Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 335–348.
https://doi.org/10.2298/AADM190406016C

q-BINOMIAL FORMULAE OF DIXON'S TYPE AND THE FIBONOMIAL SUMS

*Wenchang Chu and Emrah Kılıç**

Cubic sums of the Gaussian q-binomial coefficients with certain weight functions will be evaluated in this paper. To realize this, we will derive two remarkable formulae by means of the Carlitz–Sears transformation on terminating well–poised q-series. As consequences, several summation formulae on Fibonomial coefficients are presented by specializing the value of base q in our q-series identities.

1. INTRODUCTION

Throughout this paper we use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

When $x = q$, we sometimes use the notation $(q)_n$ instead of $(q;q)_n$. We conveniently adopt the notation that $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$ or $k > n$. With the q-shifted factorial given by $[n]! = \prod_{i=1}^{n} \frac{1-q^i}{1-q} = (q;q)_n/(1-q)^n$, the q-analogue of Dixon's identity [1,6] can be reproduced as

$$\sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} a+b \\ a+k \end{bmatrix}_q \begin{bmatrix} b+c \\ b+k \end{bmatrix}_q \begin{bmatrix} c+a \\ c+k \end{bmatrix}_q = \frac{[a+b+c]!}{[a]![b]![c]!}.$$

*Corresponding author. Emrah Kılıç

2010 Mathematics Subject Classification. 11B39, 05A30.
Keywords and Phrases. Basic hypergeometric series, Gaussian q-binomial coefficient, q-Dixon sum, Fibonomial coefficient.

Here and forth, the q-binomial sum "$\sum_k$" is finite even though the summation index k runs formally from $-\infty$ to ∞, because the summation limits (lower and upper) will be determined automatically by the q-binomial coefficients involved.

Define the $\{U_n, V_n\}$ sequences by linear recurrences for $n \geq 2$ by

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \\ V_n &= pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p. \end{aligned}$$

With $\alpha, \beta = \big(p \pm \sqrt{p^2+4}\big)/2$, they admit the expressions in the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad V_n = \alpha^n + \beta^n.$$

For $n \geq k \geq 1$, we define further the generalized Fibonomial coefficient

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k)(U_1 U_2 \dots U_{n-k})} \quad \text{with} \quad \begin{Bmatrix} n \\ 0 \end{Bmatrix}_U = \begin{Bmatrix} n \\ n \end{Bmatrix}_U = 1.$$

When $p = 1$, they reduce to the usual Fibonomial coefficient, denoted by $\left\{ {n \atop k} \right\}_F$. For more details about the Fibonomial and generalized Fibonomial coefficients, see [4, 5, 12, 14, 15].

Our approach will essentially be based on the following connection between the generalized Fibonomial and Gaussian q-binomial coefficients

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = \beta/\alpha \quad \text{or} \quad \alpha = \mathbf{i}/\sqrt{q}.$$

It will employed to transform summation formulae on Gaussian q-binomial coefficients with certain weight functions into those on generalized Fibonomial coefficients with certain generalized Fibonacci and Lucas numbers as coefficients. Therefore, we categorize such sums in the current literature according to the number of Gaussian q-binomial coefficients (or Fibonomial coefficients) in these sums:

- We can refer to [7–9, 12] where the authors compute certain sums including only one coefficient.
- In [9, 11] the authors compute various sums including products of two coefficients.
- For the sums including products of three coefficients, which are sometimes called "q-Dixon-like formula", we refer to [10].

Now we recall one result from each group. Recently the authors of [7,9] proved sum identities including certain generalized Fibonomial sums and their squares with or without generalized Fibonacci and Lucas numbers. We recall such a result: if n

and m are both *nonnegative* integers, then from [7], we have, besides three similar formulæ, that

$$\sum_{k=0}^{2n}\left\{{2n \atop k}\right\}_U U_{(2m-1)k}=T_{n,m}\sum_{k=1}^{m}\left\{{2m-1 \atop 2k-1}\right\}_U U_{(4k-2)n},$$

where

$$T_{n,m}=\begin{cases}\prod\limits_{k=0}^{n-m}V_{2k}, & \text{if } n\geq m;\\ \prod\limits_{k=1}^{m-n-1}V_{2k}^{-1}, & \text{if } n<m.\end{cases}$$

