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# THE INVERSE OF A TRIANGULAR MATRIX AND SEVERAL IDENTITIES OF THE CATALAN NUMBERS

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In the paper, the authors establish two identities to express higher order derivatives and integer powers of the generating function of the Chebyshev polynomials of the second kind in terms of integer powers and higher order derivatives of the generating function of the Chebyshev polynomials of the second kind respectively, find an explicit formula and an identity for the Chebyshev polynomials of the second kind, conclude the inverse of an integer, unit, and lower triangular matrix, derive an inversion theorem, present several identities of the Catalan numbers, and give some remarks on the closely related results including connections of the Catalan numbers with the Chebyshev polynomials of the second kind, the central Delannoy numbers, and the Fibonacci polynomials respectively.

## 1. PRELIMINARIES

It is common knowledge [**5, 11, 42**] that the generalized hypergeometric series

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} $$

is defined for complex numbers $a_i\in\mathbb{C}$ and $b_i\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, for positive integers $p,q\in\mathbb{N}$, and in terms of the rising factorials $(x)_n$ defined by

$$ (x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge1;\\ 1, & n=0.\end{cases} $$

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Specially, one calls ${}_2F_1(a,b;c;z)$ the classical hypergeometric function.

It is well known [**7, 39, 47**] that the Catalan numbers $C_n$ for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular $n$-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number $C_{n-2}$". The Catalan numbers $C_n$ can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + \cdots$$

and can be explicitly expressed as

$$C_n = \frac{1}{n+1}\binom{2n}{n} = {}_2F_1(1-n,-n;2;1) = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)},$$

where the classical Euler gamma function $\Gamma(z)$ can be defined [**5, 11, 16, 26, 42**] by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}.$$

For more information on the Catalan numbers $C_n$ and their recent developments, please refer to the monographs [**2, 7, 47**], the papers [**8, 9, 14, 18, 19, 22, 27, 28, 29, 30, 33, 35, 36, 37, 38, 39, 44, 49, 50, 51, 52**], and the closely related references therein.

The first six Chebyshev polynomials of the second kind $U_k(x)$ for $0 \le k \le 5$ are

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x,$$
$$U_4(x) = 16x^4 - 12x^2 + 1, \quad U_5(x) = 32x^5 - 32x^3 + 6x.$$

They can be generated by

$$F(t) = F(t,x) = \frac{1}{1-2xt+t^2} = \sum_{k=0}^{\infty} U_k(x)t^k$$

for $|x| < 1$ and $|t| < 1$. For more information on the Chebyshev polynomials of the second kind $U_k(x)$, please refer to the papers [**20, 32, 37**], the monographs [**5, 10, 11, 42, 43**], and the closely related references therein. For more information on other new and special polynomials, please refer to the papers [**15, 17, 31, 40**] and closely related references cited therein.

Let $\lfloor x \rfloor$ denote the floor function whose value is the largest integer less than or equal to $x$ and let $\lceil x \rceil$ stand for the ceiling function which gives the smallest integer not less than $x$. When $n \in \mathbb{Z}$, it is easy to see that

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{1}{2}\left[n - \frac{1-(-1)^n}{2}\right] \quad \text{and} \quad \left\lceil \frac{n}{2} \right\rceil = \frac{1}{2}\left[n + \frac{1-(-1)^n}{2}\right].$$

In this paper, we will establish two identities to express the generating function $F(t)$ of the Chebyshev polynomials of the second kind $U_k(x)$ and its higher order derivatives $F^{(k)}(t)$ in terms of $F^{(k)}(t)$ and $F(t)$ each other, find an explicit formula and an identity for the Chebyshev polynomials of the second kind $U_k(x)$, derive the inverse of an integer, unit, and lower triangular matrix, acquire an inversion theorem, present several identities of the Catalan numbers $C_k$, and give some remarks on the closely related results including connections of the Catalan numbers $C_k$ with the Chebyshev polynomials of the second kind $U_k(x)$, the central Delannoy numbers [**20, 21**], and the Fibonacci polynomials [**20, 24**] respectively.

## 2. LEMMAS

In order to prove our main results, we recall several lemmas below.

**Lemma 2.1** ([**2**, p. 134, Theorem A] and [**2**, p. 139, Theorem C])**.** *For $n \ge k \ge 0$, the Bell polynomials of the second kind, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i\ell_i = n \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

*The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f \circ h(t) = \sum_{k=1}^{n} f^{(k)}(h(t)) \, \mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big), \quad n \in \mathbb{N}. \tag{2.1}$$

**Lemma 2.2** ([**2**, p. 135])**.** *For complex numbers $a$ and $b$, we have*

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \, \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}). \tag{2.2}$$

**Lemma 2.3** ([**23**, Theorem 4.1], [**37**, Eq. (2.8)], and [**48**, Lemma 2.5])**.** *For $0 \le k \le n$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\mathrm{B}_{n,k}(x, 1, 0, \dots, 0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n}, \tag{2.3}$$

*where $\binom{p}{q} = 0$ for $q > p \ge 0$.*

**Lemma 2.4** ([**4**] and [**7**, pp. 112–114])**.** *Let $T(r, 1) = 1$ and*

$$T(r, c) = \sum_{i=c-1}^{r} T(i, c-1), \quad c \ge 2,$$

*or, equivalently,*

$$T(r,c)=\sum_{j=1}^{c}T(r-1,j), \quad r,c\in\mathbb{N}.$$

*Then*

$$T(r,c)=\frac{r-c+2}{r+1}\binom{r+c-1}{r}, \quad r,c\in\mathbb{N}$$

*and* $T(n,n)=C_n$ *for* $n\in\mathbb{N}$.

