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A CHAIN OF MEAN VALUE INEQUALITIES

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Let

$$G = G(x,y) = \sqrt{xy}, \quad L = L(x,y) = \frac{x-y}{\log(x) - \log(y)},$$
$$I = I(x,y) = \frac{1}{e}\Big(\frac{x^x}{y^y}\Big)^{1/(x-y)}, \quad A = A(x,y) = \frac{x+y}{2},$$

be the geometric, logarithmic, identric, and arithmetic means of x and y. We prove that the inequalities

$$L(G^2, A^2) < G(L^2, I^2) < A(L^2, I^2) < I(G^2, A^2)$$

are valid for all $x, y > 0$ with $x \neq y$. This refines a result of Seiffert [**12**].

1. Introduction and statement of the result

We study the geometric, logarithmic, identric, and arithmetic means of two positive real numbers x and y with $x \neq y$. These mean values are defined by

$$G = G(x,y) = \sqrt{xy}, \quad L = L(x,y) = \frac{x-y}{\log(x) - \log(y)}, \tag{1}$$

$$I = I(x,y) = \frac{1}{e}\Big(\frac{x^x}{y^y}\Big)^{1/(x-y)}, \quad A = A(x,y) = \frac{x+y}{2}. \tag{2}$$

They are members of Stolarsky's one-parameter mean value family

$$S_r = S_r(x,y) = \Big(\frac{x^r - y^r}{r(x-y)}\Big)^{1/(r-1)} \quad (r \neq 0, 1;\ x \neq y);$$

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see Stolarsky [**13**]. Indeed, we have

$$S_{-1} = G, \quad S_0 = \lim_{r \to 0} S_r = L, \quad S_1 = \lim_{r \to 1} S_r = I, \quad S_2 = A.$$

Since $r \mapsto S_r(x, y)$ is strictly increasing on $\mathbb{R}$, we obtain

$$G < L < I < A. \tag{3}$$

The means given in (1.1) and (1.2) have interesting applications in statistics, physics, economics, meteorology, and other fields. Numerous researchers studied intensively the properties of these and other mean values. In particular, many inequalities for means can be found in the literature. For detailed information on this subject we refer to the monographs of Bullen, Mitrinović and Vasić [**2**], Sándor [**10**], the studies of Kouba [**3**], Neuman and Sándor [**7**], Sándor [**8, 9**], and the references cited therein.

Our work was inspired by an interesting note published by Seiffert [**12**] in 1995. Among others, he proved

$$L < \sqrt{L(G^2, A^2)} < \sqrt{I(G^2, A^2)} < I. \tag{4}$$

Neuman and Sándor [**6**] offered the following remarkable refinement of (1.4):

$$L < L(G, A) < \sqrt{L(G^2, A^2)} < I(G, A) < \sqrt{\frac{G^2 + 4GA + A^2}{6}} \tag{5}$$

$$< \frac{G + A}{2} < \sqrt{\frac{G^2 + GA + A^2}{3}} \tag{6}$$

$$< \sqrt{I(G^2, A^2)} < I.$$

The elegant inequality

$$\frac{G + A}{2} L < L(G^2, A^2)$$

was given by Sándor and Bhayo [**11**]. It leads to an improvement of the left-hand side of (1.4). Moreover, they obtained a double-inequality involving $I(G^2, A^2)$, namely,

$$\frac{e}{4} I^2 < I(G^2, A^2) < I^2. \tag{7}$$

The constant factors $e/4$ and 1 are sharp.

The aim of this paper is to refine the second inequality in (1.4) as follows.

Theorem. *For all positive real numbers x and y with $x \neq y$ we have*

$$L(G^2, A^2) < G(L^2, I^2) < A(L^2, I^2) < I(G^2, A^2). \tag{8}$$

We note that the lower bounds for $I(G^2, A^2)$ as given in (1.6) and (1.7) cannot be compared. Indeed, the expression $A(L^2, I^2) - (e/4)I^2$ attains positive and negative values.

