Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **13** (2019), 746–773.
https://doi.org/10.2298/AADM190326031A

CERTAIN FEYNMAN TYPE INTEGRALS INVOLVING GENERALIZED K-MITTAG-LEFFLER FUNCTION AND GENERAL CLASS OF POLYNOMIALS

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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In the present paper, certain Feynman type integrals involving the generalized k-Mittag-Leffler function and the general class of polynomials are established and further extended these results involving Laguerre polynomials. On account of the most general nature of the functions involved therein, our main findings are capable of yielding a large number of new, interesting, and useful integrals, expansion formulas involving the generalized k-Mittag-Leffler function, and the Laguerre polynomials as their special cases.

1. INTRODUCTION AND PRELIMINARIES

Numerous integral formulae associated with a variety of special functions have been established by many authors (see, [**21, 20, 23, 22**]). Very recently, Jain et al. [**25**] gave pathway fractional integrals with the 3m-parametric type Mittag-Leffler function and and discusses some of it's particular cases in application point of view. In the present paper, integral formulae are established involving k-Mittag-Leffler

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2010 Mathematics Subject Classification. 26A33, 33C45, 33C60, 33C70.
Keywords and Phrases. Generalized k-Mittag-Leffler function, general class of polynomials, Lagurre polynomials, Pochhammer symbol, Feynman integrals.

function and are expressed in terms of generalized (Wright) hypergeometric functions.

In 2006, Diaz and Pariguan[**24**] introduced the k-Pochhemmer symbol and k-gamma function defined as follows:

$$(1) \qquad (\gamma)_{n,k} := \begin{cases} \dfrac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R}; \gamma \in \mathbb{C} \setminus \{0\}) \\ \gamma(\gamma+k)...(\gamma+(n-1)k) & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases}$$

and the relation with the classical Euler's gamma function as:

$$(2) \qquad \Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right)$$

where $\gamma \in \mathbb{C}, k \in \mathbb{R}$ and $n \in \mathbb{N}$.

when $k = 1$, (1) reduces to the classical Pochhammer symbol and Euler's gamma function respectively.

Also let $\gamma \in \mathbb{C}, k, s \in \mathbb{R}$, then the following identity holds

$$(3) \qquad \Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1}\Gamma_k\left(\frac{k\gamma}{s}\right),$$

in particular,

$$(4) \qquad \Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right),$$

Further, let $\gamma \in \mathbb{C}, k, s \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, then the following identity holds

$$(5) \qquad (\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq}\left(\frac{k\gamma}{s}\right)_{nq},$$

in particular,

$$(6) \qquad (\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq},$$

For more details of k-Pochhammer symbol, k-special function and fractional Fourier transform one can refer to the papers by Romero et. al.[**18, 19**]

Let $k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $q \in \mathbb{R}^+$, then the generalized k-Mittag-Leffler function, denoted by $E_{k,\alpha,\beta}^{\gamma,q}(z)$, is defined as

$$E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} z^n}{\Gamma_k(n\alpha+\beta)n!} \tag{7}$$

where $(\gamma)_{nq,k}$ denotes the K-Pochhammer symbol given by equation (6) and $\Gamma_k(\gamma)$ is the K-gamma function given by the equation (4) as (also see[**13**]).

Particular cases of $E_{k,\alpha,\beta}^{\gamma,q}(z)$

(i)For $q = 1$, equation(7) yields k-Mittag-Leffler function (Dorrego and Cerutti[**7**]), defined as:

$$E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k(n\alpha+\beta)n!} = E_{k,\alpha,\beta}^{\gamma}(z) \tag{8}$$

(ii) For $k = 1$, equation(7) yields Mittag-Leffler function, defined as (Shukla and Prajapati [**1**])

$$E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq} z^n}{\Gamma(n\alpha+\beta)n!} = E_{\alpha,\beta}^{\gamma,q}(z), \tag{9}$$

(iii) For $q = 1$ and $k = 1$, equation(7) gives Mittag-Leffler function, defined as (Dorrego and Cerutti [**7**])

$$E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha+\beta)n!} = E_{\alpha,\beta}^{\gamma}(z) \tag{10}$$

(iv) For $q = 1, k = 1$ and $\gamma = 1$, equation(7) gives Mittag-Leffler function (Wiman [**3**]), defined as

$$E_{1,\alpha,\beta}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z) \tag{11}$$

(v) For $q = 1, k = 1, \gamma = 1$ and $\beta = 1$, equation(7) gives Mittag-Leffler function (Mittag-Leffler [**8**]), defined as

$$E_{1,\alpha,1}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} = E_{\alpha}(z). \tag{12}$$

(vi) For $q = 1, k = 1, \gamma = 1, \alpha = 1$ and $\beta = 1$, equation(7) gives Mittag-Leffler function (Mittag-Leffler [**8**]), defined as

$$E_{1,1,1}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = exp(z). \tag{13}$$

The results given by Kiryakova [**27**], Miller and Ross [**17**], Srivastava et. al., [**11**] can be referred for some basic results on fractional calculus. The Fox-Wright function ${}_p\Psi_q$ defined as (see, for details, Srivastava and Karisson 1985, [**10**])

$$\begin{aligned} {}_p\Psi_q[z] = {}_p\Psi_q \left[\begin{matrix} (a_1, \alpha_1), ..., (a_p, \alpha_p); \\ (b_1, \beta_1), ..., (b_q, \beta_q); \end{matrix} \; z \right] \\ = {}_p\Psi_q \left[\begin{matrix} (a_i, \alpha_i)_{1,p}; \\ (b_j, \beta_j)_{1,q}; \end{matrix} \; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i n)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j n)} \frac{z^n}{n!}, \end{aligned} \tag{14}$$

where the coefficients $\alpha_1, ..., \alpha_p,\ \beta_1, ..., \beta_q \in \mathbb{R}^+$ such that

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0. \tag{15}$$

The general class of polynomials $S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}[x]$ will be defined and represented as follows [**9**, p.185, Eqn. (7)]:

$$S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[x] = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} x^{l_i} \tag{16}$$

where $n_1, ..., n_r = 0, 1, 2, ...; m_1, ..., m_r$ are arbitrary positive integers, the coefficients $A_{n_i,l_i}(n_i, l_i \geq 0)$ are arbitrary constants, real or complex. $S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}[x]$ yields a number of known polynomials as its special cases. These includes, among other, the Jacobi polynomials, the Bessel Polynomials,the Lagurre Polynomials, the Brafman Polynomials and several others [**12**, p. 158-161].

The following results and definitions are also required in our investigations.

Prabhaker and Suman [**26**] defined the polynomials $L_n^{(\alpha,\beta)}(x)$ as:

$$L_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha n + \beta + 1)}{\Gamma(n+1)} \sum_{k=0}^{n} \frac{(-n)_k \, x^k}{k! \Gamma(\alpha k + \beta + 1)} \tag{17}$$

Where $\alpha \in \mathbb{C}^{+}$, $\beta \in \mathbb{C}_{-1}^{+}$and $n \in \mathbb{N}$.

If $\alpha = 1$, then (17) reduces as:

$$L_n^{(1,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \sum_{k=0}^{n} \frac{(-n)_k x^k}{k!\Gamma(k+\beta+1)} = L_n^{\beta}(x) \tag{18}$$

Where $L_n^{\beta}(x)$is well-known generalized Laguerre polynomials (Rainville [**5**]).

