

## ASYMPTOTIC EXPANSIONS AND CONTINUED FRACTION APPROXIMATIONS FOR HARMONIC NUMBERS

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In this paper, we provide a method for constructing a continued fraction approximation based on a given asymptotic expansion. We establish some asymptotic expansions for the harmonic numbers which employ the  $n$ th triangular number. Based on these expansions, we derive the corresponding continued fraction approximations for the harmonic numbers.

### 1. INTRODUCTION

The Indian mathematician Ramanujan (see [1, p. 531] and [11, p. 276]) claimed the following asymptotic expansion for the  $n$ th harmonic number:

$$(1.1) \quad H_n := \sum_{k=1}^n \frac{1}{k} \sim \frac{1}{2} \ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\ - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \dots$$

as  $n \rightarrow \infty$ , where  $m = n(n+1)/2$  is the  $n$ th triangular number and  $\gamma$  is the Euler-Mascheroni constant.

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Ramanujan's formula (1.1) is the subject of intense investigations and has motivated a large number of research papers (see, for example, [2, 4, 5, 6, 7, 8, 9, 10, 12, 13]).

Villarino [12, Theorem 1.1] first gave a complete proof of expansion (1.1) in terms of the Bernoulli polynomials. Recently, Chen [5] gave a recursive relation for determining the coefficients of Ramanujan's asymptotic expansion (1.1), without the Bernoulli numbers and polynomials

$$(1.2) \quad H_n \sim \frac{1}{2} \ln(2m) + \gamma + \sum_{\ell=1}^{\infty} \frac{a_\ell}{m^\ell}, \quad n \rightarrow \infty,$$

where the coefficients  $a_\ell$  ( $\ell \in \mathbb{N} := \{1, 2, \dots\}$ ) are given by the recurrence relation

$$(1.3) \quad a_1 = \frac{1}{12}, \quad a_\ell = \frac{1}{2^{\ell+1}\ell} \left\{ \frac{1}{2\ell+1} - \sum_{j=1}^{\ell-1} 2^{j+1} a_j \binom{2\ell-j}{2\ell-2j+1} \right\}, \quad \ell \geq 2.$$

Mortici and Villarino [10, Theorem 2] and Chen [2, Theorem 3.3] obtained the following asymptotic expansion:

$$(1.4) \quad H_n \sim \frac{1}{2} \ln \left( 2m + \frac{1}{3} \right) + \gamma + \sum_{j=2}^{\infty} \frac{\rho_j}{(2m + \frac{1}{3})^j}, \quad n \rightarrow \infty.$$

Moreover, these authors gave a formula for determining the coefficients  $\rho_j$  in (1.4). From a computational viewpoint, (1.4) is an improvement of (1.2).

Chen [2, Theorem 3.1] obtained the following asymptotic expansion:

$$(1.5) \quad H_n \sim \gamma + \frac{1}{2} \ln \left( 2m + \frac{1}{3} + \sum_{\ell=1}^{\infty} \frac{\omega_\ell}{(2m)^\ell} \right), \quad n \rightarrow \infty,$$

with the coefficients  $\omega_\ell$  ( $\ell \in \mathbb{N}$ ) given by the recursive relation

$$(1.6) \quad \omega_1 = -\frac{1}{90}, \quad \omega_\ell = b_{2(\ell+1)} - \sum_{j=1}^{\ell-1} \binom{2\ell-j-1}{2\ell-2j} \omega_j, \quad \ell \geq 2,$$

where  $b_j$  are given by

$$(1.7) \quad b_j = \sum_{k_1+2k_2+\dots+jk_j=j} \frac{(-2)^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left( \frac{B_1}{1} \right)^{k_1} \left( \frac{B_2}{2} \right)^{k_2} \dots \left( \frac{B_j}{j} \right)^{k_j},$$

and  $B_j$  are the Bernoulli numbers and the summation is taken over all nonnegative integers  $k_j$  satisfying the equation  $k_1 + 2k_2 + \dots + jk_j = j$ .

It follows from [3, Corollary 3.1] that

$$(1.8) \quad H_n - \ln \left( n + \frac{1}{2} \right) - \gamma = \frac{\frac{1}{48}}{m + \frac{17}{80}} + O \left( \frac{1}{n^6} \right), \quad n \rightarrow \infty.$$

In this paper, we provide a method to construct a continued fraction approximation based on a given asymptotic expansion. We establish some asymptotic expansions for the harmonic numbers which employ the  $n$ th triangular number. Based on these expansions, we derive the corresponding continued fraction approximations for the harmonic numbers. All results of the present paper are motivated by (1.1), (1.4), (1.5) and (1.8).