**Lemma 2.5** ([**12**, p. 2, Eq. (10)] and [**3, 18, 19, 27, 45, 46**])**.** *For* $n\in\mathbb{N}$*, the Catalan numbers* $C_n$ *have the integral representation*

$$C_n=\frac{1}{2\pi}\int_0^4\sqrt{\frac{4-x}{x}}\,x^n\,\mathrm{d}x. \tag{2.4}$$

**Lemma 2.6.** *For* $0\ne|t|<1$ *and* $k\in\mathbb{N}$*, we have*

$${}_2F_1\left(\frac{1-k}{2},\frac{2-k}{2};1-k;\frac{1}{t^2}\right)=\frac{t}{2^k\sqrt{t^2-1}}\left[\left(1+\frac{\sqrt{t^2-1}}{t}\right)^k-\left(1-\frac{\sqrt{t^2-1}}{t}\right)^k\right].$$

*Proof.* In [**5**, pp. 999–1000] and [**11**, pp. 442 and 449, Items 18.5.10 and 18.12.4], it was listed that

$$G_n^\lambda(t)=\frac{1}{\sqrt{\pi}}\frac{\Gamma(2\lambda+n)}{n!\Gamma(2\lambda)}\frac{\Gamma\left(\frac{2\lambda+1}{2}\right)}{\Gamma(\lambda)}\int_0^\pi\left(t+\sqrt{t^2-1}\cos\phi\right)^n\sin^{2\lambda-1}\phi\,\mathrm{d}\phi, \quad |t|<1 \tag{2.5}$$

and

$$G_n^\lambda(t)=\frac{(2t)^n\Gamma(\lambda+n)}{n!\Gamma(\lambda)}{}_2F_1\left(-\frac{n}{2},\frac{1-n}{2};1-\lambda-n;\frac{1}{t^2}\right), \quad 0\ne|t|<1, \tag{2.6}$$

where $G_n^\lambda(t)$ stands for the Gegenbauer polynomials which are the coefficients of $\alpha^n$ in the power-series expansion

$$\frac{1}{(1-2t\alpha+\alpha^2)^\lambda}=\sum_{k=0}^{\infty}G_k^\lambda(t)\alpha^k, \quad |t|<1.$$

Taking $n=j-1$ and $\lambda=1$ in equalities (2.5) and (2.6), combining them, and simplifying give

$$\begin{aligned}
{}_2F_1\left(\frac{1-j}{2},\frac{2-j}{2};1-j;\frac{1}{t^2}\right)&=\frac{j}{2^j}\frac{1}{t^{j-1}}\int_0^\pi\left(t+\sqrt{t^2-1}\cos\phi\right)^{j-1}\sin\phi\,\mathrm{d}\phi\\
&=\frac{j}{2^j}\frac{(t^2-1)^{(j-1)/2}}{t^{j-1}}\int_0^\pi\left(\frac{t}{\sqrt{t^2-1}}+\cos\phi\right)^{j-1}\sin\phi\,\mathrm{d}\phi\\
&=\frac{j}{2^j}\frac{(t^2-1)^{(j-1)/2}}{t^{j-1}}\int_0^\pi\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\left(\frac{t}{\sqrt{t^2-1}}\right)^{j-1-\ell}\cos^\ell\phi\sin\phi\,\mathrm{d}\phi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{j}{2^j}\frac{(t^2-1)^{(j-1)/2}}{t^{j-1}}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\left(\frac{t}{\sqrt{t^2-1}}\right)^{j-1-\ell}\int_0^\pi \cos^\ell\phi\sin\phi\,\mathrm{d}\phi\\
&= \frac{j}{2^j}\frac{(t^2-1)^{(j-1)/2}}{t^{j-1}}\left(\frac{t}{\sqrt{t^2-1}}\right)^{j-1}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\left(\frac{\sqrt{t^2-1}}{t}\right)^{\ell}\frac{(-1)^\ell+1}{\ell+1}\\
&= \frac{j}{2^j}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\left(\frac{\sqrt{t^2-1}}{t}\right)^{\ell}\frac{(-1)^\ell+1}{\ell+1}\\
&= \frac{1}{2^j}\frac{t}{\sqrt{t^2-1}}\left[\left(1+\frac{\sqrt{t^2-1}}{t}\right)^j-\left(1-\frac{\sqrt{t^2-1}}{t}\right)^j\right]
\end{aligned}
$$

for $|t|<1$ and $t\ne0$. The proof of Lemma 2.6 is complete. □

**Lemma 2.7** ([**5**, p. 399])**.** *If* $\Re(\nu)>0$*, then*

$$
\int_0^{\pi/2}\cos^{\nu-1}x\cos(ax)\,\mathrm{d}x=\frac{\pi}{2^\nu\nu B\left(\frac{\nu+a+1}{2},\frac{\nu-a+1}{2}\right)}, \tag{2.7}
$$

*where* $B(\alpha,\beta)$ *stands for the classical beta function satisfying*

$$
B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=B(\beta,\alpha),\quad \Re(\alpha),\Re(\beta)>0.
$$

## 3. IDENTITIES OF THE CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

In this section, we establish three identities and an explicit formula for the Chebyshev polynomials of the second kind $U_k(x)$, their generating function $F(t)$, and higher order derivatives $F^{(k)}(t)$.