The Konhauser polynomials of second kind (Srivastava [**14**]) is defined as:

$$Z_n^{\beta}(x;k) = \frac{\Gamma(kn+\beta+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\beta+1)} \tag{19}$$

Where $\beta \in \mathbb{C}_{-1}^{+}, n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

It can be easily verified that:

$$L_n^{(k,\beta)}\left(x^k\right) = Z_n^{\beta}(x;k) \tag{20}$$

$$L_n^{\beta}(x) = Z_n^{\beta}(x;1) \tag{21}$$

The polynomials $Z_n^{(\alpha,\beta)}(x;k)$ is defined[**2**] as:

$$Z_n^{(\alpha,\beta)}(x;k) = \sum_{j=0}^{n} \frac{\Gamma(kn+\beta+1)(-1)^j x^{kj}}{j!\Gamma(kj+\beta+1)\Gamma(\alpha n-\alpha j+1)}, \tag{22}$$

Where $\alpha \in \mathbb{C}^{+}$, $\beta \in \mathbb{C}_{-1}^{+}, n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

From (19) and (22), we get:

$$Z_n^{(1,\beta)}(x;k) = Z_n^{\beta}(x;k) \tag{23}$$

If $\alpha \in \mathbb{N}$ then (22) can be written in the following form:

$$Z_n^{(\alpha,\beta)}(x;k) = \frac{\Gamma(kn+\beta+1)}{\Gamma(\alpha n+1)} \sum_{m=0}^{n} \frac{(-\alpha n)_{\alpha m} x^{km}}{m!\Gamma(km+\beta+1)(-1)^{(\alpha-1)m}} \tag{24}$$

The set of polynomials $L_n^{(\alpha,\beta)}(\gamma;x)$ is defined [**2**] as:

$$L_n^{(\alpha,\beta)}(\gamma;x)=\sum_{r=0}^{n}\frac{\Gamma(\alpha n+\beta+1)(-1)^r x^r}{r!\Gamma(\alpha r+\beta+1)\Gamma(\gamma n-\gamma r+1)} \tag{25}$$

Where $\alpha,\gamma\in\mathbb{C}^+$, $\beta\in\mathbb{C}^+_{-1}, n\in\mathbb{N}$.

From (25) and (17), we have:

$$L_n^{(\alpha,\beta)}(1;x)=L_n^{(\alpha,\beta)}(x) \tag{26}$$

One can easily verify that:

$$L_n^{(k,\beta)}\left(\alpha;x^k\right)=Z_n^{(\alpha,\beta)}(x;k) \tag{27}$$

$$Z_n^{(1,\beta)}(x;1)=L_n^{\beta}(\alpha;x) \tag{28}$$

$$Z_n^{(1,\beta)}(x;1)=Z_n^{\beta}(x;1)=L_n^{\beta}(x) \tag{29}$$

$$L_n^{(1,\beta)}(1;x)=L_n^{(1,\beta)}(x)=L_n^{\beta}(x) \tag{30}$$

Some facts are listed below (see Spanier and Oldham [**16**])

$$(-x)_n=(-1)^n(x-n+1)_n \quad , \tag{31}$$

$$(x+y)_n=\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}(x)_j(y)_{n-j} \quad , \tag{32}$$

$$(x)_{n+m}=(x)_n(x+n)_m \quad \text{and} \tag{33}$$

$$\begin{pmatrix} x \\ n \end{pmatrix}=\frac{(-1)^n}{n!}(-x)_n \quad . \tag{34}$$

2. INTEGRAL FORMULAS

In this section, we establish certain new double integral relations pertaining to a product involving general class of polynomials and generalized k-Mittag-Leffler function and further we extend these results involving Laguerre polynomials. These double integral relations are unified in nature and act as key formulae from which we can obtain as their special cases, double integral relations concerning a large number of simpler special function and polynomials, which are listed below.

Theorem 1. *If* $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < x, y < 1$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\lambda} \left[\frac{1-y}{1-xy}\right]^{\mu} \left[\frac{1-xy}{(1-x)(1-y)}\right] S_{n_1 \ldots n_r}^{m_1 \ldots m_r} \left[\frac{(1-x)y}{1-xy}\right] \\
&\qquad \times E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right) dxdy \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda + l_i) \\
&\quad \times {}_2\Psi_2\left[\begin{array}{c} (\gamma/k, q), (\mu, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu, 1) \end{array} \middle| k^{(q-\alpha/k)} v\right].
\end{aligned} \tag{35}$$

Proof. For convenience, we denote the left-hand side of the result (35) by $\mathscr{I}$. Using (7) and (16), then changing the order of integration and summation, which are valid under the conditions of Theorem 1, then

$$\begin{aligned}
\mathscr{I} &= \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} v^n}{\Gamma_k(n\alpha + \beta) n!} \\
&\times \int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\lambda + l_i} \left[\frac{1-y}{1-xy}\right]^{\mu + n} \left[\frac{1-xy}{(1-x)(1-y)}\right] dxdy,
\end{aligned} \tag{36}$$

further using the known result [**15**, p.145], we have:

$$\begin{aligned}
\mathscr{I} &= \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \\
&\times \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} v^n}{\Gamma_k(n\alpha + \beta) n!} \frac{\Gamma(\lambda + l_i)\Gamma(\mu + n)}{\Gamma(\lambda + l_i + \mu + n)}
\end{aligned} \tag{37}$$

applying the results (1) and (2), the above equation (37) reduced to

$$\mathscr{I} = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda+l_i) \times \sum_{n=0}^{\infty} \frac{\Gamma(\gamma/k+nq)}{\Gamma(n\alpha/k+\beta/k)} \frac{\Gamma(\mu+n)}{\Gamma(\lambda+l_i+\mu+n)} \frac{k^{n(q-\alpha/k)} v^n}{n!} \tag{38}$$

interpret the above equation with the help of (14), we have the required result. □

Theorem 2. *If $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ and $q \in \mathbb{R}^+$ such that $0 < u, v < \infty$. Then the following formulas hold:*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda-1} S_{n_1 \ldots n_r}^{m_1 \ldots m_r}[u] \, E_{k,\alpha,\beta}^{\gamma,q}(v) \, du dv \\ &= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda+l_i) \\ &\times \int_0^\infty \phi(z) z^{\lambda+l_i+\mu-1} {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu, 1) \end{matrix} \Big| k^{(q-\alpha/k)} z \right] \end{aligned} \tag{39}$$

Proof. For convenience, we denote the left-hand side of the result (39) by $\mathscr{I}$. Using (7) and (16), then changing the order of integration and summation, which are valid under the conditions of Theorem 2, then

$$\begin{aligned} \mathscr{I} = &\sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(n\alpha+\beta) n!} \\ &\times \int_0^\infty \int_0^\infty \phi(u+v) v^{\mu+n-1} u^{\lambda+l_i-1} du dv, \end{aligned} \tag{40}$$

further making the use of known result [**15**, p.177], after simplification the above equation (40) reduced to

$$\begin{aligned} \mathscr{I} = &\sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(n\alpha+\beta) n!} \\ &\times \int_0^\infty \phi(z) z^{\lambda+l_i+\mu+n-1} \frac{\Gamma(\lambda+l_i) \Gamma(\mu+n)}{\Gamma(\lambda+l_i+\mu+n)} dz, \end{aligned} \tag{41}$$

applying the results (1) and (2), the above equation (37) reduced to

$$
\begin{aligned}
\mathscr{I} = & \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda + l_i) \\
& \times \int_0^\infty \phi(z) z^{\lambda+l_i+\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma/k + nq)}{\Gamma(n\alpha/k + \beta/k)} \\
& \times \frac{\Gamma(\mu+n)}{\Gamma(\lambda + l_i + \mu + n)} \frac{k^{n(q-\alpha/k)} z^n}{n!} dz,
\end{aligned} \tag{42}
$$

interpret the above equation with the help of (14), we have the required result. □