A short calculation reveals that the first inequality in (1.7) is equivalent to

$$\frac{L(G,A)}{L} < \frac{I}{A(G,A)}. \tag{9}$$

This is a counterpart of the left-hand side of (1.5) written as

$$1 < \frac{L(G,A)}{L}.$$

Using (1.3) and (1.8) leads to the double-inequality

$$GA < L(G,A)A(G,A) < LI$$

which improves the well-known inequality $GA < LI$ given by Alzer [**1**].

In the next section, we present a proof of our Theorem. The difficult part consists of proving the right-hand side of (1.7). We show that $A(L^2, I^2) < I(G^2, A^2)$ is equivalent to an inequality involving the hyperbolic functions cosh, sinh, and coth. To settle this inequality we need numerous algebraic computations which have been carried out by using the computer software MAPLE 13. In this context, we mention the interesting papers by Makragić [**4**] and Malešević and Makragić [**5**], who presented a method of proving inequalities of the form

$$\sum_{j=1}^{n} a_j x^{p_j} \cosh^{q_j}(x) \sinh^{r_j}(x) > 0.$$

2. Proof of the theorem

In view of the arithmetic mean - geometric mean inequality, it is enough to prove the first and the third inequality. We may assume that $x > y > 0$. Let $t = (1/2)\log(x/y) > 0$. Since the mean values given in (1.1) and (1.2) are homogeneous of degree 1, we obtain

$$\begin{aligned}
L\big(G^2(x,y), A^2(x,y)\big) &= xyL\big(G^2(e^t, e^{-t}), A^2(e^t, e^{-t})\big),\\
G\big(L^2(x,y), I^2(x,y)\big) &= xyG\big(L^2(e^t, e^{-t}), I^2(e^t, e^{-t})\big),\\
A\big(L^2(x,y), I^2(x,y)\big) &= xyA\big(L^2(e^t, e^{-t}), I^2(e^t, e^{-t})\big),\\
I\big(G^2(x,y), A^2(x,y)\big) &= xyI\big(G^2(e^t, e^{-t}), A^2(e^t, e^{-t})\big).
\end{aligned}$$

This implies that it suffices to prove

$$L\big(G^2(e^t, e^{-t}), A^2(e^t, e^{-t})\big) < G\big(L^2(e^t, e^{-t}), I^2(e^t, e^{-t})\big) \tag{10}$$

and

$$A\big(L^2(e^t,e^{-t}),I^2(e^t,e^{-t})\big) < I\big(G^2(e^t,e^{-t}),A^2(e^t,e^{-t})\big). \tag{11}$$

We have

$$G(e^t,e^{-t}) = 1, \quad A(e^t,e^{-t}) = \cosh(t),$$

$$L(e^t,e^{-t}) = \frac{\sinh(t)}{t}, \quad I(e^t,e^{-t}) = \exp\big(-1+t\coth(t)\big).$$

It follows that (2.1) and (2.2) can be written as

$$\frac{\cosh^2(t)-1}{2\log\big(\cosh(t)\big)} < \frac{\sinh(t)}{t}\exp\big(-1+t\coth(t)\big) \tag{12}$$

and

$$\frac{1}{2}\Big[\Big(\frac{\sinh(t)}{t}\Big)^2 + \exp\big(-2+2t\coth(t)\big)\Big] < \frac{1}{e}(\cosh(t))^{2+2/(\cosh^2(t)-1)}, \tag{13}$$

respectively.

Proof of (2.3): Let $t>0$. We define

$$u(t) = \log\Big(\frac{\cosh^2(t)-1}{2\log\big(\cosh(t)\big)}\Big), \quad u(0) = \lim_{t\to 0} u(t) = 0,$$

and

$$v(t) = \log\Big(\frac{\sinh(t)}{t}\exp\big(-1+t\coth(t)\big)\Big), \quad v(0) = \lim_{t\to 0} v(t) = 0.$$

