{2-\lambda} \sum_{n=0}^{\infty} \mathcal{B}_n\left(x;\frac{\lambda}{2-\lambda}\right) \frac{u^n}{n!} = \sum_{m=0}^{\infty} \lambda^{-m} Y_{m,2}(x;\lambda) \sum_{n=0}^{\infty} S(n,m) \frac{u^{n+1}}{n!}.$$

Since $S(n,m)=0$ if $m>n$, we obtain

$$\frac{2}{2-\lambda} \sum_{n=0}^{\infty} \mathcal{B}_n\left(x;\frac{\lambda}{2-\lambda}\right) \frac{u^n}{n!} = \sum_{n=0}^{\infty} n \sum_{m=0}^{n-1} \lambda^{-m} Y_{m,2}(x;\lambda)\, S(n-1,m) \frac{u^n}{n!}.$$

Comparing the coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we get the desired result. $\square$

Substituting $x = 0$ into (25) and the final equation combining with (21) and (8), we get the following explicit formula for the Apostol-Bernoulli numbers.

Corollary 1. *Let $n \in \mathbb{N}$. Then we have*

$$\mathcal{B}_n\left(\frac{\lambda}{2-\lambda}\right) = \frac{2-\lambda}{2\lambda-2} n \sum_{m=0}^{n-1} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^{n-1} \left(\frac{\lambda}{2\lambda-2}\right)^m .$$

Theorem 8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{E}_n\left(x; \frac{\lambda}{\lambda-2}\right) = (\lambda-2) \sum_{m=0}^{n} \lambda^{-m} Y_{m,2}(x;\lambda) S(n,m) . \tag{26}$$

Proof. Substituting $\lambda t = e^u - 1$ into (12), we get the following functional equation:

$$(\lambda-2) F_{Y_2}\left(\frac{e^u-1}{\lambda}, x; \lambda\right) = F_E\left(u, x; \frac{\lambda}{\lambda-2}\right)$$

Combining the above equation with (4), (12), and (7), after some elementary calculations, we obtain

$$\sum_{n=0}^{\infty} \mathcal{E}_n\left(x; \frac{\lambda}{\lambda-2}\right) \frac{u^n}{n!} = (\lambda-2) \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda^{-m} Y_{m,2}(x;\lambda) S(n,m) \frac{u^n}{n!}$$

since $S(n,m) = 0$ if $m > n$. Comparing the coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we get the desired result. $\square$

Substituting $x = 0$ into (26) and the final equation combining with (21) and (8), we get the following explicit formula for the Apostol-Euler numbers.

Corollary 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{E}_n\left(\frac{\lambda}{\lambda-2}\right) = \sum_{m=0}^{n} \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^n \left(\frac{\lambda}{2}\right)^m \frac{\lambda-2}{(\lambda-1)^{m+1}} .$$

Theorem 9.

$$\mathcal{G}_n\left(x; \frac{\lambda}{\lambda-2}\right) = (\lambda-2) n \sum_{m=0}^{n-1} \lambda^{-m} Y_{m,2}(x;\lambda) S(n-1,m) . \tag{27}$$

Proof. Substituting $\lambda t = e^u - 1$ into (12), we get the following functional equation:

$$(\lambda-2) u F_{Y_2}\left(\frac{e^u-1}{\lambda}, x; \lambda\right) = F_G\left(u, x; \frac{\lambda}{\lambda-2}\right)$$

Combining the above equation with (5) and (12), and (7), after some elementary calculations, we get

$$\sum_{n=0}^{\infty}\mathcal{G}_n\left(x;\frac{\lambda}{\lambda-2}\right)\frac{u^n}{n!}=(\lambda-2)\sum_{n=0}^{\infty}n\sum_{m=0}^{n-1}\lambda^{-m}Y_{m,2}\left(x;\lambda\right)S\left(n-1,m\right)\frac{u^n}{n!}.$$

since $S\left(n,m\right)=0$ if $m>n$. Comparing the coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we get the desired result. □

Substituting $x=0$ into (27) and the final equation combining with (21), we get the following explicit formula for the Apostol-Genocchi numbers.