The following lemma will be useful in our present investigation.

**Lemma 1.1.** *Let  $a_1 \neq 0$  and*

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \quad x \rightarrow \infty$$

*be a given asymptotic expansion. Define the function  $B$  by*

$$A(x) = \frac{a_1}{B(x)}.$$

*Then the function  $B(x) = a_1/A(x)$  has asymptotic expansion of the following form*

$$B(x) \sim x + \sum_{j=0}^{\infty} \frac{b_j}{x^j}, \quad x \rightarrow \infty,$$

*where*

$$(1.9) \quad b_0 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1} \left( a_{j+2} + \sum_{k=1}^j a_{k+1} b_{j-k} \right), \quad j \geq 1.$$

*Proof.* We can let

$$(1.10) \quad \frac{a_1}{A(x)} \sim x + \sum_{j=0}^{\infty} \frac{b_j}{x^j}, \quad x \rightarrow \infty,$$

where  $b_j$  (for  $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ) are real numbers to be determined. Write (1.10) as

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{a_j}{x^j} \left( x + \sum_{k=0}^{\infty} \frac{b_k}{x^k} \right) &\sim a_1, \\ \sum_{j=0}^{\infty} \frac{a_{j+2}}{x^j} &\sim - \sum_{j=0}^{\infty} \frac{a_{j+1}}{x^j} \sum_{k=0}^{\infty} \frac{b_k}{x^k}, \end{aligned}$$

$$(1.11) \quad \sum_{j=0}^{\infty} a_{j+2} x^{-j} \sim \sum_{j=0}^{\infty} \left( \sum_{k=0}^j (-a_{k+1} b_{j-k}) \right) x^{-j}.$$

Equating the coefficients of equal powers of  $x$  in (1.11), we obtain

$$a_{j+2} = - \sum_{k=0}^j a_{k+1} b_{j-k}, \quad j \geq 0,$$

For  $j = 0$  we obtain  $b_0 = -a_2/a_1$ , and for  $j \geq 1$  we have

$$a_{j+2} = - \sum_{k=1}^j a_{k+1} b_{j-k} - a_1 b_j, \quad j \geq 1,$$

which gives the desired formula (1.9).  $\square$

Lemma 1.1 provides a method to construct a continued fraction approximation based on a given asymptotic expansion. We state this method as a consequence of Lemma 1.1.

**Corollary 1.1.** *Let  $a_1 \neq 0$  and*

$$(1.12) \quad A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \quad x \rightarrow \infty$$

*be a given asymptotic expansion. Then the function  $A$  has the following continued fraction approximation of the form*

$$(1.13) \quad A(x) \approx \frac{a_1}{x + b_0 + \frac{b_1}{x + c_0 + \frac{c_1}{x + d_0 + \ddots}}}, \quad x \rightarrow \infty,$$

*where the constants in the right-hand side of (1.13) are given by the following recurrence relations:*

$$(1.14) \quad \begin{cases} b_0 = -\frac{a_2}{a_1}, & b_j = -\frac{1}{a_1} \left( a_{j+2} + \sum_{k=1}^j a_{k+1} b_{j-k} \right) \\ c_0 = -\frac{b_2}{b_1}, & c_j = -\frac{1}{b_1} \left( b_{j+2} + \sum_{k=1}^j b_{k+1} c_{j-k} \right) \\ d_0 = -\frac{c_2}{c_1}, & d_j = -\frac{1}{c_1} \left( c_{j+2} + \sum_{k=1}^j c_{k+1} d_{j-k} \right) \\ \dots & \dots \end{cases}$$

**Remark 1.1.** *Clearly,  $a_j \implies b_j \implies c_j \implies d_j \implies \dots$ . Thus, the asymptotic expansion (1.12)  $\implies$  the continued fraction approximation (1.13). Corollary 1.1 transforms the asymptotic expansion (1.12) into a corresponding continued fraction of the form (1.13), and provides the system (1.14) to determine the constants in the right-hand side of (1.13).*