Differentiation yields

$$\frac{t\cosh(t)\sinh^2(t)\log\big(\cosh(t)\big)}{\big(t^2+\sinh^2(t)\big)\cosh(t)}\big(v'(t)-u'(t)\big) = \frac{t\sinh^3(t)}{\big(t^2+\sinh^2(t)\big)\cosh(t)} - \log\big(\cosh(t)\big)$$
$$= p(t), \text{ say,}$$

and

$$-\frac{\cosh^2(t)\big(t^2+\sinh^2(t)\big)^2}{t\sinh(t)}p'(t) = -3t^2\sinh(t) - (1+2t^2)\sinh^3(t)$$
$$+\big(t^3+3t\sinh^2(t)\big)\cosh(t) = q(t), \text{ say.}$$

Since

$$-\frac{1}{t\sinh(t)}q'(t) = \frac{5}{2} - t^2 - \frac{5}{2}\cosh(2t) + 3t\sinh(2t) = \frac{1}{2}\sum_{k=2}^{\infty}(6k-5)\frac{(2t)^{2k}}{(2k)!} > 0,$$

we obtain $q'(t)<0$ and $q(t)<q(0)=0$. It follows that $p'(t)>0$ and $p(t)>\lim_{s\to 0}p(s)=0$. This implies $v'(t)-u'(t)>0$. Thus, $v(t)-u(t)>v(0)-u(0)=0$. This leads to (2.3).

Proof of (2.4): Let $t > 0$ and

$$U(t) = \log\Big(\frac{1}{2}\Big[\Big(\frac{\sinh(t)}{t}\Big)^2 + \exp\big(-2 + 2t\coth(t)\big)\Big]\Big), \quad U(0) = \lim_{t\to 0} U(t) = 0,$$

$$V(t) = \log\Big(\frac{1}{e}\big(\cosh(t)\big)^{2+2/(\cosh^2(t)-1)}\Big), \quad V(0) = \lim_{t\to 0} V(t) = 0.$$

To prove that $V'(t) > U'(t)$ we define

$$P(t, x) = \big(\sinh^4(t) + t^4 x\big)\sinh(t) - 2t\cosh(t)\big(\sinh^2(t) + t^2 x\big)\log\big(\cosh(t)\big).$$

Then,

$$t\sinh^3(t)\big(\sinh^2(t) + t^2\phi(t)\big)\big(V'(t) - U'(t)\big) = 2P(t, \phi(t)), \tag{14}$$

where

$$\phi(t) = \exp\big(-2 + 2t\coth(t)\big).$$

Next, we provide a lower bound for $P(t, \phi(t))$. Partial differentiation yields

$$\frac{\partial}{\partial x} P(t, x) = t^3\cosh(t)\sigma(t)$$

with

$$\sigma(t) = t\tanh(t) - 2\log\big(\cosh(t)\big).$$

Since

$$\sigma(0) = 0 \quad \text{and} \quad \sigma'(t) = \frac{2t - \sinh(2t)}{2\cosh^2(t)} < 0,$$

we conclude that $x \mapsto P(t, x)$ is strictly decreasing on $\mathbb{R}$. Using

$$\phi(t) = I^2(e^t, e^{-t}) < A^2(e^t, e^{-t}) = \cosh^2(t)$$

gives

$$P(t, \phi(t)) > P(t, \cosh^2(t)). \tag{15}$$

To prove that $P(t, \cosh^2(t)) > 0$ we define

$$Q(t) = \frac{P(t, \cosh^2(t))}{2t\cosh(t)\big(\sinh^2(t) + t^2\cosh^2(t)\big)}. \tag{16}$$

Then,

$$2t^2\cosh^2(t)\big(\sinh^2(t) + t^2\cosh^2(t)\big)^2 Q'(t) = R(t) \tag{17}$$

with

$$R(t) = -t\sinh^8(t) - \cosh(t)\sinh^7(t) + (3t - 3t^3)\cosh^2(t)\sinh^6(t)$$

$$-\left[3t^2\cosh^3(t)+2t^2\cosh(t)\right]\sinh^5(t)+\left[5t^3\cosh^4(t)+t^5\cosh^2(t)\right]\sinh^4(t)$$
$$-t^4\cosh^3(t)\sinh^3(t)-(t^5+t^7)\cosh^4(t)\sinh^2(t)-t^6\cosh^5(t)\sinh(t)+t^7\cosh^6(t).$$