**Theorem 3.1.** *Let* $n\in\mathbb{N}$*. Then*

1. *the nth derivatives of the generating function* $F(t)$ *of the Chebyshev polynomials of the second kind* $U_k(x)$ *satisfy*

$$
F^{(n)}(t)=\frac{n!}{[2(t-x)]^n}\sum_{k=\lceil n/2\rceil}^{n}(-1)^k\binom{k}{n-k}[2(t-x)]^{2k}F^{k+1}(t) \tag{3.8}
$$

*and*

$$
F^{n+1}(t)=\frac{1}{n}\frac{1}{[2(t-x)]^{2n}}\sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}[2(t-x)]^kF^{(k)}(t); \tag{3.9}
$$

2. *the equations* (3.8) *and* (3.9) *are equivalent to each other.*

*Consequently,*

1. *the Chebyshev polynomials of the second kind* $U_n(x)$ *satisfy*

$$U_n(x)=\frac{(-1)^n}{(2x)^n}\sum_{k=\lceil n/2\rceil}^{n}(-1)^k\binom{k}{n-k}(2x)^{2k} \tag{3.10}$$

*and*

$$\sum_{k=1}^{n}k\binom{2n-k-1}{n-1}(2x)^kU_k(x)=n(2x)^{2n}; \tag{3.11}$$

2. *the equations* (3.10) *and* (3.11) *are equivalent to each other.*

*Proof.* By the formulas (2.1), (2.2), and (2.3) in sequence, we have

$$\begin{aligned}
F^{(n)}(t)&=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{1-2tx+t^2}\right)\\
&=\sum_{k=1}^{n}\left(\frac{1}{u}\right)^{(k)}\mathrm{B}_{n,k}(-2x+2t,2,0,\dots,0)\\
&=\sum_{k=1}^{n}\frac{(-1)^kk!}{u^{k+1}}2^k\,\mathrm{B}_{n,k}(t-x,1,0,\dots,0)\\
&=\sum_{k=1}^{n}\frac{(-1)^kk!}{u^{k+1}}2^k\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}(t-x)^{2k-n}\\
&=(-1)^nn!\sum_{k=1}^{n}(-1)^k2^{2k-n}\binom{k}{n-k}\frac{(x-t)^{2k-n}}{(1-2tx+t^2)^{k+1}}\\
&=(-1)^nn!\sum_{k=1}^{n}(-1)^k2^{2k-n}\binom{k}{n-k}(x-t)^{2k-n}F^{k+1}(t)
\end{aligned}$$

for $n\in\mathbb{N}$, where $u=u(t,x)=1-2tx+t^2$. This can be rewritten as the formula (3.8).

We can reformulate the formula (3.8) as

$$\begin{pmatrix}
\frac{[2(t-x)]^1}{1!}F'(t)\\
\frac{[2(t-x)]^2}{2!}F''(t)\\
\frac{[2(t-x)]^3}{3!}F^{(3)}(t)\\
\vdots\\
\frac{[2(t-x)]^{n-2}}{(n-2)!}F^{(n-2)}(t)\\
\frac{[2(t-x)]^{n-1}}{(n-1)!}F^{(n-1)}(t)\\
\frac{[2(t-x)]^{n}}{n!}F^{(n)}(t)
\end{pmatrix}=A_n\begin{pmatrix}
(-1)^1[2(x-t)]^2F^2(t)\\
(-1)^2[2(x-t)]^4F^3(t)\\
(-1)^3[2(x-t)]^6F^4(t)\\
\vdots\\
(-1)^{n-2}[2(x-t)]^{2(n-2)}F^{n-1}(t)\\
(-1)^{n-1}[2(x-t)]^{2(n-1)}F^{n}(t)\\
(-1)^{n}[2(x-t)]^{2n}F^{n+1}(t)
\end{pmatrix}$$

for $n \in \mathbb{N}$, where $A_n = (a_{i,j})_{n\times n}$ with

$$a_{i,j} = \begin{cases} 0, & i<j \\ \dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i>2j \end{cases} \tag{3.12}$$

for $i,j \in \mathbb{N}$. This means that

$$\begin{pmatrix} (-1)^1[2(x-t)]^2F^2(t) \\ (-1)^2[2(x-t)]^4F^3(t) \\ (-1)^3[2(x-t)]^6F^4(t) \\ \vdots \\ (-1)^{n-2}[2(x-t)]^{2(n-2)}F^{n-1}(t) \\ (-1)^{n-1}[2(x-t)]^{2(n-1)}F^{n}(t) \\ (-1)^{n}[2(x-t)]^{2n}F^{n+1}(t) \end{pmatrix} = A_n^{-1} \begin{pmatrix} \frac{[2(t-x)]^1}{1!}F'(t) \\ \frac{[2(t-x)]^2}{2!}F''(t) \\ \frac{[2(t-x)]^3}{3!}F^{(3)}(t) \\ \vdots \\ \frac{[2(t-x)]^{n-2}}{(n-2)!}F^{(n-2)}(t) \\ \frac{[2(t-x)]^{n-1}}{(n-1)!}F^{(n-1)}(t) \\ \frac{[2(t-x)]^{n}}{n!}F^{(n)}(t) \end{pmatrix} \tag{3.13}$$

for $n \in \mathbb{N}$, where $A_n^{-1} = (b_{i,j})_{n\times n}$ denotes the inverse matrix of $A_n$.