## 2. MAIN RESULTS

**Theorem 2.1.** Let  $m = \frac{1}{2}n(n+1)$ . As  $n \rightarrow \infty$ , we have

$$(2.15) \quad H_n \approx \frac{1}{2} \ln(2m) + \gamma + \frac{a_1}{m + b_0 + \frac{b_1}{m + c_0 + \frac{c_1}{m + d_0 + \ddots}}},$$

where

$$(2.16) \quad a_1 = \frac{1}{12}, \quad b_0 = \frac{1}{10}, \quad b_1 = -\frac{19}{2100}, \quad c_0 = \frac{91}{190}, \quad c_1 = -\frac{16585}{83391}, \quad d_0 = \frac{2357167}{1638598}, \quad \dots$$

*Proof.* Denote

$$A(m) = H_n - \frac{1}{2} \ln(2m) - \gamma.$$

It follows from (1.2) that

$$(2.17) \quad A(m) \sim \sum_{\ell=1}^{\infty} \frac{a_\ell}{m^\ell} = \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \dots$$

as  $m \rightarrow \infty$ , where the coefficients  $a_\ell$  ( $\ell \in \mathbb{N}$ ) are given in (1.3). Then,  $A(m)$  has the continued fraction approximation of the form

$$(2.18) \quad A(m) = H_n - \frac{1}{2} \ln(2m) - \gamma \approx \frac{a_1}{m + b_0 + \frac{b_1}{m + c_0 + \frac{c_1}{m + d_0 + \ddots}}}, \quad m \rightarrow \infty,$$

where the constants in the right-hand side of (2.18) can be determined using (1.14). Noting that

$$a_1 = \frac{1}{12}, \quad a_2 = -\frac{1}{120}, \quad a_3 = \frac{1}{630}, \quad a_4 = -\frac{1}{1680}, \quad a_5 = \frac{1}{2310}, \quad a_6 = -\frac{191}{360360}, \quad \dots,$$

we obtain from the first recurrence relation in (1.14) that

$$\begin{aligned} b_0 &= -\frac{a_2}{a_1} = \frac{1}{10}, \\ b_1 &= -\frac{a_3 + a_2 b_0}{a_1} = -\frac{19}{2100}, \\ b_2 &= -\frac{a_4 + a_2 b_1 + a_3 b_0}{a_1} = \frac{13}{3000}, \\ b_3 &= -\frac{a_5 + a_2 b_2 + a_3 b_1 + a_4 b_0}{a_1} = -\frac{187969}{48510000}, \\ b_4 &= -\frac{a_6 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0}{a_1} = \frac{3718037}{700700000}, \quad \dots \end{aligned}$$

We obtain from the second recurrence relation in (1.14) that

$$\begin{aligned} c_0 &= -\frac{b_2}{b_1} = \frac{91}{190}, \\ c_1 &= -\frac{b_3 + b_2 c_0}{b_1} = -\frac{16585}{83391}, \\ c_2 &= -\frac{b_4 + b_2 c_1 + b_3 c_0}{b_1} = \frac{11785835}{41195154}, \quad \dots \end{aligned}$$

Continuing the above process, we find

$$d_0 = -\frac{c_2}{c_1} = \frac{2357167}{1638598}, \quad \dots$$

The proof is complete.  $\square$

**Remark 2.2.** *It is well known that*

$$\begin{aligned} H_n - \ln n - \gamma &\sim -\sum_{k=1}^{\infty} \frac{B_k}{kn^k} \\ (2.19) \quad &= \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \dots, \quad n \rightarrow \infty, \end{aligned}$$

where  $B_k$  are the Bernoulli numbers. Following the method used in the proof of Theorem 2.1, we derive

$$(2.20) \quad H_n \approx \ln n + \gamma + \frac{\frac{1}{2}}{n + \frac{1}{6} + \frac{\frac{1}{36}}{n + \frac{13}{30} + \frac{\frac{9}{25}}{n + \frac{17}{630} + \dots}}}, \quad n \rightarrow \infty.$$