We show that R is positive on $(0,\infty)$. Let

$$F_1(t)=256e^{8t}R(t). \tag{18}$$

Then, we obtain

$$F_1(t)=2t^3+3t^2+2t+1+(4t^6+4t^4+12t^3+2t^2-4t-6)e^{2t}$$
$$+(16t^7+16t^6-16t^5-32t^3-38t^2-16t+14)e^{4t}$$
$$+(64t^7+20t^6-12t^4-12t^3+58t^2+68t-14)e^{6t}$$
$$+(96t^7+32t^5+60t^3-100t)e^{8t}+(64t^7-20t^6+12t^4-12t^3-58t^2+68t+14)e^{10t}$$
$$+(16t^7-16t^6-16t^5-32t^3+38t^2-16t-14)e^{12t}$$
$$+(-4t^6-4t^4+12t^3-2t^2-4t+6)e^{14t}+(2t^3-3t^2+2t-1)e^{16t}$$

$$=\sum_{k=0}^{8}S_k(t)\exp(2kt), \tag{19}$$

where each $S_k(t)$ is a polynomial in t. Differentiating F_1 four times we conclude that $S_0(t)$ vanishes in the summation, while the remaining terms keep the format,

$$F_1^{(4)}(t)=\sum_{k=1}^{8}\tilde{S}_k(t)\exp(2kt).$$

When compared with (2.10), the length of the summation is reduced by 1, from 9 to 8. When this process is repeated, we will, after 8 iterations, arrive at a summation of only 1 term. Therefore, we introduce the following functions:

$$F_2(t)=\frac{1}{32}e^{-2t}F_1^{(4)}(t),\quad F_3(t)=\frac{1}{128}e^{-2t}F_2^{(7)}(t),$$
$$F_4(t)=\frac{1}{3}e^{-2t}F_3^{(8)}(t),\quad F_5(t)=\frac{1}{301989888}e^{-2t}F_4^{(8)}(t),$$
$$F_6(t)=\frac{1}{256}e^{-2t}F_5^{(8)}(t),\quad F_7(t)=\frac{1}{26542080}e^{-2t}F_6^{(8)}(t),$$

$$F_8(t)=e^{-2t}F_7^{(8)}(t), \tag{20}$$

$$F_9(t)=e^{-2t}F_8^{(8)}(t). \tag{21}$$

By direct computation we find

$$F_1^{(k)}(0) = 0 \quad (k = 0, 1, 2, 3), \quad F_2^{(k)}(0) = 0 \quad (k = 0, ..., 6),$$

$$F_3^{(k)}(0) = 0 \quad (k = 0, 1, 2, 3), \quad F_3^{(k)}(0) > 0 \quad (k = 4, 5, 6, 7),$$

$$F_4^{(k)}(0) > 0 \quad (k = 0, ..., 7), \quad F_5^{(k)}(0) > 0 \quad (k = 0, ..., 7), \quad F_6^{(k)}(0) > 0 \quad (k = 0, ..., 7),$$

(22)
$$F_7^{(k)}(0) > 0 \quad (k = 0, ..., 7),$$

(23)
$$F_8^{(k)}(0) = 0 \quad (k = 0, ..., 7),$$

and we obtain the representation

$$\frac{1}{10^4} F_9(t) = 1113928758834531139584000t^3 + 335849520788611138584576000t^2$$

$$+3276678798181796672647987200t + 10340006209427872449130659840.$$

From (2.12) and (2.14) we conclude that $F_8(t) > 0$. Applying (2.11) and (2.13) reveals that $F_7(t) > 0$. Successively, we get that $F_6(t)$, ..., $F_1(t)$ are positive. From (2.9) we obtain that $R(t) > 0$. Using (2.8) gives $Q(t) > \lim_{s \to 0} Q(s) = 0$, so that (2.6) and (2.7) yield $P(t, \phi(t)) > 0$. Finally, we use (2.5) to obtain $V(t) - U(t) > V(0) - U(0) = 0$. This implies (2.4).

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