By the software Mathematica or by hands, we can obtain immediately that

$$A_7^{-1} = \begin{pmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&2&1&0&0&0 \\ 0&1&3&1&0&0 \\ 0&0&3&4&1&0 \\ 0&0&1&6&5&1 \end{pmatrix}^{-1} = \begin{pmatrix} 1&0&0&0&0&0 \\ -1&1&0&0&0&0 \\ 2&-2&1&0&0&0 \\ -5&5&-3&1&0&0 \\ 14&-14&9&-4&1&0 \\ -42&42&-28&14&-5&1 \end{pmatrix}. \tag{3.14}$$

The first few values of the sequence $T(r,c)$ can be listed as Table 1, where $T(r,c)$ denote the $r$th element in column $c$ for $r,c \ge 1$, see [**7**, p. 113]. Comparing Table 1

Table 1: Definition of $T(r,c)$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | 1 | 2 | | | |
| 3 | 1 | 3 | 5 | | |
| 4 | 1 | 4 | 9 | 14 | |
| 5 | 1 | 5 | 14 | 28 | 42 |

and the inverse matrix (3.14) should infer that

$$T(k+m,k) = (-1)^{k+1} b_{k+m+1,m+2}, \quad k \ge 1, \quad m \ge 0.$$

Hence, by Lemma 2.4, we should obtain

$$b_{p,q} = (-1)^{p-q}T(p-1,p-q+1) = (-1)^{p-q}\frac{q}{p}\binom{2p-q-1}{p-1}, \quad p \ge q \ge 2.$$

It is easy to see that the formula

$$b_{p,q} = (-1)^{p-q}\frac{q}{p}\binom{2p-q-1}{p-1}$$

should be valid for all $p \ge q \ge 1$. This should imply that

$$(-1)^n[2(x-t)]^{2n}F^{n+1}(t) = \sum_{k=1}^{n} b_{n,k}\frac{[2(t-x)]^k}{k!}F^{(k)}(t), \quad n \in \mathbb{N}. \tag{3.15}$$

We now start out to inductively verify the equation (3.15). When $n = 1, 2$, the equation (3.15) are

$$-[2(x-t)]^2F^2(t) = b_{1,1}\frac{2(t-x)}{1!}F'(t) = b_{1,1}\frac{2(t-x)}{1!}\frac{2x-2t}{(1-2tx+t^2)^2}$$

and

$$\begin{aligned}
[2(x-t)]^4F^3(t) &= \sum_{k=1}^{2} b_{2,k}\frac{[2(t-x)]^k}{k!}F^{(k)}(t) \\
&= b_{2,1}\frac{2(t-x)}{1!}F'(t) + b_{2,2}\frac{[2(t-x)]^2}{2!}F''(t) \\
&= b_{2,1}\frac{2(t-x)}{1!}\frac{2x-2t}{(1-2tx+t^2)^2} + b_{2,2}\frac{[2(t-x)]^2}{2!}\frac{2\bigl(3t^2-6tx+4x^2-1\bigr)}{(t^2-2tx+1)^3}
\end{aligned}$$

which are clearly valid. When $n \ge 3$, we rewrite (3.15) as

$$(-1)^nF^{n+1}(t) = \sum_{k=1}^{n} b_{n,k}\frac{[2(t-x)]^{k-2n}}{k!}F^{(k)}(t). \tag{3.16}$$

Differentiating with respect to $t$ on both sides of (3.16) yields

$$\begin{aligned}
&(-1)^n(n+1)F^n(t)F'(t) \\
&= \sum_{k=1}^{n}\frac{b_{n,k}}{k!}\bigl\{2(k-2n)[2(t-x)]^{k-2n-1}F^{(k)}(t) + [2(t-x)]^{k-2n}F^{(k+1)}(t)\bigr\} \\
&= \sum_{k=1}^{n}\frac{b_{n,k}}{k!}2(k-2n)[2(t-x)]^{k-2n-1}F^{(k)}(t) + \sum_{k=1}^{n}\frac{b_{n,k}}{k!}[2(t-x)]^{k-2n}F^{(k+1)}(t) \\
&= \sum_{k=1}^{n}\frac{2(k-2n)b_{n,k}}{k!}[2(t-x)]^{k-2n-1}F^{(k)}(t) + \sum_{k=2}^{n+1}\frac{b_{n,k-1}}{(k-1)!}[2(t-x)]^{k-1-2n}F^{(k)}(t) \\
&= \frac{b_{n,1}}{1!}\frac{2(1-2n)}{[2(t-x)]^{2n}}F'(t) + \frac{b_{n,n}}{n!}\frac{1}{[2(t-x)]^n}F^{(n+1)}(t) \\
&\quad + \sum_{k=2}^{n}\left[\frac{b_{n,k}}{k!}2(k-2n) + \frac{b_{n,k-1}}{(k-1)!}\right][2(t-x)]^{k-2n-1}F^{(k)}(t)
\end{aligned}$$