**Theorem 2.2.** Let  $m = \frac{1}{2}n(n+1)$ . The harmonic numbers have the following asymptotic expansion:

$$\begin{aligned}
 H_n &\sim \ln\left(n + \frac{1}{2}\right) + \gamma + \sum_{\ell=1}^{\infty} \frac{r_\ell}{m^\ell} \\
 &= \ln\left(n + \frac{1}{2}\right) + \gamma + \frac{1}{48m} - \frac{17}{3840m^2} + \frac{407}{322560m^3} - \frac{1943}{3440640m^4} \\
 (2.21) \quad &+ \frac{32537}{75694080m^5} - \frac{25019737}{47233105920m^6} + \dots
 \end{aligned}$$

as  $n \rightarrow \infty$ , where the coefficients  $r_\ell$  ( $\ell \in \mathbb{N}$ ) are given by the recurrence relation

$$(2.22) \quad r_1 = \frac{1}{48}, \quad r_\ell = \frac{1}{2^{\ell+1}\ell} \left\{ \frac{1}{2^{2\ell}(2\ell+1)} - \sum_{j=1}^{\ell-1} 2^{j+1} r_j \binom{2\ell-j}{2\ell-2j+1} \right\}, \quad \ell \geq 2.$$

*Proof.* Denote

$$I_n = H_n - \ln\left(n + \frac{1}{2}\right) - \gamma \quad \text{and} \quad J_n = \sum_{\ell=1}^{\infty} \frac{r_\ell}{m^\ell}.$$

Let  $I_n \sim J_n$  and

$$\Delta I_n := I_{n+1} - I_n \sim \Delta J_n := J_{n+1} - J_n$$

as  $n \rightarrow \infty$ , where  $r_\ell$  ( $\ell \in \mathbb{N}$ ) are real numbers to be determined.

It is easy to see that

$$\begin{aligned}
 \Delta I_n &= \frac{1}{n+1} - \left\{ \ln\left(1 + \frac{1}{2(n+1)}\right) - \ln\left(1 - \frac{1}{2(n+1)}\right) \right\} \\
 (2.23) \quad &= \frac{1}{n+1} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} + 1}{2^k k (n+1)^k} = - \sum_{\ell=1}^{\infty} \frac{1}{2^{2\ell}(2\ell+1)} (n+1)^{-2\ell-1}
 \end{aligned}$$

and

$$(2.24) \quad \Delta J_n = \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 + \frac{1}{n+1}\right)^{-k} - \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 - \frac{1}{n+1}\right)^{-k}.$$

Direct computation yields

$$\begin{aligned}
 \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 + \frac{1}{n+1}\right)^{-k} &= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{(n+1)^j} \\
 &= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{(n+1)^j} \\
 &= \sum_{k=1}^{\infty} \sum_{j=1}^k 2^j r_j (-1)^{k-j} \binom{k-1}{k-j} \frac{1}{(n+1)^{k+j}} \\
 (2.25) \qquad &= \sum_{\ell=2}^{\infty} \sum_{j=1}^{\lfloor \frac{\ell}{2} \rfloor} 2^j r_j (-1)^{\ell} \binom{\ell-j-1}{\ell-2j} \frac{1}{(n+1)^{\ell}}
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 - \frac{1}{n+1}\right)^{-k} &= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{(-1)^j}{(n+1)^j} \\
 &= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} \binom{k+j-1}{j} \frac{1}{(n+1)^j} \\
 &= \sum_{k=1}^{\infty} \sum_{j=1}^k 2^j r_j \binom{k-1}{k-j} \frac{1}{(n+1)^{k+j}} \\
 (2.26) \qquad &= \sum_{\ell=2}^{\infty} \sum_{j=1}^{\lfloor \frac{\ell}{2} \rfloor} 2^j r_j \binom{\ell-j-1}{\ell-2j} \frac{1}{(n+1)^{\ell}}.
 \end{aligned}$$

Substituting (2.25) and (2.26) into (2.24) yields

$$(2.27) \qquad \Delta J_n = \sum_{\ell=2}^{\infty} \sum_{j=1}^{\lfloor \frac{\ell}{2} \rfloor} \left( (-1)^{\ell} - 1 \right) 2^j r_j \binom{\ell-j-1}{\ell-2j} \frac{1}{(n+1)^{\ell}}.$$

Replacement of  $\ell$  by  $2\ell + 1$  in (2.27) yields

$$(2.28) \qquad \Delta J_n = - \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} 2^{j+1} r_j \binom{2\ell-j}{2\ell-2j+1} (n+1)^{-2\ell-1}.$$

Equating the coefficients of the term  $(n+1)^{-2\ell-1}$  on the right-hand sides of (2.23) and (2.28) yields

$$(2.29) \qquad \sum_{j=1}^{\ell} 2^{j+1} r_j \binom{2\ell-j}{2\ell-2j+1} = \frac{1}{2^{2\ell}(2\ell+1)}, \quad \ell \geq 1.$$

For  $\ell = 1$  in (2.29) we obtain  $r_1 = \frac{1}{48}$ , and for  $\ell \geq 2$  we have

$$\sum_{j=1}^{\ell-1} 2^{j+1} r_j \binom{2\ell-j}{2\ell-2j+1} + 2^{\ell+1} l r_\ell = \frac{1}{2^{2\ell}(2\ell+1)},$$

which gives the desired formula (2.22).  $\square$

**Remark 2.3.** We here give the recursive relation (2.22) for determining the coefficients  $r_\ell$  in expansion (2.21), without the Bernoulli numbers and polynomials.