From [9], we have that for any positive integer n,

$$\sum_{k=0}^{2n}\mathbf{i}^{\pm k}\left\{{2n \atop k}\right\}_U=\mathbf{i}^{\pm n}\prod_{k=1}^{n}V_{2k-1} \text{ and } \sum_{k=0}^{2n}\left\{{2n \atop k}\right\}_U^2=\prod_{k=1}^{n}\frac{V_{2k}U_{2(2k-1)}}{U_{2k}}.$$

Recently Kılıç and Prodinger [8] computed the following Gaussian q-binomial sums with a parametric rational weight function: For any positive integer w, any nonzero real number a, nonnegative integer n, integers t and r such that $t+n\geq 0$ and $r\geq -1$,

$$\begin{aligned}\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q\frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j;q^w)_{r+1}}=(q;q)_n\Bigg\{&\sum_{j=0}^{r}\frac{(-1)^j a^{-t}q^{w\binom{j+1}{2}-twj}}{(q^w;q^w)_j\,(q^w;q^w)_{r-j}\,(aq^{wj};q)_{n+1}}\\ &-(-1)^r\sum_{j=0}^{t-r-1}\begin{bmatrix}n+j\\ n\end{bmatrix}_q\begin{bmatrix}t-1-j\\ r\end{bmatrix}_{q^w}q^{w\binom{r+1}{2}+(j-t)rw}a^{j-t}\Bigg\}.\end{aligned}$$

For the sums including certain triples of Fibonomial coefficients, with or without extra Fibonacci numbers, Kılıç and Prodinger [10] investigated them systematically, discussed their proofs and recorded a long list of summation formulæ as corollaries. We recall one of them as an example. For nonnegative integer n:

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\ k\end{bmatrix}_q^2\begin{bmatrix}2n+1\\ k\end{bmatrix}_q q^{\frac{k}{2}(3k-6n-3)}(1+q^{2k})&\\ =2(-1)^n q^{-\frac{n}{2}(3n+1)}\begin{bmatrix}2n\\ n\end{bmatrix}_q\begin{bmatrix}3n+1\\ n\end{bmatrix}_q&.\end{aligned}$$

As corollaries of previous identities, by specializing the value of q, the authors showed that each identity corresponds to two identities which have slightly different forms. In fact, the last identity corresponds to the following two identities:

$$\sum_{k=0}^{4n}(-1)^{\binom{k+1}{2}}\left\{{4n \atop k}\right\}_U^2\left\{{4n+1 \atop k}\right\}_U V_{2k}=2(-1)^n\left\{{4n \atop 2n}\right\}_U\left\{{6n+1 \atop 2n}\right\}_U,$$

$$\sum_{k=0}^{4n+2}(-1)^{\binom{k}{2}}\left\{{4n+2 \atop k}\right\}_U^2\left\{{4n+3 \atop k}\right\}_U V_{2k}=2(-1)^{n+1}\left\{{4n+2 \atop 2n+1}\right\}_U\left\{{6n+4 \atop 2n+1}\right\}_U.$$

As mentioned above, the authors [10] consider some sum formulæ including certain triple Gaussian q-binomial or Fibonomial coefficients, where one of them at least has different upper index, that is, upper indices of these coefficients are not same.

In this paper, we shall investigate cubic sums of Gaussian q-binomial coefficients and their applications to sums of Fibonomial coefficients. By means of the Carlitz–Sears transformation, we shall first prove two main theorems about the q-binomial sums of Dixon's type and apply them to derive several closed formulae. Then these q-binomial identities will be transformed into Fibonomial sums as consequences.

2. EVALUATION OF Q-BINOMIAL SUMS OF DIXON'S TYPE

In the theory of basic hypergeometric series (briefly as q-series), there are numerous useful summation and transformation formulae. We record here the following transformation due to Carlitz and Sears (cf. Gasper–Rahman [3, III-14], where the reader may also refer for the q-series notation).