which can be rearranged as

$$
\begin{aligned}
(-1)^{n+1}F^{n+2}(t) = {} & \frac{2(1-2n)b_{n,1}}{n+1}\frac{[2(t-x)]^{1-2(n+1)}}{1!}F'(t) \\
& + b_{n,n}\frac{[2(t-x)]^{(n+1)-2(n+1)}}{(n+1)!}F^{(n+1)}(t) \\
& + \sum_{k=2}^{n}\frac{2(k-2n)b_{n,k}+kb_{n,k-1}}{n+1}\frac{[2(t-x)]^{k-2(n+1)}}{k!}F^{(k)}(t).
\end{aligned}
$$

It is easy to see that

$$
\frac{2(1-2n)b_{n,1}}{n+1} = \frac{2(1-2n)}{n+1}(-1)^{n-1}\frac{1}{n}\binom{2n-2}{n-1} = (-1)^n\frac{1}{n+1}\binom{2n}{n} = b_{n+1,1}.
$$

Since $b_{k,k}=1$ for all $1\le k\le n\in\mathbb{N}$, it is sufficient to show

$$
\frac{2(k-2n)b_{n,k}+kb_{n,k-1}}{n+1} = b_{n+1,k} \tag{3.17}
$$

for $2\le k\le n$. This is equivalent to

$$
\begin{aligned}
\frac{2(k-2n)}{n+1}(-1)^{n-k}\frac{k}{n}\binom{2n-k-1}{n-1} + \frac{k}{n+1}(-1)^{n-k+1}\frac{k-1}{n}\binom{2n-k}{n-1} \\
= (-1)^{n+1-k}\frac{k}{n+1}\binom{2n-k+1}{n}
\end{aligned}
$$

which can be verified straightforwardly. The equation (3.15), which can be reformulated as (3.9) for $n\in\mathbb{N}$, is thus proved.

The formulas (3.10) and (3.11) follow readily from taking $t\to 0$ on both sides of (3.8) and (3.9) respectively. The proof of Theorem 3.1 is complete. □

## 4. THE INVERSE OF A TRIANGULAR MATRIX AND AN INVERSION THEOREM

In this section, we will conclude the inverse $A_n^{-1}$ of the integer, unit, and lower triangular matrix $A_n$ defined by (3.12) and derive an inversion theorem from $A_n^{-1}$.

### 4.1 The inverse of a triangular matrix

Basing on equations (3.8) and (3.9), we first derive the inverse of an integer, unit, and lower triangular matrix.

**Theorem 4.2.** *For $n \in \mathbb{N}$, let*

$$A_n = (a_{i,j})_{n\times n} = \begin{pmatrix} \binom{1}{0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \binom{1}{1} & \binom{2}{0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \binom{2}{1} & \binom{3}{0} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \binom{2}{2} & \binom{3}{1} & \binom{4}{0} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \binom{3}{2} & \binom{4}{1} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \binom{3}{3} & \binom{4}{2} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \binom{4}{3} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-3}{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-3}{1} & \binom{n-2}{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-3}{2} & \binom{n-2}{1} & \binom{n-1}{0} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-3}{3} & \binom{n-2}{2} & \binom{n-1}{1} & \binom{n}{0} \end{pmatrix}_{n\times n},$$

*where*

$$a_{i,j} = \begin{cases} 0, & i < j \\ \dbinom{j}{i-j}, & j \le i \le 2j \\ 0, & i > 2j \end{cases}$$

*for $1 \le i, j \le n$. Then*

$$\begin{aligned} & A_n^{-1} = (b_{i,j})_{n\times n} \\ &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 2 & -2 & 1 & \cdots & 0 & 0 \\ -5 & 5 & -3 & \cdots & 0 & 0 \\ 14 & -14 & 9 & \cdots & 0 & 0 \\ -42 & 42 & -28 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^{n-1}}{n-2}\binom{2n-6}{n-3} & \frac{(-1)^{n}2}{n-2}\binom{2n-7}{n-3} & \frac{(-1)^{n-1}3}{n-2}\binom{2n-8}{n-3} & \cdots & 0 & 0 \\ \frac{(-1)^{n}}{n-1}\binom{2n-4}{n-2} & \frac{(-1)^{n-1}2}{n-1}\binom{2n-5}{n-2} & \frac{(-1)^{n}3}{n-1}\binom{2n-6}{n-2} & \cdots & 1 & 0 \\ \frac{(-1)^{n-1}}{n}\binom{2n-2}{n-1} & \frac{(-1)^{n}2}{n}\binom{2n-3}{n-1} & \frac{(-1)^{n-1}3}{n}\binom{2n-4}{n-1} & \cdots & -(n-1) & 1 \end{pmatrix}_{n\times n}, \end{aligned}$$

*where*

$$b_{i,j} = \begin{cases} 0, & 1 \le i < j \le n; \\ (-1)^{i-j}\dfrac{j}{i}\dbinom{2i-j-1}{i-1}, & n \ge i > j \ge 1. \end{cases} \tag{4.18}$$

*Proof.* This follows straightforwardly from combining (3.13) with (3.9). The proof of Theorem 4.2 is complete. $\square$

## 4.2 An inversion theorem

In [**6**, p. 4, Eq. (1.1.9d)], it was given that

$$\sum_{k=\ell}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{\ell}=\begin{cases}1, & \ell=n;\\ 0, & 1\le \ell<n.\end{cases} \tag{4.19}$$

We now deduce a similar result to (4.19) from Theorem 4.2 as follows.