**Remark 2.4.** Denote

$$A^*(m) = H_n - \ln\left(n + \frac{1}{2}\right) - \gamma.$$

It follows from (2.21) that

$$(2.30) \quad A^*(m) \sim \sum_{\ell=1}^{\infty} \frac{r_\ell}{m^\ell} = \frac{1}{48m} - \frac{17}{3840m^2} + \frac{407}{322560m^3} - \frac{1943}{3440640m^4} \\ + \frac{32537}{75694080m^5} - \frac{25019737}{47233105920m^6} + \dots$$

as  $m \rightarrow \infty$ , where the coefficients  $r_\ell$  ( $\ell \in \mathbb{N}$ ) are given in (2.22). Following the method used in the proof of Theorem 2.1, we derive

$$(2.31) \quad H_n \approx \ln\left(n + \frac{1}{2}\right) + \gamma + \frac{\lambda_1}{m + u_1 + \frac{\lambda_2}{m + \mu_2 + \frac{\lambda_3}{m + \mu_3 + \dots}}},$$

where

$$(2.32) \quad \lambda_1 = \frac{1}{48}, \quad \mu_1 = \frac{17}{80}, \quad \lambda_2 = -\frac{2071}{134400}, \quad \mu_2 = \frac{117863}{165680}, \\ \lambda_3 = -\frac{15685119025}{63409182144}, \quad \mu_3 = \frac{2312217133079747}{1351329470432240}, \dots$$

Thus, we develop the approximation formula (1.8) to produce a continued fraction approximation.

**Theorem 2.3.** Let  $m = \frac{1}{2}n(n+1)$ . The harmonic numbers have the following asymptotic expansion:

$$(2.33) \quad H_n \sim \frac{1}{2} \ln\left(2m + \frac{1}{3}\right) + \gamma + \sum_{\ell=2}^{\infty} \frac{s_\ell}{m^\ell} \\ = \frac{1}{2} \ln\left(2m + \frac{1}{3}\right) + \gamma - \frac{1}{720m^2} + \frac{37}{45360m^3} - \frac{181}{362880m^4} + \frac{503}{1197504m^5} \\ - \frac{1480211}{2802159360m^6} + \frac{2705333}{2802159360m^7} - \frac{793046533}{326651719680m^8} \\ + \frac{470463477509}{58650316268544m^9} - \dots$$

as  $n \rightarrow \infty$ , with the coefficients  $s_\ell$  given by

$$(2.34) \quad s_\ell = a_\ell - \frac{(-1)^{\ell-1}}{6^\ell 2^\ell}, \quad \ell \geq 2,$$

where  $a_\ell$  are given in (1.3).

*Proof.* We find by (1.2) that, as  $n \rightarrow \infty$ ,

$$\begin{aligned} H_n - \frac{1}{2} \ln \left( 2m + \frac{1}{3} \right) - \gamma &= H_n - \frac{1}{2} \ln(2m) - \gamma - \frac{1}{2} \ln \left( 1 + \frac{1}{6m} \right) \\ &\sim \sum_{\ell=1}^{\infty} \frac{a_\ell}{m^\ell} - \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell(6m)^\ell}. \end{aligned}$$

Noting that  $a_1 = \frac{1}{12}$ , we obtain, as  $n \rightarrow \infty$ ,

$$H_n \sim \frac{1}{2} \ln \left( 2m + \frac{1}{3} \right) + \gamma + \sum_{\ell=2}^{\infty} \left\{ a_\ell - \frac{(-1)^{\ell-1}}{6^\ell 2^\ell} \right\} \frac{1}{m^\ell}.$$