Corollary 3. *Let $n\in\mathbb{N}$. Then we have*

$$\mathcal{G}_n\left(\frac{\lambda}{\lambda-2}\right)=n\sum_{m=0}^{n-1}\sum_{j=0}^{m}(-1)^j\left(\begin{array}{c}m\\j\end{array}\right)j^{n-1}\left(\frac{\lambda}{2}\right)^m\frac{\lambda-2}{\left(\lambda-1\right)^{m+1}}.$$

Substituting $t=\frac{2\lambda-2}{\lambda^2}\left(e^u-1\right)$ into (14), after some elementary calculations, we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{n}Y_{n,2}\left(\lambda\right)\left(\frac{2\lambda-2}{\lambda^2}\right)^n S(m,n)\frac{u^m}{m!}=\frac{1}{\lambda-1}\sum_{m=0}^{\infty}(-1)^m\frac{u^m}{m!}.$$

Comparing the coefficients of $\frac{u^m}{m!}$ on both sides of the above equation, we get the following Theorem:

Theorem 10.

$$\sum_{n=0}^{m}Y_{n,2}\left(\lambda\right)S\left(m,n\right)\left(\frac{2\lambda-2}{\lambda^2}\right)^n=\frac{(-1)^m}{\lambda-1}.$$

We now give relations between the polynomials $Y_{n,2}\left(x;\lambda\right)$ and the Peters polynomials.

Theorem 11. *Let $n\in\mathbb{N}$. Then we have*

$$\begin{aligned}s_n\left(x;\lambda,\mu\right) &= \frac{n}{2}\sum_{j=0}^{n-1}\binom{n-1}{j}\theta^{j+2-n}s_j\left(\lambda,\mu\right)Y_{n-1-j,2}\left(x,\theta\right)\\ &\quad+(\theta-1)\sum_{j=0}^{n}\binom{n}{j}\theta^{j-n}s_j\left(\lambda,\mu\right)Y_{n-j,2}\left(x,\theta\right).\end{aligned}$$

Proof. Replacing t by θt in equation (9) and combining this equation with (12) yields the following functional equation:

$$F_P\left(\theta t,x;\lambda,\mu\right)=\left(\frac{\theta^2}{2}t+\theta-1\right)F_{Y_2}\left(t,x;\theta\right)F_P\left(\theta t,0;\lambda,\mu\right).$$

From the above equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\,\theta^n \frac{t^n}{n!} &= \frac{\theta^2}{2}\sum_{n=0}^{\infty} n\sum_{j=0}^{n-1}\binom{n-1}{j}\theta^j s_j(\lambda,\mu)\,Y_{n-1-j,2}(x,\theta)\,\frac{t^n}{n!}\\ &\quad+(\theta-1)\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\theta^j s_j(\lambda,\mu)\,Y_{n-j,2}(x,\theta)\,\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

Lemma 2. *Let $\mu \in \mathbb{N}$. Then we have*

$$(x)_n = \sum_{v=0}^{n}\sum_{j=0}^{\mu}\binom{\mu}{j}\binom{n}{v}(\lambda j)_v\, s_{n-v}(x;\lambda,\mu). \tag{28}$$

Proof. For $\mu \in \mathbb{N}$, we can modify (9) as follows:

$$(1+t)^x = F_P(t,x;\lambda,\mu)\sum_{j=0}^{\mu}\binom{\mu}{j}(1+t)^{\lambda j}.$$

Under the assumption $|t| < 1$ and with the help of binomial theorem, the above equation yields to the following relation:

$$\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!} = \sum_{j=0}^{\mu}\binom{\mu}{j}\sum_{n=0}^{\infty}\left(\sum_{v=0}^{n}\binom{n}{v}s_{n-v}(x;\lambda,\mu)\,(\lambda j)_v\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

Theorem 12. *Let $\mu \in \mathbb{Z}^+$. Then we have*

$$(x)_n = \sum_{v=0}^{n}\sum_{k=0}^{v}\binom{n}{v}\lambda^k B(k,\mu)\, s(v,k)\, s_{n-v}(x;\lambda,\mu).$$

Proof. Substituting $(x)_n = \sum_{k=0}^{n} s(n,k)\,x^k$ (*cf.* [**10**]) into the right-hand side of (28), after some elementary calculations, we obtain

$$(x)_n = \sum_{v=0}^{n}\sum_{k=0}^{v}\binom{n}{v}\lambda^k s_{n-v}(x;\lambda,\mu)\, s(v,k)\sum_{j=0}^{\mu}\binom{\mu}{j}j^k.$$

Combining the above equation with (11) yields the assertion of theorem. □

3. Combinatorial identities including the numbers $Y_{n,2}(\lambda)$, Fibonacci and Lucas numbers

In this section, we prove two very interesting and novel combinatorial identities containing the Fibonacci numbers and the Lucas numbers. The first of these formulas is the well-known formula as the Lucas formula proven by Lucas in 1876. We give a proof of this formula with the help of the generating function for the numbers $Y_{n,2}(\lambda)$ given in the equation (14). On the other hand, we prove the second formula, which includes a relation between the Lucas numbers and the Fibonacci numbers, with similar method.