**Theorem 4.3.** *For $\ell,n\in\mathbb{N}$ with $\ell\le n$, we have*

$$\sum_{k=\ell}^{n}(-1)^{k-\ell}k\binom{2n-k-1}{n-1}\binom{\ell}{k-\ell}=\begin{cases}n, & \ell=n;\\ 0, & 0<\ell<n.\end{cases}$$

*Proof.* Since $A_n^{-1}A=I_n$, using the last row of $A_n^{-1}$ to multiply every column of $A_n$ gives the desired conclusion. The proof of Theorem 4.3 is complete. □

It is well known [**2**, pp. 143–144] that the binomial inversion theorem reads that

$$s_n=\sum_{k=0}^{n}\binom{n}{k}S_k \quad\text{if and only if}\quad S_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}s_k$$

for $n\ge 0$, where $\{s_n, n\ge 0\}$ and $\{S_n, n\ge 0\}$ are sequences of complex numbers. The formula (4.19) plays a central role in proving the above binomial inversion theorem. Now we use Theorem 4.3 to deduce an inversion theorem similar to the binomial inversion theorem.

**Theorem 4.4.** *For $k\ge 1$, let $s_k$ and $S_k$ be two sequences independent of $n$ such that $n\ge k\ge 1$. Then*

$$\frac{s_n}{n!}=\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}S_k \quad\textit{if and only if}\quad nS_n=\sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}s_k$$

*or, equivalently,*

$$s_n=\sum_{k=1}^{n}\binom{k}{n-k}S_k \quad\textit{if and only if}\quad (-1)^n nS_n=\sum_{k=1}^{n}\binom{2n-k-1}{n-1}(-1)^k k s_k.$$

*First proof.* By standard argument, we have

$$\begin{aligned}nS_n&=\sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}\left[k!\sum_{\ell=1}^{k}(-1)^\ell\binom{\ell}{k-\ell}S_\ell\right]\\&=\sum_{k=1}^{n}\sum_{\ell=1}^{k}(-1)^{k-\ell}k\binom{2n-k-1}{n-1}\binom{\ell}{k-\ell}S_\ell\end{aligned}$$

$$= \sum_{\ell=1}^{n}\left[\sum_{k=\ell}^{n}(-1)^{k-\ell}k\binom{2n-k-1}{n-1}\binom{\ell}{k-\ell}\right]S_\ell$$
$$= nS_n,$$

where we used Theorem 4.3 in the last step.

Similarly, we can prove the converse direction. The first proof of Theorem 4.4 is complete. □

*Second proof.* Let $\vec{s}_n = (s_1, s_2, \dots, s_n)^T$ and $\vec{S}_n = (S_1, S_2, \dots, S_n)^T$, where $T$ stands for the transpose of a matrix. Theorem 4.2 means that $\vec{s}_n = A_n \vec{S}_n$ if and only if $\vec{S}_n = A_n^{-1}\vec{s}_n$. This necessary and sufficient condition is equivalent to the one that

$$s_n = \sum_{k=1}^{n} a_{n,k}S_k = \sum_{k=1}^{n}\binom{k}{n-k}S_k$$

if and only if

$$S_n = \sum_{k=1}^{n} b_{n,k}s_k = \sum_{k=1}^{n}(-1)^{n-k}\frac{k}{n}\binom{2n-k-1}{n-1}s_k$$

for all $n \in \mathbb{N}$. In other words,

$$s_n = \sum_{k=1}^{n}\binom{k}{n-k}S_k \quad \text{if and only if} \quad (-1)^n nS_n = \sum_{k=1}^{n}\binom{2n-k-1}{n-1}(-1)^k k s_k.$$

Further replacing $S_k$ by $(-1)^k S_k$ and $s_k$ by $\frac{s_k}{k!}$ reveals that

$$\frac{s_n}{n!} = \sum_{k=1}^{n}\binom{k}{n-k}(-1)^k S_k$$

if and only if

$$(-1)^n n(-1)^n S_n = \sum_{k=1}^{n}(-1)^k k\binom{2n-k-1}{n-1}\frac{s_k}{k!}$$

for all $n \in \mathbb{N}$. The second proof of Theorem 4.4 is thus complete. □

## 5. IDENTITIES OF THE CATALAN NUMBERS

In this section, we present several identities of the Catalan numbers $C_k$.