The proof is complete.  $\square$

**Theorem 2.4.** Let  $m = \frac{1}{2}n(n+1)$ . As  $n \rightarrow \infty$ , we have

$$(2.35) \quad H_n \approx \frac{1}{2} \ln \left( 2m + \frac{1}{3} \right) + \gamma + \frac{p_1}{m^2 + \frac{37}{63}m + q_1 + \frac{p_2}{m + q_2 + \frac{p_3}{m + q_3 + \dots}}},$$

where

$$(2.36) \quad \begin{aligned} p_1 &= -\frac{1}{720}, & q_1 &= -\frac{451}{31752}, & p_2 &= \frac{228764}{2750517}, & q_2 &= \frac{21448004509}{11990893824}, \\ p_3 &= -\frac{36637398233630775}{36226136230395904}, & q_3 &= \frac{86442719924955272247584297}{26612223343933404862713600}, & \dots & \end{aligned}$$

*Proof.* Denote

$$F(m) = H_n - \frac{1}{2} \ln \left( 2m + \frac{1}{3} \right) - \gamma.$$

It follows from (2.33) that

$$(2.37) \quad \begin{aligned} F(m) \sim \sum_{\ell=2}^{\infty} \frac{s_\ell}{m^\ell} &= -\frac{1}{720m^2} + \frac{37}{45360m^3} - \frac{181}{362880m^4} + \frac{503}{1197504m^5} \\ &\quad - \frac{1480211}{2802159360m^6} + \frac{2705333}{2802159360m^7} - \frac{793046533}{326651719680m^8} \\ &\quad + \frac{470463477509}{58650316268544m^9} - \dots \end{aligned}$$

as  $m \rightarrow \infty$ , where the coefficients  $s_\ell$  ( $\ell \in \mathbb{N}$ ) are given in (2.34).

Define the function  $G(m)$  by

$$F(m) = \frac{s_2}{G(m)}.$$

We obtain by (2.37) and Lemma 1.1 that

$$\begin{aligned} G(m) = \frac{s_2}{F(m)} &\sim \frac{s_2}{\sum_{\ell=2}^{\infty} s_\ell m^{-\ell}} = m \left( \frac{s_2}{\sum_{\ell=1}^{\infty} s_{\ell+1} m^{-\ell}} \right) \\ &= m^2 + \frac{37}{63}m - \frac{451}{31752} + A^{**}(m), \end{aligned}$$

where

$$(2.38) \quad \begin{aligned} A^{**}(m) &\sim \frac{228764}{2750517m} - \frac{21448004509}{144171099072m^2} + \frac{3180925176497}{9082779241536m^3} \\ &- \frac{898929405728511653}{856033777956284928m^4} + \frac{2008288563825356198279}{512336216106836529408m^5} - \dots, \end{aligned}$$

We then obtain

$$(2.39) \quad F(m) \sim \frac{-\frac{1}{720}}{m^2 + \frac{37}{63}m - \frac{451}{31752} + A^{**}(m)}.$$

Following the method used in the proof of Theorem 2.1, we derive the continued fraction approximation of  $A^{**}(m)$  (we here omit the derivation of (2.40))

$$(2.40) \quad A^{**}(m) \approx \frac{p_2}{m + q_2 + \frac{p_3}{m + q_3 + \ddots}}$$

as  $m \rightarrow \infty$ , where  $p_j$  and  $q_j$  (for  $j \geq 2$ ) are given in (2.36). Substituting (2.40) into (2.39) yields (2.35).  $\square$

**Theorem 2.5.** Let  $m = \frac{1}{2}n(n+1)$ . As  $n \rightarrow \infty$ , we have

$$(2.41) \quad H_n \approx \frac{1}{2} \ln \left( 2m + \frac{1}{3} + \frac{\alpha_1}{m + \beta_1 + \frac{\alpha_2}{m + \beta_2 + \frac{\alpha_3}{m + \beta_3 + \ddots}}} \right) + \gamma,$$

where

$$(2.42) \quad \begin{aligned} \alpha_1 &= -\frac{1}{180}, \quad \beta_1 = \frac{53}{126}, \quad \alpha_2 = -\frac{26329}{317520}, \quad \beta_2 = \frac{42684239}{36491994}, \\ \alpha_3 &= -\frac{487447163992501}{785108985906960}, \quad \beta_3 = \frac{2049473595024948803087}{847043761130064882714}, \quad \dots \end{aligned}$$