Lemma 1 (Carlitz [2] and Sears [13]: $a = q^{-m}$)**.**

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix} a, & b, & d \\ & qa/b, & qa/d \end{matrix} \,\middle|\, q; \frac{qax}{bd}\right] &= \frac{(ax;q)_\infty}{(x;q)_\infty} \\
\times\, {}_5\phi_4 &\left[\begin{matrix} qa/bd, & \sqrt{a}, & -\sqrt{a}, & \sqrt{qa}, & -\sqrt{qa} \\ & ax, & q/x, & qa/b, & qa/d \end{matrix} \,\middle|\, q; q\right].
\end{aligned}
$$

Performing the replacement $k \to i - n$ on the summation index k, we can express the q-binomial sum as the following well–poised series

$$
\begin{aligned}
\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} x^k &= \sum_{i=0}^{2n} (-1)^{n-i} \begin{bmatrix} 2n \\ i \end{bmatrix}_q^3 q^{\frac{(n-i)(1+3n-3i)}{2}} x^{i-n} \\
&= q^{3\binom{n}{2}+2n} \Big(\frac{-1}{x}\Big)^n \times {}_3\phi_2\left[\begin{matrix} q^{-2n}, & q^{-2n}, & q^{-2n} \\ & q, & q \end{matrix} \,\middle|\, q; q^{1+3n}x\right].
\end{aligned}
$$

Now applying Lemma 1, we can rewrite the last ${}_3\phi_2$-series as

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{-2n}, & q^{-2n}, & q^{-2n} \\ & q, & q \end{matrix} \,\middle|\, q; q^{1+3n}x\right] \\
&= \frac{(q^{-n}x;q)_\infty}{(q^n x;q)_\infty} {}_5\phi_4\left[\begin{matrix} q^{1+2n}, & q^{-n}, & -q^{-n}, & q^{\frac{1}{2}-n}, & -q^{\frac{1}{2}-n} \\ & q, & q, & q^{-n}x, & q^{1-n}/x \end{matrix} \,\middle|\, q; q\right].
\end{aligned}
$$

Then we can reformulate the ${}_5\phi_4$-series, by its reversal, as

$$
\begin{aligned}
&{}_5\phi_4\left[\begin{matrix} q^{1+2n}, & q^{-n}, & -q^{-n}, & q^{\frac{1}{2}-n}, & -q^{\frac{1}{2}-n} \\ & q, & q, & q^{-n}x, & q^{1-n}/x \end{matrix} \,\middle|\, q; q\right] \\
&= \frac{q^n (q^{1+2n};q)_n)(q^{-2n};q)_{2n}}{(q;q)_n^3 (q^{-n}x;q)_n (q^{1-n}/x;q)_n} {}_5\phi_4\left[\begin{matrix} q^{-n}, & q^{-n}, & q^{-n}, & x, & q/x \\ & -q, & q^{\frac{1}{2}}, & -q^{\frac{1}{2}}, & q^{-3n} \end{matrix} \,\middle|\, q; q\right].
\end{aligned}
$$

By combining the last and simplifying the quotient of q-shifted factorials

$$q^n\frac{(q^{-n}x;q)_\infty}{(q^nx;q)_\infty}\frac{(q^{1+2n};q)_n(q^{-2n};q)_{2n}}{(q;q)_n^3(q^{-n}x;q)_n(q^{1-n}/x;q)_n}=q^{-2n-3\binom{n}{2}}\frac{(q;q)_{3n}}{(q;q)_n^3}(-x)^n.$$

We establish the following theorem.

Theorem 2.

$$\sum_k(-1)^k\begin{bmatrix}2n\\n+k\end{bmatrix}_q^3q^{\frac{k(3k-1)}{2}}x^k=\frac{(q;q)_{3n}}{(q;q)_n^3}$$
$$\times{}_5\phi_4\left[\begin{matrix}q^{-n}, & q^{-n}, & q^{-n}, & x, & q/x\\ & -q, & q^{\frac12}, & -q^{\frac12}, & q^{-3n}\end{matrix}\,\Big|\,q;q\right].$$

Theorem 3.

$$\sum_k(-1)^k\begin{bmatrix}2n+1\\n+k\end{bmatrix}_q^3q^{\frac{3k(k-1)}{2}}x^k=\frac{(1-x)(q;q)_{3n+1}}{(1-q)(q;q)_n^3}$$
$$\times\,{}_5\phi_4\left[\begin{matrix}q^{-n}, & q^{-n}, & q^{-n}, & qx, & q/x\\ & -q, & q^{\frac32}, & -q^{\frac32}, & q^{-1-3n}\end{matrix}\,\Big|\,q;q\right].$$