**Theorem 5.5.** *For $i \ge j \ge 1$, we have*

$$\sum_{\ell=0}^{\lfloor (j-1)/2\rfloor}(-1)^\ell\binom{j-\ell-1}{\ell}C_{i-\ell-1} = \frac{j}{i}\binom{2i-j-1}{i-1}. \tag{5.20}$$

*Proof.* Observing the special result (3.14) again, we guess that the elements $b_{i,j}$ of the inverse of the triangular matrix $A_n$ should satisfy the following relations:

1. for $i < j$, the elements in the upper triangle are $b_{i,j} = 0$;
2. for all $i \in \mathbb{N}$, the elements on the main diagonal are $b_{i,i} = 1$;
3. the elements in the first two columns satisfy $b_{i,1} = -b_{i,2}$ for $i \geq 2$;
4. the elements in the first column are $b_{i,1} = (-1)^{i-1} C_{i-1}$;
5. for $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$,
$$b_{i+1,j+2} = b_{i,j} - b_{i+1,j+1};$$
6. for $i \geq j \geq 2$,
$$b_{i,j} = \sum_{k=-1}^{i-j-1} (-1)^{k+1} b_{i-1,j+k}.$$

Basing on these observations, we guess out that the elements $b_{i,j}$ should alternatively satisfy

$$b_{i,j} = (-1)^{i-j} \sum_{\ell=0}^{\lfloor (j-1)/2 \rfloor} (-1)^{\ell} \binom{j-\ell-1}{\ell} C_{i-\ell-1}, \quad i \geq j \geq 1. \tag{5.21}$$

Combining this with (4.18) and simplifying should yield the identity (5.20).

We now start off to verify the identity (5.20). By virtue of the integral representation (2.4), Lemma 2.6, and the integral (2.7) in Lemma 2.7, we acquire

$$\begin{aligned}
&\sum_{\ell=0}^{\lfloor (j-1)/2 \rfloor} (-1)^{\ell} \binom{j-\ell-1}{\ell} C_{i-\ell-1} \\
&= \frac{1}{2\pi} \int_0^4 \sqrt{\frac{4-x}{x}} \left[ \sum_{\ell=0}^{\lfloor (j-1)/2 \rfloor} (-1)^{\ell} \binom{j-\ell-1}{\ell} x^{i-\ell-1} \right] \mathrm{d} x \\
&= \frac{1}{2\pi} \int_0^4 x^{i-3/2} (4-x)^{1/2} \left[ \sum_{\ell=0}^{\lfloor (j-1)/2 \rfloor} \frac{(j-1-\ell)!}{(j-1-2\ell)!} \frac{1}{\ell!} \left(-\frac{1}{x}\right)^{\ell} \right] \mathrm{d} x \\
&= \frac{1}{2\pi} \int_0^4 x^{i-3/2} (4-x)^{1/2} \left[ \sum_{\ell=0}^{\lfloor (j-1)/2 \rfloor} \frac{\left(\frac{1-j}{2}\right)_{\ell} \left(\frac{2-j}{2}\right)_{\ell}}{(1-j)_{\ell}} \frac{1}{\ell!} \left(\frac{4}{x}\right)^{\ell} \right] \mathrm{d} x \\
&= \frac{1}{2\pi} \int_0^4 x^{i-3/2} (4-x)^{1/2} {}_2F_1\left(\frac{1-j}{2}, \frac{2-j}{2}; 1-j; \frac{4}{x}\right) \mathrm{d} x \\
&= \frac{4^i}{2\pi} \int_0^1 t^{i-3/2} (1-t)^{1/2} {}_2F_1\left(\frac{1-j}{2}, \frac{2-j}{2}; 1-j; \frac{1}{t}\right) \mathrm{d} t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4^i}{2\pi}\int_0^1 t^{i-3/2}(1-t)^{1/2}\frac{1}{2^j}\frac{\sqrt{t}}{\sqrt{t-1}}\left[\left(1+\frac{\sqrt{t-1}}{\sqrt{t}}\right)^j-\left(1-\frac{\sqrt{t-1}}{\sqrt{t}}\right)^j\right]\mathrm{d}t\\
&= \frac{2^{2i-j}}{2\pi}\boldsymbol{i}\int_0^1 t^{i-1}\left[\left(1+\sqrt{1-\frac{1}{t}}\right)^j-\left(1-\sqrt{1-\frac{1}{t}}\right)^j\right]\mathrm{d}t \qquad \left(\boldsymbol{i}=\sqrt{-1}\,\right)\\
&= \frac{2^{2i-j}}{\pi}\boldsymbol{i}\int_0^\infty \frac{s}{(1+s^2)^{i+1}}\left[(1-\boldsymbol{i}s)^j-(1+\boldsymbol{i}s)^j\right]\mathrm{d}s\\
&= \frac{2^{2i-j}}{\pi}\boldsymbol{i}\int_0^\infty \frac{s}{(1+s^2)^{i+1}}\left[\left(\sqrt{1+s^2}\,e^{-\boldsymbol{i}\arctan s}\right)^j-\left(\sqrt{1+s^2}\,e^{\boldsymbol{i}\arctan s}\right)^j\right]\mathrm{d}s\\
&= \frac{2^{2i-j}}{\pi}\boldsymbol{i}\int_0^\infty \frac{s}{(1+s^2)^{i-j/2+1}}\left(e^{-\boldsymbol{i}j\arctan s}-e^{\boldsymbol{i}j\arctan s}\right)\mathrm{d}s\\
&= \frac{2^{2i-j}}{\pi}\int_0^\infty \frac{s}{(1+s^2)^{i-j/2+1}}\sin(j\arctan s)\,\mathrm{d}s\\
&= \frac{2^{2i-j}}{\pi}\int_0^{\pi/2} \frac{\tan t}{(1+\tan^2 t)^{i-j/2+1}}\sin(jt)\sec^2 t\,\mathrm{d}t\\
&= \frac{2^{2i-j}}{\pi}\int_0^{\pi/2} \frac{\tan t}{\sec^{2i-j} t}\sin(jt)\,\mathrm{d}t\\
&= \frac{2^{2i-j}}{\pi}\int_0^{\pi/2} \sin t\cos^{2i-j-1} t\sin(jt)\,\mathrm{d}t\\
&= \frac{2^{2i-j}}{\pi}\int_0^{\pi/2} [\cos((j-1)t)-\cos((j+1)t)]\cos^{2i-j-1} t\,\mathrm{d}t\\
&= \frac{2^{2i-j}}{\pi}\left[\frac{\pi}{2^{2i-j}(2i-j)B(i,i-j+1)}-\frac{\pi}{2^{2i-j}(2i-j)B(i+1,i-j)}\right]\\
&= \frac{1}{2i-j}\left[\frac{1}{B(i,i-j+1)}-\frac{1}{B(i+1,i-j)}\right]\\
&= \frac{1}{2i-j}\left[\frac{\Gamma(2i-j+1)}{\Gamma(i)\Gamma(i-j+1)}-\frac{\Gamma(2i-j+1)}{\Gamma(i+1)\Gamma(i-j)}\right]\\
&= (2i-j-1)!\left[\frac{1}{\Gamma(i)\Gamma(i-j+1)}-\frac{1}{\Gamma(i+1)\Gamma(i-j)}\right]\\
&= (2i-j-1)!\left[\frac{1}{(i-1)!(i-j)!}-\frac{1}{i!(i-j-1)!}\right]\\
&= \frac{j}{i}\binom{2i-j-1}{i-1}.
\end{aligned}
$$