Some infinite series representations involving the numbers $Y_n(\lambda)$ were studied in [**15**]. Infinite series representations involving the numbers $Y_{n,2}(\lambda)$ and the Fibonacci numbers are given as follows: Assume that $\left|\frac{\lambda^2}{\lambda-1}\right| < 1$. Then we have

$$\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{2^n}{n!} = -\frac{1}{\lambda}F_f(\lambda) \tag{29}$$

and on the other hand, if we assume that $\left|\frac{\lambda-1}{\lambda^2}\right| < 1$, then we have

$$\sum_{n=0}^{\infty} \frac{n!2^{-n}}{Y_{n,2}(\lambda)} = \lambda F_f(\lambda) - \lambda^2 F_f(\lambda).$$

Theorem 13. *Let $n \in \mathbb{N}$. Then we have*

$$f_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{n-2k-1}, \tag{30}$$

where $[x]$ denotes the greatest integer function.

In order to prove Theorem 13, we need the following Lemma.

Lemma 3 (*cf.* [**9**, p. 57, Lemma 11, Eq-(7)]).

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} A(k,n-2k), \tag{31}$$

where $[x]$ denotes the greatest integer function.

Proof of Theorem 13. By substituting (21) into (29), we obtain

$$\sum_{n=0}^{\infty} Y_{n,2}(\lambda)\frac{2^n}{n!} = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \binom{n+1+m-1}{m} \lambda^{2n+m}. \tag{32}$$

Combining (32) with (31), we get

$$\sum_{n=0}^{\infty} Y_{n,2}(\lambda) \frac{2^n}{n!} = -\sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{n-2k} \lambda^n$$

Combining the above equation with (29) and (1) yields the following relation:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{n-2k} \lambda^{n+1} = \sum_{n=0}^{\infty} f_n \lambda^n$$

Since $f_0 = 0$, comparing the coefficients of λ^n on both sides of the above equation, we arrive at the desired result. □

Remark 1. *There are many different proof of the Lucas Formula*

$$f_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k}$$

where $n \in \mathbb{N}$ ([**4**, p. 50], [**7**, p. 155, Theorem 12.4. (Lucas Formula, 1876)], [**16**]). *We give few examples for proof of the Lucas Formula. With the help of Pascal's triangle, Koshy* [**7**, p. 155, Theorem 12.4. (Lucas Formula, 1876)] *gave proof of Lucas formula, which discovered by Lucas in 1876. On the other hand, in* [**16**], *using the following sum of binomial coefficients for* $n \in \mathbb{N}_0$, $\sum_{m \geq 0} \binom{m}{n-m}$, *Wilf proved Lucas Formula.*

Theorem 14. *Let* $n \in \mathbb{N}_0$. *Then we have*

$$L_n = 2f_{n+1} - f_n \tag{33}$$

Proof. Multiplying both sides of equation (29) by $(\lambda - 2)$, we get infinite series representation involving the numbers $Y_{n,2}(\lambda)$ and the Lucas numbers:

$$(\lambda - 2) \sum_{n=0}^{\infty} Y_{n,2}(\lambda) \frac{2^n}{n!} = F_L(\lambda).$$

Combining the above equation with (32) and (2), we obtain

$$-(\lambda - 2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+1+m-1}{m} \lambda^{2n+m} = \sum_{n=0}^{\infty} L_n \lambda^n.$$

Applying (31) to the above equation yields

$$(\lambda - 2) \left(-\sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n-j}{n-2j} \lambda^n \right) = \sum_{n=0}^{\infty} L_n \lambda^n.$$

Combining the above equation with (30), we get

$$\sum_{n=0}^{\infty}\left(2f_{n+1}-f_n\right)\lambda^n=\sum_{n=0}^{\infty}L_n\lambda^n.$$

Comparing the coefficients of λ^n on both sides of the above equation, we arrive at the desired result. □

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Yilmaz Simsek
Department of Mathematics,
Faculty of Science
University of Akdeniz
TR-07058 Antalya, Turkey.
E-mail: *ysimsek@akdeniz.edu.tr*

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