Theorem 3. *If* $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < 1$. *Then the following formulas hold:*

$$
\begin{aligned}
& \int_0^1 \int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1} v^{\lambda} S_{n_1 \dots n_r}^{m_1 \dots m_r}[v(1-u)] \\
& \times E_{k,\alpha,\beta}^{\gamma,q}(1-v)\, du dv \\
& = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda + l_i) \\
& \times \int_0^\infty f(z)(1-z)^{\lambda+l_i+\mu-1} \\
& \times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu, 1) \end{matrix} \middle| k^{(q-\alpha/k)}(1-z) \right].
\end{aligned} \tag{43}
$$

Proof. For convenience, we denote the left-hand side of the result (39) by $\mathscr{I}$. Using (7) and (16), then changing the order of integration and summation, which are valid under the conditions of Theorem 3, then

$$
\begin{aligned}
\mathscr{I} = & \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(n\alpha + \beta) n!} \\
& \times \int_0^1 \int_0^1 f(uv)(1-u)^{\lambda+l_i-1}(1-v)^{\mu+n-1} v^{\lambda+l_i} du dv,
\end{aligned} \tag{44}
$$

further using the known result [**15**, p.243], the above equation (44) reduced to the following form:

$$
\mathscr{I} = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(n\alpha+\beta)n!}
\times \int_0^{\infty} f(z)(1-z)^{\lambda+l_i+\mu+n-1} \frac{\Gamma(\lambda+l_i)\Gamma(\mu+n)}{\Gamma(\lambda+l_i+\mu+n)} dz, \tag{45}
$$

applying the results (1) and (2), the above equation (37) reduced to

$$
\begin{aligned}
\mathscr{I} = {} & \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda+l_i) \\
& \times \int_0^{\infty} f(z)(1-z)^{\lambda+l_i+\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma/k+nq)}{\Gamma(n\alpha/k+\beta/k)} \\
& \times \frac{\Gamma(\mu+n)\, k^{n(q-\alpha/k)}(1-z)^n}{\Gamma(\lambda+l_i+\mu+n)\, n!} dz,
\end{aligned} \tag{46}
$$

interpret the above equation with the help of (14), we have the required result.

□

In the sequel of our paper the following lemma is important to establish the main results.

Lemma 1. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}^+_{-1}, m, n \in \mathbb{N}$ *[**21**], then*

$$
\begin{aligned}
& L_n^{(a,b)}(\xi; x)\, L_m^{(c,d)}(\zeta; x) \\
& \quad = \sum_{h=0}^{m+n} \sum_{k=0}^{h} \frac{\Gamma(an+b+1)\Gamma(cm+d+1)}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)} \\
& \quad \times \frac{(-x)^h}{\Gamma(\xi(n-k)+1)\Gamma(ak+b+1)\Gamma(c(h-k)+d+1)}
\end{aligned} \tag{47}
$$

Theorem 4. *If* $a, c, \zeta, \xi \in \mathbb{C}^+, b, d \in \mathbb{C}^+_{-1}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < x, y < 1$. *Then the following formulas hold:*

$$\begin{aligned}&\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\lambda}\left[\frac{1-y}{1-xy}\right]^{\mu}\left[\frac{1-xy}{(1-x)(1-y)}\right]\\&\quad\times S_{n_1\ldots n_r}^{m_1\ldots m_r}\left[\frac{(1-x)y}{1-xy}\right]L_n^{(a,b)}\left[\xi;\sigma\left(\frac{1-y}{1-xy}\right)\right]\\&\quad\times L_m^{(c,d)}\left[\zeta;\sigma\left(\frac{1-y}{1-xy}\right)\right]E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right)dxdy\\&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\sum_{h=0}^{m+n}\Delta_{a,b,c,d}^{n,m,\xi,\zeta}(\sigma)^h\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\\&\quad\times\Gamma(\lambda+l_i)\,{}_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}v\right].\end{aligned}\tag{48}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by*

$$\begin{aligned}\Delta_{a,b,c,d}^{n,m,\xi,\zeta}&=\sum_{k=0}^{h}\binom{h}{k}\frac{\Gamma(an+b+1)\,\Gamma(cm+d+1)\,(-1)^h}{\Gamma(\zeta(m-h+k)+1)\,\Gamma(\xi(n-k)+1)}\\&\quad\times\frac{1}{\Gamma(h+1)\,\Gamma(ak+b+1)\,\Gamma(c(h-k)+d+1)}\end{aligned}\tag{49}$$

Proof. For convenience, we denote the left-hand side of the result (48) by $\mathscr{I}$, we have

$$\begin{aligned}\mathscr{I}&=\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\lambda}\left[\frac{1-y}{1-xy}\right]^{\mu}\left[\frac{1-xy}{(1-x)(1-y)}\right]\\&\quad\times S_{n_1\ldots n_r}^{m_1\ldots m_r}\left[\frac{(1-x)y}{1-xy}\right]L_n^{(a,b)}\left[\xi;\sigma\left(\frac{1-y}{1-xy}\right)\right]\\&\quad\times L_m^{(c,d)}\left[\zeta;\sigma\left(\frac{1-y}{1-xy}\right)\right]E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right)dxdy\end{aligned}\tag{50}$$

and upon using the result from equation (47), the equation (50) reduced to

$$\begin{aligned}\mathscr{I}&=\sum_{h=0}^{m+n}\sum_{k=0}^{h}\frac{\Gamma(an+b+1)\,\Gamma(cm+d+1)}{\Gamma(h-k+1)\,\Gamma(\zeta(m-h+k)+1)\,\Gamma(k+1)}\\&\quad\times\frac{(-\sigma)^h}{\Gamma(\xi(n-k)+1)\,\Gamma(ak+b+1)\,\Gamma(c(h-k)+d+1)}\\&\quad\times\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\lambda}\left[\frac{1-y}{1-xy}\right]^{\mu+h}\left[\frac{1-xy}{(1-x)(1-y)}\right]\\&\quad\times S_{n_1\ldots n_r}^{m_1\ldots m_r}\left[\frac{(1-x)y}{1-xy}\right]E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right)dxdy\end{aligned}\tag{51}$$

applying the result from equation (35), the equation (51) yields

$$\begin{aligned}\mathscr{I} = & \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{h=0}^{m+n} \sum_{k=0}^{h} \frac{\Gamma(an+b+1)\Gamma(cm+d+1)}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)} \\ & \times \frac{(-\sigma)^h}{\Gamma(\xi(n-k)+1)\Gamma(ak+b+1)\Gamma(c(h-k)+d+1)} \\ & \times \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda+l_i) \\ & \times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{array} \middle| k^{(q-\alpha/k)} v \right].\end{aligned} \tag{52}$$

The above equation (52) can be written as:

$$\begin{aligned}\mathscr{I} = & \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{h=0}^{m+n} \sum_{k=0}^{h} \binom{h}{k} \frac{\Gamma(an+b+1)\Gamma(cm+d+1)(\sigma)^h}{\Gamma(h+1)\Gamma(\zeta(m-h+k)+1)} \\ & \times \frac{(-1)^h}{\Gamma(\xi(n-k)+1)\Gamma(ak+b+1)\Gamma(c(h-k)+d+1)} \\ & \times \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda+l_i) \\ & \times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{array} \middle| k^{(q-\alpha/k)} v \right].\end{aligned} \tag{53}$$

Thus, we have

$$\begin{aligned}\mathscr{I} = & \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{h=0}^{m+n} \Delta_{a,b,c,d}^{n,m,\xi,\zeta} (\sigma)^h \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \\ & \times \Gamma(\lambda+l_i) {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{array} \middle| k^{(q-\alpha/k)} v \right].\end{aligned} \tag{54}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ is given by the equation (49).