Proof. Making the replacement $k\to 1+n-i$ on the summation index k, we can express the q-binomial sum as the following well–poised series

$$\sum_k(-1)^k\begin{bmatrix}2n+1\\n+k\end{bmatrix}_q^3q^{\frac{3k(k-1)}{2}}x^k=\sum_{i=0}^{2n+1}(-1)^{1+n-i}\begin{bmatrix}2n+1\\i\end{bmatrix}_q^3q^{3\binom{1+n-i}{2}}x^{1+n-i}$$
$$=q^{3\binom{n+1}{2}}(-x)^{n+1}\times{}_3\phi_2\left[\begin{matrix}q^{-1-2n}, & q^{-1-2n}, & q^{-1-2n}\\ & q, & q\end{matrix}\,\Big|\,q;q^{3+3n}/x\right].$$

The last ${}_3\phi_2$-series can be expressed, according to Lemma 1, as

$${}_3\phi_2\left[\begin{matrix}q^{-1-2n}, & q^{-1-2n}, & q^{-1-2n}\\ & q, & q\end{matrix}\,\Big|\,q;q^{3+3n}/x\right]$$
$$=\frac{(q^{-n}/x;q)_\infty}{(q^{1+n}/x;q)_\infty}{}_5\phi_4\left[\begin{matrix}q^{2+2n}, & q^{-n}, -q^{-n}, & q^{-\frac12-n}, -q^{-\frac12-n}\\ & q, \quad q, & q^{-n}x, \quad q^{-n}/x\end{matrix}\,\Big|\,q;q\right].$$

Taking into account of the reversal series, we have also

$${}_5\phi_4\left[\begin{matrix}q^{2+2n}, & q^{-n}, -q^{-n}, & q^{-\frac12-n}, -q^{-\frac12-n}\\ & q, \quad q, & q^{-n}x, \quad q^{-n}/x\end{matrix}\,\Big|\,q;q\right]$$
$$=q^n\frac{(q^{2+2n};q)_n(q^{-1-2n};q)_{2n}}{(q;q)_n^3(q^{-n}x;q)_n(q^{-n}/x;q)_n}$$
$$\times{}_5\phi_4\left[\begin{matrix}q^{-n}, & q^{-n}, & q^{-n}, & qx, & q/x\\ & -q, & q^{\frac32}, & -q^{\frac32}, & q^{-1-3n}\end{matrix}\,\Big|\,q;q\right].$$

Finally, summing up and then simplifying the quotient

$$\begin{aligned} & q^n \frac{(q^{-n}/x;q)_\infty}{(q^{1+n}/x;q)_\infty} \frac{(q^{2+2n};q)_n (q^{-1-2n};q)_{2n}}{(q;q)_n^3 (q^{-n}x;q)_n (q^{-n}x;q)_n} \\ =& q^{-3\binom{n+1}{2}} \Big(\frac{-1}{x}\Big)^{n+1} \frac{(1-x)(q;q)_{3n+1}}{(1-q)(q;q)_n^3}. \end{aligned}$$

We prove the formula stated in Theorem 3. □

By specifying x to powers of the base q in Theorems 2 and 3, we get immediately the following elegant formulae.

Example 1 (Theorems 2 and 3: $x = 1$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} = \frac{(q;q)_{3n}}{(q;q)_n^3},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} = 0.$$

Example 2 (Theorems 2 and 3: $x = q$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k+1)}{2}} = \frac{(q;q)_{3n}}{(q;q)_n^3},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} = \frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

Example 3 (Theorems 2 and 3: $x = q^{-1}$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{3q^n(1-q^n)}{1-q^{3n}},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-5)}{2}} = -\frac{(q;q)_{3n+1}}{q(q;q)_n^3}.$$

Example 4 (Theorems 2 and 3: $x = q^2$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k+1)}{2}} = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{3q^n(1-q^n)}{1-q^{3n}},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k+1)}{2}} = \frac{(q;q)_{3n+1}}{(q;q)_n^3} \times \frac{A(q) - q^{3n+2}A(q^{-1})}{1-q^{3n+1}},$$

where $A(q) = 1 + 3q^{n+1}$.