*Proof.* Write (1.5) as

$$e^{2(H_n-\gamma)} \sim 2m + \frac{1}{3} + \sum_{\ell=1}^{\infty} \frac{d_\ell}{m^\ell}, \quad n \rightarrow \infty,$$

with the coefficients  $d_\ell$  given by

$$(2.43) \quad d_1 = -\frac{1}{180}, \quad d_\ell = \frac{\omega_\ell}{2^\ell}, \quad \ell \geq 2,$$

where  $\omega_\ell$  are given in (1.6). Denote

$$A^{***}(m) = e^{2(H_n-\gamma)} - 2m - \frac{1}{3}.$$

We have, as  $m \rightarrow \infty$ ,

$$(2.44) \quad \begin{aligned} A^{***}(m) &\sim \sum_{\ell=1}^{\infty} \frac{d_\ell}{m^\ell} \\ &= -\frac{1}{180m} + \frac{53}{22680m^2} - \frac{3929}{2721600m^3} + \frac{240673}{179625600m^4} - \frac{488481881}{267478848000m^5} \\ &\quad + \frac{8834570273}{2521943424000m^6} - \frac{652512638837083}{72026704189440000m^7} + \dots \end{aligned}$$

Following the method used in the proof of Theorem 2.1, we derive

$$(2.45) \quad A^{***}(m) \approx \frac{\alpha_1}{m + \beta_1 + \frac{\alpha_2}{m + \beta_2 + \frac{\alpha_3}{m + \beta_3 + \dots}}}$$

as  $m \rightarrow \infty$ , where  $\alpha_j$  and  $\beta_j$  are given in (2.42). We here omit the derivation of (2.45). Formula (2.45) can be written as (2.41). □

### 3. CONJECTURE

In view (1.1), (2.21), (2.33) and (2.44), we propose the following conjecture.

**Conjecture 3.1.** (i) Let  $a_\ell$  ( $\ell \in \mathbb{N}$ ) be given in (1.2). Then we have

$$(3.46) \quad (-1)^{\ell-1} a_\ell > 0, \quad \ell \in \mathbb{N}$$

and

$$(3.47) \quad \sum_{\ell=1}^{2p} \frac{a_\ell}{m^\ell} < H_n - \frac{1}{2} \ln(2m) - \gamma < \sum_{\ell=1}^{2p+1} \frac{a_\ell}{m^\ell},$$

where  $m = n(n+1)/2$ ,  $n \in \mathbb{N}$  and  $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ .

(ii) Let  $r_\ell$  ( $\ell \in \mathbb{N}$ ) be given in (2.22). Then we have

$$(3.48) \quad (-1)^{\ell-1} r_\ell > 0, \quad \ell \in \mathbb{N}$$

and

$$(3.49) \quad \sum_{\ell=1}^{2p} \frac{r_\ell}{m^\ell} < H_n - \ln \left( n + \frac{1}{2} \right) - \gamma < \sum_{\ell=1}^{2p+1} \frac{r_\ell}{m^\ell},$$

where  $m = n(n+1)/2$ ,  $n \in \mathbb{N}$  and  $p \in \mathbb{N}_0$ .

(iii) Let  $s_\ell$  ( $\ell \geq 2$ ) be given in (2.34). Then we have

$$(3.50) \quad (-1)^{\ell-1} s_\ell > 0, \quad \ell \geq 2$$

and

$$(3.51) \quad \sum_{\ell=2}^{2q} \frac{s_\ell}{m^\ell} < H_n - \frac{1}{2} \ln \left( 2m + \frac{1}{3} \right) - \gamma < \sum_{\ell=2}^{2q+1} \frac{s_\ell}{m^\ell},$$

where  $m = n(n+1)/2$ ,  $n \in \mathbb{N}$  and  $q \in \mathbb{N}$ .

(iv) Let  $d_\ell$  ( $\ell \in \mathbb{N}$ ) be given in (2.43). Then we have

$$(3.52) \quad (-1)^\ell d_\ell > 0, \quad \ell \geq 1$$

and

$$(3.53) \quad \frac{1}{2} \ln \left( 2m + \frac{1}{3} + \sum_{\ell=1}^{2q-1} \frac{\omega_\ell}{m^\ell} \right) < H_n - \gamma < \frac{1}{2} \ln \left( 2m + \frac{1}{3} + \sum_{\ell=1}^{2q} \frac{\omega_\ell}{m^\ell} \right),$$

where  $m = n(n+1)/2$ ,  $n \in \mathbb{N}$  and  $q \in \mathbb{N}$ .

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