This completes the proof of the equation (48).

□

Theorem 5. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}_{-1}^+, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that*

$0 < u, v < \infty$. *Then the following formulas hold:*

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda-1} S_{n_1 \dots n_r}^{m_1 \dots m_r}[u] \, L_n^{(a,b)}[\xi; \sigma(u)] \\
&\quad \times L_m^{(c,d)}[\zeta; \sigma(u)] \, E_{k,\alpha,\beta}^{\gamma,q}(v) \, du dv \\
&\quad = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{h=0}^{m+n} \Delta_{a,b,c,d}^{n,m,\xi,\zeta} (\sigma)^h \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i, l_i} \\
&\quad \times \Gamma(\lambda + l_i) \int_0^\infty \phi(z) z^{\lambda + l_i + \mu + h - 1} \\
&\quad \times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu + h, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu + h, 1) \end{matrix} \middle| k^{(q-\alpha/k)} z \right]
\end{aligned} \tag{55}
$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by equation* (49).

Proof. The proof of the equation (55) is same as the proof (48), so we omit the detail. □

Theorem 6. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}_{-1}^+, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < 1$. *Then the following formulas hold:*

$$
\begin{aligned}
&\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1} v^{\lambda} S_{n_1 \dots n_r}^{m_1 \dots m_r}[v(1-u)] \\
&\quad \times L_n^{(a,b)}[\xi; \sigma(1-u)] \, L_m^{(c,d)}[\zeta; \sigma(1-u)] \, E_{k,\alpha,\beta}^{\gamma,q}(1-v) \, du dv \\
&\quad = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{h=0}^{m+n} \Delta_{a,b,c,d}^{n,m,\xi,\zeta} (\sigma)^h \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i, l_i} \\
&\quad \times \Gamma(\lambda + l_i) \int_0^\infty f(z)(1-z)^{\lambda + l_i + \mu + h - 1} \\
&\quad \times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu + h, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu + h, 1) \end{matrix} \middle| k^{(q-\alpha/k)} (1-z) \right].
\end{aligned} \tag{56}
$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by equation* (49).

Proof. The proof of the equation (56) is same as the proof (48), so we omit the detail. □

Theorem 7. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}_{-1}^+, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < x, y < 1$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\lambda}\left[\frac{1-y}{1-xy}\right]^{\mu}\left[\frac{1-xy}{(1-x)(1-y)}\right]\\
&\times S_{n_1\dots n_r}^{m_1\dots m_r}\left[\frac{(1-x)y}{1-xy}\right]L_n^{(a,b)}\left[\xi;\sigma\left(\frac{1-y}{1-xy}\right)\right]\\
&\times L_m^{(c,d)}\left[\zeta;\sigma\left(\frac{1-y}{1-xy}\right)\right]E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right)dxdy \qquad (57)\\
&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\sum_{l_1=0}^{[n_1/m_1]}\dots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\sum_{h=0}^{m+n}(\sigma)^h\,{}_1\Delta_{a,b,c,d}^{n,m,\xi,\zeta}\\
&\times\Gamma(\lambda+l_i)_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}v\right].
\end{aligned}$$

where ${}_1\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by*

$$\begin{aligned}
{}_1\Delta_{a,b,c,d}^{n,m,\xi,\zeta}&=\frac{\Gamma(an+b+1)\,\Gamma(cm+d+1)}{\Gamma(\zeta m+1)\,\Gamma(\xi n+1)}\\
&\times\sum_{k=0}^{h}\left[\binom{h}{k}\frac{(-1)^{h-\zeta(h-k)-\xi k}(-\zeta m)_{\zeta(h-k)}(-\xi n)_{\xi k}}{\Gamma(ak+b+1)\,\Gamma(c(h-k)+d+1)}\right] \qquad (58)
\end{aligned}$$

Proof. Let $\xi,\zeta\in\mathbb{N}$ and using (31), the equation (53) reduced to

$$\begin{aligned}
\mathscr{I}&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\frac{\Gamma(an+b+1)\,\Gamma(cm+d+1)}{\Gamma(\zeta m+1)\,\Gamma(\xi n+1)}\sum_{h=0}^{m+n}(\sigma)^h\\
&\times\sum_{k=0}^{h}\left[\binom{h}{k}\frac{(-1)^{h-\zeta(h-k)-\xi k}(-\zeta m)_{\zeta(h-k)}(-\xi n)_{\xi k}}{\Gamma(ak+b+1)\,\Gamma(c(h-k)+d+1)}\right] \qquad (59)\\
&\times\sum_{l_1=0}^{[n_1/m_1]}\dots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\Gamma(\lambda+l_i)\\
&\times{}_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}v\right].
\end{aligned}$$

Thus we have:

$$\begin{aligned}
\mathscr{I}&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\sum_{h=0}^{m+n}(\sigma)^h\,{}_1\Delta_{a,b,c,d}^{n,m,\xi,\zeta}\sum_{l_1=0}^{[n_1/m_1]}\dots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i} \qquad (60)\\
&\times\Gamma(\lambda+l_i)_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}v\right].
\end{aligned}$$

where ${}_1\Delta_{a,b,c,d}^{n,m,\,\xi,\,\zeta}$ is given by (58). □

Theorem 8. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}_{-1}^+, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < \infty$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda-1} S_{n_1 \dots n_r}^{m_1 \dots m_r}[u]\, L_n^{(a,b)}[\xi; \sigma(u)] \\
&\times L_m^{(c,d)}[\zeta; \sigma(u)]\, E_{k,\alpha,\beta}^{\gamma,q}(v)\, dudv \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i, l_i} \\
&\times \sum_{h=0}^{m+n} (\sigma)^h\, \Delta_{a,b,c,d}^{n,m,\,\xi,\,\zeta} \Gamma(\lambda + l_i) \int_0^\infty \phi(z) z^{\lambda + l_i + \mu + h - 1} \\
&\times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu + h, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu + h, 1) \end{array} \middle| k^{(q-\alpha/k)} z \right]
\end{aligned} \tag{61}$$

where ${}_1\Delta_{a,b,c,d}^{n,m,\,\xi,\,\zeta}$ *is given by equation* (58).