Example 5 (Theorems 2 and 3: $x = q^{-2}$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-5)}{2}} = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{B(q) - q^{5n+1}B(q^{-1})}{(1+q^n+q^{2n})(q^{3n}-q)},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-7)}{2}} = \frac{(q;q)_{3n+1}}{(q;q)_n^3} \times \frac{A(q) - q^{3n+2}A(q^{-1})}{q^2(q^{3n+1}-1)},$$

where $A(q) = 1 + 3q^{n+1}$ *and* $B(q) = 1 - 2q^n - 2q^{2n} + 6q^{3n}$.

For two integers $\sigma \leq \tau$, define the Laurent polynomial by

$$p(y) := \sum_{\lambda=\sigma}^{\tau} \Omega_\lambda y^\lambda.$$

Then according to Theorems 2 and 3, we have the following corollaries.

Corollary 4.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3}$$

$$\times \sum_{\lambda=\sigma}^{\tau} \Omega_\lambda \cdot {}_5\phi_4 \left[\begin{matrix} q^{-n}, & q^{-n}, & q^{-n}, & q^\lambda, & q^{1-\lambda} \\ & -q, & q^{\frac{1}{2}}, & -q^{\frac{1}{2}}, & q^{-3n} \end{matrix} \middle| q; q \right].$$

Corollary 5.

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = \frac{(q;q)_{3n+1}}{(1-q)(q;q)_n^3}$$

$$\times \sum_{\lambda=\sigma}^{\tau} (1-q^\lambda)\Omega_\lambda {}_5\phi_4 \left[\begin{matrix} q^{-n}, & q^{-n}, & q^{-n}, & q^{1+\lambda}, & q^{1-\lambda} \\ & -q, & q^{\frac{3}{2}}, & -q^{\frac{3}{2}}, & q^{-1-3n} \end{matrix} \middle| q; q \right].$$

Performing the replacements $k \to -k$ and $k \to 1-k$, respectively, on summation indices, we get the following reciprocal relations

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}+k\lambda} = \sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k+1)}{2}-k\lambda},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}+k\lambda} = \sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}-k\lambda+\lambda}.$$

They lead directly to the two general annihilated q-binomial sums:

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} \left\{p(q^k) - q^k p(q^{-k})\right\} = 0,$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} \left\{p(q^k) + p(q^{1-k})\right\} = 0.$$

The two formulae displayed in Corollaries 4 and 5 are remarkably useful in evaluating the q-binomial sums of Dixon's type weighted by Laurent polynomial factors, because the sums on the right hand side contain a limited number of terms independent of n. By appropriately devised *Mathematica* commands, we are able to deduce a number of closed formulae. A selection of those "nice" ones are given below as examples, where the specific polynomials are highlighted in the headers.

Example 6 ($p(y) = 1 + y$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = 2\frac{(q;q)_{3n}}{(q;q)_n^3},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = \frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

Example 7 ($p(y) = 1 - y$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = 0,$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = -\frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

Example 8 ($p(y) = 1 + y^{-1}$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = -\frac{(q;q)_{3n+1}}{q(q;q)_n^3}.$$

Example 9 ($p(y) = 1 - y^{-1}$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^3}{1-q^{3n}},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = \frac{(q;q)_{3n+1}}{q(q;q)_n^3}.$$

Example 10 ($p(y) = (1 + y)^2$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{3\left(1-q^{2n}\right)\left(1+q^n\right)}{1-q^{3n}}.$$

Example 11 ($p(y) = (1 - y)^2$)**.**

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = -\frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^3}{1-q^{3n}}.$$

Example 12 $(p(y) = y^{-1}(1+y)^2)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{3(1+q^n)\left(1-q^{2n}\right)}{(1-q^{3n})}.$$

Example 13 $(p(y) = y^{-1}(1-y)^2)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = -\frac{(q;q)_{3n}}{(q;q)_n^3} \frac{(1-q^n)^3}{(1-q^{3n})}.$$

Example 14 $(p(y) = y(1+y^{-1})^2)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{3\left(1-q^{2n}\right)(1+q^n)}{(1-q^{3n})}.$$