Proof. The proof of the equation (61) is same as the proof (57), so we omit the detail. □

Theorem 9. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}_{-1}^+, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < 1$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1} v^{\lambda} S_{n_1 \dots n_r}^{m_1 \dots m_r}[v(1-u)] \\
&\times L_n^{(a,b)}[\xi; \sigma(1-u)]\, L_m^{(c,d)}[\zeta; \sigma(1-u)]\, E_{k,\alpha,\beta}^{\gamma,q}(1-v)\, dudv \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i, l_i} \\
&\times \sum_{h=0}^{m+n} (\sigma)^h\, \Delta_{a,b,c,d}^{n,m,\,\xi,\,\zeta} \Gamma(\lambda + l_i) \int_0^\infty f(z)(1-z)^{\lambda + l_i + \mu + h - 1} \\
&\times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu + h, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu + h, 1) \end{array} \middle| k^{(q-\alpha/k)}(1-z) \right].
\end{aligned} \tag{62}$$

where ${}_1\Delta_{a,b,c,d}^{n,m,\,\xi,\,\zeta}$ *is given by equation* (58).

Proof. The proof of the equation (62) is same as the proof (57), so we omit the detail. □

3. SPECIAL CASES

In this section we briefly consider certain special cases of the derivation of the results obtained in the preceding sections by suitable substitutions.

(I) By applying the our results given in (35),(39),(43),(48),(55)and (56) to the case of Hermite polynomials [**4, 6**] by setting $S_n^2(x) \to x^{n/2} H_n \left[\frac{1}{2\sqrt{x}}\right]$ in which $m_1, ..., m_r = 2; n_1, ..., n_r = n; r = 1; A_{n_i, l_i} = (-1)^l$, we have the following interesting results.

Corollary 1. *If $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ and $q \in \mathbb{R}^+$ such that $0 < x, y < 1$. Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\lambda} \left[\frac{1-y}{1-xy}\right]^{\mu} \left[\frac{1-xy}{(1-x)(1-y)}\right] \left[\frac{(1-x)y}{1-xy}\right]^{n/2} \\
&\quad \times H_n \left[2\sqrt{\frac{1-xy}{(1-x)y}}\right] E_{k,\alpha,\beta}^{\gamma,q} \left(\frac{(1-y)v}{1-xy}\right) dxdy \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!} (-1)^l \Gamma(\lambda + l_i) \\
&\qquad \times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu, 1) \end{matrix} \middle| k^{(q-\alpha/k)} v \right].
\end{aligned} \tag{63}$$

Corollary 2. *If $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ and $q \in \mathbb{R}^+$ such that $0 < u, v < \infty$. Then the following formulas hold:*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda+n/2-1} H_n \left[\frac{1}{2\sqrt{u}}\right] E_{k,\alpha,\beta}^{\gamma,q}(v)\, dudv \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!} (-1)^l \Gamma(\lambda + l_i) \int_0^\infty \phi(z) z^{\lambda+l_i+\mu-1} \\
&\qquad \times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu, 1) \\ (\beta/k, \alpha/k), (\lambda + l_i + \mu, 1) \end{matrix} \middle| k^{(q-\alpha/k)} z \right].
\end{aligned} \tag{64}$$

Corollary 3. *If $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) >$*

$0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < 1$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda+n/2-1}(1-v)^{\mu-1}v^{\lambda+n/2} \\
&\quad \times H_n\left[\frac{1}{2\sqrt{v(1-u)}}\right] E_{k,\alpha,\beta}^{\gamma,q}(1-v)\,dudv \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!}(-1)^l \Gamma(\lambda+l_i) \int_0^\infty f(z)(1-z)^{\lambda+l_i+\mu-1} \\
&\quad \times {}_2\Psi_2\left[\begin{matrix}(\gamma/k,q),(\mu,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu,1)\end{matrix}\,\middle|\,k^{(q-\alpha/k)}(1-z)\right].
\end{aligned} \tag{65}$$

Corollary 4. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}^+_{-1}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < x, y < 1$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^\lambda \left[\frac{1-y}{1-xy}\right]^\mu \left[\frac{1-xy}{(1-x)(1-y)}\right] \left[\frac{(1-x)y}{1-xy}\right]^{n/2} \\
&\quad \times H_n\left[2\sqrt{\frac{1-xy}{(1-x)y}}\right] L_n^{(a,b)}\left[\xi;\sigma\left(\frac{1-y}{1-xy}\right)\right] \\
&\quad \times L_m^{(c,d)}\left[\zeta;\sigma\left(\frac{1-y}{1-xy}\right)\right] E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right) dxdy \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!}(-1)^l \sum_{h=0}^{m+n} (\sigma)^h\, \Delta_{a,b,c,d}^{n,m,\xi,\zeta} \Gamma(\lambda+l_i) \\
&\quad \times {}_2\Psi_2\left[\begin{matrix}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{matrix}\,\middle|\,k^{(q-\alpha/k)}v\right].
\end{aligned} \tag{66}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by equation (49).*

Corollary 5. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}^+_{-1}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < \infty$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \phi(u+v)v^{\mu-1}u^{\lambda+n/2-1} H_n\left[\frac{1}{2\sqrt{u}}\right] L_n^{(a,b)}[\xi;\sigma(u)] \\
&\quad \times L_m^{(c,d)}[\zeta;\sigma(u)]\, E_{k,\alpha,\beta}^{\gamma,q}(v)\,dudv \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!}(-1)^l \sum_{h=0}^{m+n} (\sigma)^h\, \Delta_{a,b,c,d}^{n,m,\xi,\zeta} \\
&\quad \times \Gamma(\lambda+l_i) \int_0^\infty \phi(z) z^{\lambda+l_i+\mu+h-1} \\
&\quad \times {}_2\Psi_2\left[\begin{matrix}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{matrix}\,\middle|\,k^{(q-\alpha/k)}z\right]
\end{aligned} \tag{67}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by equation* (49).

Corollary 6. *If* $a,c,\zeta,\xi \in \mathbb{C}^{+}$, $b,d \in \mathbb{C}_{-1}^{+}, m,n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha,\beta,\gamma,\lambda,\mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^{+}$ *such that* $0 < u, v < 1$. *Then the following formulas hold:*

$$\begin{aligned} &\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda+n/2-1}(1-v)^{\mu-1}v^{\lambda+n/2} H_n\left[\frac{1}{2\sqrt{v(1-u)}}\right] \\ &\quad \times L_n^{(a,b)}\left[\xi;\sigma\left(1-u\right)\right] L_m^{(c,d)}\left[\zeta;\sigma\left(1-u\right)\right] E_{k,\alpha,\beta}^{\gamma,q}\left(1-v\right) dudv \\ &= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!}(-1)^l \sum_{h=0}^{m+n} \left(\sigma\right)^h \Delta_{a,b,c,d}^{n,m,\xi,\zeta} \\ &\quad \times \Gamma\left(\lambda+l_i\right) \int_0^{\infty} f(z)(1-z)^{\lambda+l_i+\mu+h-1} \\ &\quad \times {}_2\Psi_2\left[\begin{array}{c} \left(\gamma/k,q\right),\left(\mu+h,1\right) \\ \left(\beta/k,\alpha/k\right),\left(\lambda+l_i+\mu+h,1\right)\end{array} \middle| k^{(q-\alpha/k)}(1-z)\right]. \end{aligned} \tag{68}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by equation* (49).

(II) By applying the our results given in (35),(39),(43),(48),(55)and (56) to the case of Lagurre polynomials [**4, 6**] by setting $S_n^2(x) \to L_n^{(\alpha)}[x]$ in which $m_1, ..., m_r = 1; n_1, ..., n_r = n; r = 1; A_{n_i,l_i} = \left(\begin{array}{c} n+\alpha \\ n \end{array}\right) \frac{1}{(\alpha+1)_l}$, we have the following interst-ing results.

Corollary 7. *If* $k \in \mathbb{R}; \alpha,\beta,\gamma,\lambda,\mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^{+}$ *such that* $0 < x, y < 1$. *Then the following formulas hold:*

$$\begin{aligned} &\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\lambda} \left[\frac{1-y}{1-xy}\right]^{\mu} \left[\frac{1-xy}{(1-x)(1-y)}\right] \\ &\quad \times L_n^{(\alpha)}\left[\frac{(1-x)y}{1-xy}\right] E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right) dxdy \\ &= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{n} \frac{(-n)_l}{l!} \left(\begin{array}{c} n+\alpha \\ n \end{array}\right) \frac{\Gamma(\lambda+l_i)}{(\alpha+1)_l} \\ &\quad \times {}_2\Psi_2\left[\begin{array}{c} \left(\gamma/k,q\right),\left(\mu,1\right) \\ \left(\beta/k,\alpha/k\right),\left(\lambda+l_i+\mu,1\right)\end{array} \middle| k^{(q-\alpha/k)}v\right]. \end{aligned} \tag{69}$$

Corollary 8. *If $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ and $q \in \mathbb{R}^+$ such that $0 < u, v < \infty$. Then the following formulas hold:*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda-1} L_n^{(\alpha)}[u] \, E_{k,\alpha,\beta}^{\gamma,q}(v) \, dudv \\ &= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{n} \frac{(-n)_l}{l!} \left(\begin{array}{c} n+\alpha \\ n \end{array} \right) \frac{\Gamma(\lambda+l_i)}{(\alpha+1)_l} \int_0^\infty \phi(z) z^{\lambda+l_i+\mu-1} \\ &\quad \times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu, 1) \end{array} \middle| k^{(q-\alpha/k)} z \right] \end{aligned} \tag{70}$$

Corollary 9. *If $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ and $q \in \mathbb{R}^+$ such that $0 < u, v < 1$. Then the following formulas hold:*

$$\begin{aligned} &\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1} v^{\lambda} L_n^{(\alpha)}[v(1-u)] \\ &\quad \times E_{k,\alpha,\beta}^{\gamma,q}(1-v) \, dudv = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{n} \frac{(-n)_l}{l!} \left(\begin{array}{c} n+\alpha \\ n \end{array} \right) \\ &\quad \times \frac{\Gamma(\lambda+l_i)}{(\alpha+1)_l} \int_0^\infty f(z)(1-z)^{\lambda+l_i+\mu-1} \\ &\quad \times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu, 1) \end{array} \middle| k^{(q-\alpha/k)}(1-z) \right]. \end{aligned} \tag{71}$$

Corollary 10. *If $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}_{-1}^+, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ and $q \in \mathbb{R}^+$ such that $0 < x, y < 1$. Then the following formulas hold:*

$$\begin{aligned} &\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\lambda} \left[\frac{1-y}{1-xy}\right]^{\mu} \left[\frac{1-xy}{(1-x)(1-y)}\right] \\ &\quad \times L_n^{(\alpha)} \left[\frac{(1-x)y}{1-xy}\right] L_n^{(a,b)} \left[\xi; \sigma \left(\frac{1-y}{1-xy}\right)\right] \\ &\quad \times L_m^{(c,d)} \left[\zeta; \sigma \left(\frac{1-y}{1-xy}\right)\right] E_{k,\alpha,\beta}^{\gamma,q} \left(\frac{(1-y)v}{1-xy}\right) dxdy \\ &\quad = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{n} \frac{(-n)_l}{l!} \left(\begin{array}{c} n+\alpha \\ n \end{array} \right) \frac{\Gamma(\lambda+l_i)}{(\alpha+1)_l} \sum_{h=0}^{m+n} (\sigma)^h \, \Delta_{a,b,c,d}^{n,m,\xi,\zeta} \\ &\quad \times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{array} \middle| k^{(q-\alpha/k)} v \right]. \end{aligned} \tag{72}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ is given by equation (49).

Corollary 11. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}^+_{-1}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < \infty$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda-1} L_n^{(\alpha)}[u]\, L_n^{(a,b)}[\xi; \sigma(u)] \\
&\quad \times L_m^{(c,d)}[\zeta; \sigma(u)]\, E_{k,\alpha,\beta}^{\gamma,q}(v)\, dudv \\
&\quad = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{n} \frac{(-n)_l}{l!} \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \frac{\Gamma(\lambda+l_i)}{(\alpha+1)_l} \\
&\quad \times \sum_{h=0}^{m+n} (\sigma)^h\, \Delta_{a,b,c,d}^{n,m,\xi,\zeta} \int_0^\infty \phi(z) z^{\lambda+l_i+\mu+h-1} \\
&\quad \times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{matrix} \middle| k^{(q-\alpha/k)} z \right]
\end{aligned} \tag{73}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by equation* (49)

Corollary 12. *If* $a, c, \zeta, \xi \in \mathbb{C}^+$, $b, d \in \mathbb{C}^+_{-1}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u, v < 1$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1} v^{\lambda} L_n^{(\alpha)}[v(1-u)]\, L_n^{(a,b)}[\xi; \sigma(1-u)] \\
&\quad \times L_m^{(c,d)}[\zeta; \sigma(1-u)]\, E_{k,\alpha,\beta}^{\gamma,q}(1-v)\, dudv \\
&\quad = \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l=0}^{n} \frac{(-n)_l}{l!} \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \frac{\Gamma(\lambda+l_i)}{(\alpha+1)_l} \\
&\quad \times \sum_{h=0}^{m+n} (\sigma)^h\, \Delta_{a,b,c,d}^{n,m,\xi,\zeta} \int_0^\infty f(z)(1-z)^{\lambda+l_i+\mu+h-1} \\
&\quad \times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{matrix} \middle| k^{(q-\alpha/k)}(1-z) \right].
\end{aligned} \tag{74}$$

where $\Delta_{a,b,c,d}^{n,m,\xi,\zeta}$ *is given by equation* (49).