Example 15 $(p(y) = y(1-y^{-1})^2)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = -\frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^3}{(1-q^{3n})}.$$

Example 16 $(p(y) = y^{-1}(1+y)^3)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{6(1+q^n)\left(1-q^{2n}\right)}{(1-q^{3n})}.$$

Example 17 $(p(y) = y^{-1}(1-y)^3)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = 0.$$

Example 18 $(p(y) = y^2(1+y^{-1})^3)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{6(1+q^n)\left(1-q^{2n}\right)}{1-q^{3n}}.$$

Example 19 $(p(y) = y^2(1-y^{-1})^3)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = 0.$$

Example 20 $(p(y) = (1 \pm q^n y))$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = (1 \pm q^n) \frac{(q;q)_{3n}}{(q;q)_n^3}.$$

Example 21 ($p(y) = (1 - q^n y)^2$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^2(1-q^{2n})}{(1-q^{3n})}.$$

Example 22 ($p(y) = y^{-1}(1 - q^n y)^2$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = q^n \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^2(1-q^{2n})}{(1-q^{3n})}.$$

Example 23 ($p(y) = y(1 + q^n/y)^2$).

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = (1 - q^{2n-1}) \frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

Example 24 ($p(y) = y(1 - q^n/y)^2$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^2(1-q^{2n})}{(1-q^{3n})},$$

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = (1 - q^{2n-1}) \frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

Example 25 ($p(y) = (1 - q^n y)^3$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^3(1-q^{2n})^3}{(1-q^{3n})(1-q^{3n-1})}.$$

Example 26 ($p(y) = y^{-1}(1 + q^n y)^3$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{6q^n(1-q^{4n})}{(1-q^{3n})}.$$

Example 27 ($p(y) = y^{-1}(1 - q^n y)^3$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = 0.$$

Example 28 ($p(y) = y(1 - q^n/y)^3$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{(1-q^n)^3(1-q^{2n})^3}{(1-q^{3n})(1-q^{3n-1})}.$$

Example 29 ($p(y) = y^2(1 + q^n/y)^3$).

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = \frac{(q;q)_{3n}}{(q;q)_n^3} \times \frac{6q^n(1-q^{4n})}{(1-q^{3n})}.$$

Example 30 $(p(y) = y^2(1-q^n/y)^3)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = 0.$$

Example 31 $(p(y) = y(1 \pm q^{2n}/y^2))$.

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = \left(1 \mp q^{2n-1}\right) \frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

Example 32 $(p(y) = (1 \pm q^{2n+1}y))$.

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{\frac{k(3k-1)}{2}} p(q^k) = (1 \pm q^{2n+1}) \frac{(q;q)_{3n}}{(q;q)_n^3}.$$

Example 33 $(p(y) = y(1 \pm q^{2n+1}/y)^2)$.

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q^3 q^{\frac{3k(k-1)}{2}} p(q^k) = (1 - q^{4n+1}) \frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

More identities could be derived similarly. However, we shall not enlarge this list of examples due to the space limitation.

3. APPLICATIONS TO FIBONOMIAL SUMS IDENTITIES

As described in the introduction, we give, in this section, some applications to the generalized Fibonomial sums identities by specializing the value of $q = \beta/\alpha$, in the examples established in the last section. For each consequent identity, the corresponding example used to derive it will be specified. Furthermore, we point out that all identities displayed below hold for all the nonnegative integers n with $\Delta = p^2 + 4$.

1. Examples 6 and 7:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 V_k = 2 \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U,$$

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 U_k = 0.$$

2. Example 16: $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$ otherwise.

$$\sum_k (-1)^{\binom{k+1}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 V_k^3 = 2(3 - \delta_{0n}) \frac{V_n U_{2n}}{U_{3n}} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

3. Example: 20:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 V_{n+k} = V_n \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U,$$
$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 U_{n+k} = U_n \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

4. Example 23:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n+1 \\ n+k \end{Bmatrix}_U^3 V_{n-k}^2 = -\Delta U_{2n-1} U_{3n+1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

5. Example 24:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n+1 \\ n+k \end{Bmatrix}_U^3 U_{n-k}^2 = U_{3n+1} U_{2n-1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