(III) On setting $\zeta = \xi = 1$, the results in equation (57),(61)and (62) reduced to the following form:

Corollary 13. *If* $a, c \in \mathbb{C}^+$, $b, d \in \mathbb{C}^+_{-1}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < x, y < 1$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\lambda}\left[\frac{1-y}{1-xy}\right]^{\mu}\left[\frac{1-xy}{(1-x)(1-y)}\right]\\
&\quad\times S_{n_1\ldots n_r}^{m_1\ldots m_r}\left[\frac{(1-x)y}{1-xy}\right]L_n^{(a,b)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right]\\
&\quad\times L_m^{(c,d)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right]E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right)dxdy\\
&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\\
&\quad\times\Gamma(\lambda+l_i)\sum_{h=0}^{m+n}(\sigma)^h\,{}_2\Delta_{a,b,c,d}^{n,m,\,1,\,1}\\
&\quad\times{}_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}v\right],
\end{aligned}\tag{75}$$

where

$$\begin{aligned}
{}_2\Delta_{a,b,c,d}^{n,m,\,1,\,1}&=\frac{\Gamma(an+b+1)\,\Gamma(cm+d+1)}{\Gamma(m+1)\,\Gamma(n+1)}\\
&\times\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(ak+b+1)\,\Gamma(c(h-k)+d+1)}\right].
\end{aligned}\tag{76}$$

Corollary 14. *If* $a,c\in\mathbb{C}^+$, $b,d\in\mathbb{C}_{-1}^+, m,n\in\mathbb{N}$; $k\in\mathbb{R};\alpha,\beta,\gamma,\lambda,\mu\in\mathbb{C};\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\lambda)>0,\Re(\mu)>0$ *and* $q\in\mathbb{R}^+$ *such that* $0<u,v<\infty$. *Then the following formulas hold:*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\phi(u+v)v^{\mu-1}u^{\lambda-1}S_{n_1\ldots n_r}^{m_1\ldots m_r}[u]\,L_n^{(a,b)}[\sigma(u)]\\
&\quad\times L_m^{(c,d)}[\sigma(u)]\,E_{k,\alpha,\beta}^{\gamma,q}(v)\,dudv\\
&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\Gamma(\lambda+l_i)\\
&\quad\times\sum_{h=0}^{m+n}(\sigma)^h\,{}_2\Delta_{a,b,c,d}^{n,m,\,1,\,1}\int_0^\infty\phi(z)z^{\lambda+l_i+\mu+h-1}\\
&\quad\times{}_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}z\right],
\end{aligned}\tag{77}$$

where ${}_2\Delta_{a,b,c,d}^{n,m,\,1,\,1}$ *is given by equation* (76).

Corollary 15. *If* $a,c\in\mathbb{C}^+$, $b,d\in\mathbb{C}_{-1}^+, m,n\in\mathbb{N}$; $k\in\mathbb{R};\alpha,\beta,\gamma,\lambda,\mu\in\mathbb{C};\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\lambda)>0,\Re(\mu)>0$ *and* $q\in\mathbb{R}^+$ *such that* $0<u,v<1$.

Then the following formulas hold:

$$\begin{aligned}
&\int_0^1\int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1}v^{\lambda}S_{n_1\dots n_r}^{m_1\dots m_r}\left[v(1-u)\right]\\
&\times L_n^{(a,b)}\left[\sigma\left(1-u\right)\right]L_m^{(c,d)}\left[\sigma\left(1-u\right)\right]E_{k,\alpha,\beta}^{\gamma,q}\left(1-v\right)dudv\\
&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\Gamma\left(\lambda+l_i\right)\\
&\times\sum_{h=0}^{m+n}(\sigma)^h\,{}_2\Delta_{a,b,c,d}^{n,m,\,1,\,1}\int_0^{\infty}f(z)(1-z)^{\lambda+l_i+\mu+h-1}\\
&\times{}_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}(1-z)\right],
\end{aligned}\tag{78}$$

where ${}_2\Delta_{a,b,c,d}^{n,m,\,1,\,1}$ *is given by equation* (76).

(IV) On setting $a=c=\zeta=\xi=1$, the results in equation (57),(61)and (62) reduced to the following form:

Corollary 16. *If* $b,d\in\mathbb{C}_{-1}^{+},m,n\in\mathbb{N}$*;* $k\in\mathbb{R};\alpha,\beta,\gamma,\lambda,\mu\in\mathbb{C};\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\Re(\lambda)>0,\Re(\mu)>0$ *and* $q\in\mathbb{R}^{+}$ *such that* $0<x,y<1$*. Then the following formulas hold:*

$$\begin{aligned}
&\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\lambda}\left[\frac{1-y}{1-xy}\right]^{\mu}\left[\frac{1-xy}{(1-x)(1-y)}\right]\\
&\times S_{n_1\dots n_r}^{m_1\dots m_r}\left[\frac{(1-x)y}{1-xy}\right]Z_n^{(1,b)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right]\\
&\times Z_m^{(1,d)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right]E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right)dxdy\\
&=\frac{k^{1-\beta/k}}{\Gamma(\gamma/k)}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\\
&\times\Gamma(\lambda+l_i)\sum_{h=0}^{m+n}(\sigma)^h\,{}_3\Delta_{1,b,1,d}^{n,m,\,1,\,1}\\
&\times{}_2\Psi_2\left[\begin{array}{c}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{array}\Big|k^{(q-\alpha/k)}v\right],
\end{aligned}\tag{79}$$

where

$$\begin{aligned}
{}_3\Delta_{1,b,1,d}^{n,m,\,1,\,1}&=\frac{\Gamma\left(n+b+1\right)\Gamma\left(m+d+1\right)\left(-1\right)^h}{\Gamma\left((m-h+k)+1\right)\Gamma\left((n-k)+1\right)}\\
&\times\sum_{k=0}^{h}\left(\begin{array}{c}h\\k\end{array}\right)\frac{1}{\Gamma\left(h+1\right)\Gamma\left(k+b+1\right)\Gamma\left((h-k)+d+1\right)}.
\end{aligned}\tag{80}$$

Corollary 17. *If* $b, d \in \mathbb{C}_{-1}^{+}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^{+}$ *such that* $0 < u, v < \infty$. *Then the following formulas hold:*

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda-1} S_{n_1\dots n_r}^{m_1\dots m_r}[u]\, Z_n^{(1,b)}[\sigma(u)] \\
&\times Z_m^{(1,d)}[\sigma(u)]\, E_{k,\alpha,\beta}^{\gamma,q}(v)\, du dv \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \\
&\times \Gamma(\lambda+l_i) \sum_{h=0}^{m+n} (\sigma)^h\, {}_3\Delta_{1,b,1,d}^{n,m,\,1,\,1} \int_0^\infty \phi(z) z^{\lambda+l_i+\mu+h-1} \\
&\times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{matrix} \middle| k^{(q-\alpha/k)} z \right],
\end{aligned}
\tag{81}
$$

where ${}_3\Delta_{1,b,1,d}^{n,m,\,1,\,1}$ *is given by equation* (80).

Corollary 18. *If* $b, d \in \mathbb{C}_{-1}^{+}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^{+}$ *such that* $0 < u, v < 1$. *Then the following formulas hold:*

$$
\begin{aligned}
&\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1} v^{\lambda} S_{n_1\dots n_r}^{m_1\dots m_r}[v(1-u)] \\
&\times Z_n^{(1,b)}[\sigma(1-u)]\, Z_m^{(1,d)}[\sigma(1-u)]\, E_{k,\alpha,\beta}^{\gamma,q}(1-v)\, du dv \\
&= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \Gamma(\lambda+l_i) \\
&\times \sum_{h=0}^{m+n} (\sigma)^h\, {}_3\Delta_{1,b,1,d}^{n,m,\,1,\,1} \int_0^\infty f(z)(1-z)^{\lambda+l_i+\mu+h-1} \\
&\times {}_2\Psi_2 \left[\begin{matrix} (\gamma/k, q), (\mu+h, 1) \\ (\beta/k, \alpha/k), (\lambda+l_i+\mu+h, 1) \end{matrix} \middle| k^{(q-\alpha/k)}(1-z) \right],
\end{aligned}
\tag{82}
$$

where ${}_3\Delta_{1,b,1,d}^{n,m,\,1,\,1}$ *is given by equation* (80).

(V) On setting $a = c = 0; \zeta = \xi = 1$, the results in equation (57),(61)and (62) reduced to the following form:

Corollary 19. *If* $b, d \in \mathbb{C}_{-1}^{+}, m, n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^{+}$ *such that* $0 < x, y < 1$. *Then the following formulas hold:*

$$
\begin{aligned}
&\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^\lambda \left[\frac{1-y}{1-xy}\right]^\mu \left[\frac{1-xy}{(1-x)(1-y)}\right] S_{n_1\ldots n_r}^{m_1\ldots m_r}\left[\frac{(1-x)y}{1-xy}\right] \\
&\quad\times \left[1-\sigma\left(\frac{1-y}{1-xy}\right)\right]^n E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right) dxdy \\
&\quad= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i}\Gamma(\lambda+l_i) \\
&\quad\times \sum_{h=0}^{n} (\sigma)^h (-n)_h \; {}_2\Psi_2\left[\begin{matrix}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{matrix}\,\middle|\, k^{(q-\alpha/k)}v\right].
\end{aligned}
\tag{83}
$$

Proof. Let $\xi,\zeta \in \mathbb{N}$ and using (31), the equation (53) reduced to

$$
\begin{aligned}
&\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^\lambda \left[\frac{1-y}{1-xy}\right]^\mu \left[\frac{1-xy}{(1-x)(1-y)}\right] S_{n_1\ldots n_r}^{m_1\ldots m_r}\left[\frac{(1-x)y}{1-xy}\right] \\
&\quad\times L_n^{(0,b)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right] L_m^{(0,d)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right] E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right) dxdy \\
&\quad= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i}\Gamma(\lambda+l_i) \\
&\quad\times \sum_{h=0}^{m+n} (\sigma)^h \frac{1}{\Gamma(m+1)\Gamma(n+1)} \sum_{k=0}^{h}\left[\begin{pmatrix} h \\ k\end{pmatrix} (-m)_{(h-k)} (-n)_k\right] \\
&\quad\times {}_2\Psi_2\left[\begin{matrix}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{matrix}\,\middle|\, k^{(q-\alpha/k)}v\right].
\end{aligned}
\tag{84}
$$

Applying the result (32), the above equation (84) reduced to

$$
\begin{aligned}
&\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^\lambda \left[\frac{1-y}{1-xy}\right]^\mu \left[\frac{1-xy}{(1-x)(1-y)}\right] S_{n_1\ldots n_r}^{m_1\ldots m_r}\left[\frac{(1-x)y}{1-xy}\right] \\
&\quad\times L_n^{(0,b)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right] L_m^{(0,d)}\left[\sigma\left(\frac{1-y}{1-xy}\right)\right] E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right) dxdy \\
&\quad= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \\
&\quad\times \Gamma(\lambda+l_i) \sum_{h=0}^{m+n} (\sigma)^h \frac{(-m-n)_h}{\Gamma(m+1)\Gamma(n+1)} \\
&\quad\times {}_2\Psi_2\left[\begin{matrix}(\gamma/k,q),(\mu+h,1)\\(\beta/k,\alpha/k),(\lambda+l_i+\mu+h,1)\end{matrix}\,\middle|\, k^{(q-\alpha/k)}v\right].
\end{aligned}
\tag{85}
$$

From equation (17) and using the results (31) -(34), we get

$$L_n^{(0,\beta)} = \frac{1}{\Gamma(n+1)} \sum_{k=0}^{h} \left(\begin{array}{c} h \\ k \end{array} \right) (-x)^k = \frac{1}{\Gamma(n+1)} (1-x)^n. \tag{86}$$

Employing the result (86), the (85) reduced to

$$\begin{aligned} &\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{\lambda} \left[\frac{1-y}{1-xy}\right]^{\mu} \left[\frac{1-xy}{(1-x)(1-y)}\right] \\ &\times S_{n_1 \dots n_r}^{m_1 \dots m_r} \left[\frac{(1-x)y}{1-xy}\right] \left[1-\sigma\left(\frac{1-y}{1-xy}\right)\right]^n \\ &\times \left[1-\sigma\left(\frac{1-y}{1-xy}\right)\right]^m E_{k,\alpha,\beta}^{\gamma,q}\left(\frac{(1-y)v}{1-xy}\right) dxdy \\ &= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \\ &\times \Gamma(\lambda+l_i) \sum_{h=0}^{m+n} (\sigma)^h (-m-n)_h \\ &\times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k,q), (\mu+h,1) \\ (\beta/k,\alpha/k), (\lambda+l_i+\mu+h,1) \end{array} \Big| k^{(q-\alpha/k)} v \right], \end{aligned} \tag{87}$$

replace $m+n$ by n, then we have the required result.

□

Corollary 20. *If* $b,d \in \mathbb{C}_{-1}^{+}, m,n \in \mathbb{N}$; $k \in \mathbb{R}; \alpha,\beta,\gamma,\lambda,\mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^+$ *such that* $0 < u,v < \infty$. *Then the following formulas hold:*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \phi(u+v) v^{\mu-1} u^{\lambda-1} S_{n_1 \dots n_r}^{m_1 \dots m_r}[u] \left[1-\sigma(u)\right]^n E_{k,\alpha,\beta}^{\gamma,q}(v)\, dudv \\ &= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \dots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \\ &\times \Gamma(\lambda+l_i) \sum_{h=0}^{n} (\sigma)^h (-n)_h \int_0^\infty \phi(z) z^{\lambda+l_i+\mu+h-1} \\ &\times {}_2\Psi_2 \left[\begin{array}{c} (\gamma/k,q), (\mu+h,1) \\ (\beta/k,\alpha/k), (\lambda+l_i+\mu+h,1) \end{array} \Big| k^{(q-\alpha/k)} z \right]. \end{aligned} \tag{88}$$

Proof. The proof of the equation (88) is same as the proof (83), so we omit the detail. □

Corollary 21. *If* $b, d \in \mathbb{C}^{+}_{-1}, m, n \in \mathbb{N}$*;* $k \in \mathbb{R}; \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$ *and* $q \in \mathbb{R}^{+}$ *such that* $0 < u, v < 1$*. Then the following formulas hold:*

$$\begin{aligned} &\int_0^1 \int_0^1 f(uv)(1-u)^{\lambda-1}(1-v)^{\mu-1} v^{\lambda} S_{n_1 \ldots n_r}^{m_1 \ldots m_r}\left[v(1-u)\right] \\ &\quad \times \left[1-\sigma\left(1-u\right)\right]^n E_{k,\alpha,\beta}^{\gamma,q}\left(1-v\right) dudv \\ &= \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} \\ &\quad \times \Gamma\left(\lambda+l_i\right) \sum_{h=0}^{n} \left(\sigma\right)^h \left(-n\right)_h \int_0^{\infty} f(z)(1-z)^{\lambda+l_i+\mu+h-1} \\ &\quad \times {}_2\Psi_2 \left[\begin{array}{c} \left(\gamma/k, q\right), \left(\mu+h, 1\right) \\ \left(\beta/k, \alpha/k\right), \left(\lambda+l_i+\mu+h, 1\right) \end{array} \middle| k^{(q-\alpha/k)}(1-z) \right]. \end{aligned} \tag{89}$$

Proof. The proof of the equation (89) is same as the proof (83), so we omit the detail. □

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(Received 26.03.2019)
(Revised 23.09.2019)

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