6. Example 29:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 V_{n-k}^3 = 6\,(-1)^n \frac{U_{4n}}{U_{3n}} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

7. Example 31:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n+1 \\ n+k \end{Bmatrix}_U^3 U_{2n-2k} = V_{2n+1} U_{3n+1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U,$$
$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n+1 \\ n+k \end{Bmatrix}_U^3 V_{2n-2k} = -\Delta U_{2n-1} U_{3n+1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

8. Example 32:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 V_{2n+1+k} = V_{2n+1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U,$$
$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U^3 U_{2n+1+k} = U_{2n+1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

9. Example 33:

$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n+1 \\ n+k \end{Bmatrix}_U^3 U_{2n+1-k}^2 = U_{4n+1} U_{3n+1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U,$$
$$\sum_k (-1)^{\binom{k}{2}} \begin{Bmatrix} 2n+1 \\ n+k \end{Bmatrix}_U^3 V_{2n+1-k}^2 = \Delta U_{4n+1} U_{3n+1} \begin{Bmatrix} 3n \\ n \end{Bmatrix}_U \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U.$$

REFERENCES

1. W. N. Bailey, *A note on certain q-identities*, Quart. J. Math. (Oxford) **12** (1941), 173–175.
2. L. Carlitz, *Some formulas of F. H. Jackson*, Monatshefte für Math. **73** (1969), 193–198.
3. G. Gasper and M. Rahman, "*Basic Hypergeometric Series (2nd Edition)*", Cambridge University Press, Cambridge, 2004.
4. H. W. Gould, *The bracket function and Fountené-Ward generalized binomial coefficients with application to fibonomial coefficients*, Fibonacci Quart. **7** (1969), 23–40.
5. V. E. Hoggatt Jr., *Fibonacci numbers and generalized binomial coefficients*, Fibonacci Quart. **5** (1967), 383–400.
6. F. H. Jackson, *Certain q-identities*, Quart. J. Math. (Oxford) **12** (1941), 167–172.
7. E. Kılıç, H. Prodinger, I. Akkuş and H. Ohtsuka, *Formulas for Fibonomial sums with generalized Fibonacci and Lucas coefficients*, Fibonacci Quart. **49**(4) (2011), 320–329.
8. E. Kılıç and H. Prodinger, *Evaluation of sums involving Gaussian q-binomial coefficients with rational weight functions*, Int. J. Number Theory **12**(2) (2016), 495–504.
9. E. Kılıç, H. Ohtsuka and I. Akkuş, *Some generalized Fibonomial sums related with the Gaussian q-binomial sums*, Bull. Math. Soc. Sci. Math. Roumanie **55**(1) (103) (2012), 51–61.
10. E. Kılıç and H. Prodinger, *Formulae related to the q-Dixon formula with applications to Fibonomial sums*, Periodica Math. Hungarica **70** (2015), 216–226.
11. N. N. Li and W. Chu, *q-Derivative operator proof for a conjecture of Melham*, Discrete Applied Mathematics, 177 (2014), 158–164.
12. D. Marques and P. Trojovsky, *On some new sums of Fibonomial coefficients*, Fibonacci Quart. **50**(2) (2012), 155–162.
13. D. B. Sears, *Transformations of basic hypergeometric functions of special type*, Proc. London Math. Soc. **2**(52) (1951), 467–483.
14. J. Seibert and P. Trojovsky, *On some identities for the Fibonomial coefficients*, Math. Slovaca **55** (2005), 9–19.
15. P. Trojovsky, *On some identities for the Fibonomial coefficients via generating function*, Discrete Appl. Math. **155**(15) (2007), 2017–2024.

(Received 06.04.2019)
(Revised 10.02.2020)

Wenchang Chu
School of Mathematics and Statistics
Zhoukou Normal University
Zhoukou (Henan), P. R. China
Corresponding address:
Department of Mathematics and Physics
University of Salento (P. O. Box 193)
73100 Lecce, Italy
E-mail: *chu.wenchang@unisalento.it*

Emrah Kılıç
TOBB University of Economics and Technology
Mathematics Department 06560
Sögütözü Ankara Turkey
Email: *ekilic@etu.edu.tr*