The identity (5.20) is thus proved. The proof of Theorem 5.5 is complete. □

**Theorem 5.6.** *For $i,j,n\in\mathbb{N}$, the Catalan numbers $C_n$ satisfy*

$$
\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}C_{n-k}=1, \tag{5.22}
$$

$$\sum_{\substack{i\le 2\ell\le 2i\\ \ell\ge j}}\sum_{k=0}^{\lfloor (j-1)/2\rfloor}(-1)^{\ell-k}\binom{\ell}{i-\ell}\binom{j-k-1}{k}C_{\ell-k-1}=0, \tag{5.23}$$

*and*

$$\sum_{\substack{i\ge \ell\ge j\\ \ell\le 2j}}\sum_{k=0}^{\lfloor (\ell-1)/2\rfloor}(-1)^{\ell-k}\binom{j}{\ell-j}\binom{\ell-k-1}{k}C_{i-k-1}=0. \tag{5.24}$$

*Proof.* This follows from expanding the matrix equation

$$A_nA_n^{-1}=A_n^{-1}A_n=I_n \tag{5.25}$$

and utilizing the expression (5.21), where $I_n$ stands for the identity matrix of $n$ orders. This can be written in details as follows.

The matrix equation (5.25) is equivalent to

$$\sum_{\ell=1}^{n}a_{i,\ell}b_{\ell,j}=\begin{cases}0, & i<j\\ \displaystyle\sum_{\ell=j}^{i}a_{i,\ell}b_{\ell,j}, & i\ge j\end{cases}=\begin{cases}0, & i\ne j\\ 1, & i=j\end{cases}$$

and

$$\sum_{\ell=1}^{n}b_{i,\ell}a_{\ell,j}=\begin{cases}0, & i<j\\ \displaystyle\sum_{\ell=j}^{i}b_{i,\ell}a_{\ell,j}, & i\ge j\end{cases}=\begin{cases}0, & i\ne j\\ 1, & i=j\end{cases}$$

which can be rearranged as

$$\sum_{\ell=j}^{i}a_{i,\ell}b_{\ell,j}=\begin{cases}0, & i>j\\ 1, & i=j\end{cases}\quad\text{and}\quad \sum_{\ell=j}^{i}b_{i,\ell}a_{\ell,j}=\begin{cases}0, & i>j\\ 1, & i=j\end{cases}$$

for $1\le i,j\le n$.

When $1\le i=j\le n$, it follows that

$$1=\sum_{\ell=j}^{i}a_{i,\ell}b_{\ell,j}=\sum_{\ell=j}^{i}b_{i,\ell}a_{\ell,j}=a_{i,i}b_{i,i}=b_{i,i}=\sum_{k=0}^{\lfloor (i-1)/2\rfloor}(-1)^{k}\binom{i-k-1}{k}C_{i-k-1}.$$

The identity (5.22) is thus concluded.

When $1\le j<i\le n$, it follows that

$$0=\sum_{\ell=j}^{i}a_{i,\ell}b_{\ell,j}=\sum_{\substack{i/2\le \ell\le i\\ \ell\ge j}}a_{i,\ell}b_{\ell,j}$$

$$= \sum_{\substack{i/2\le \ell\le i\\ \ell\ge j}} \binom{\ell}{i-\ell}(-1)^{\ell-j} \sum_{k=0}^{\lfloor (j-1)/2\rfloor} (-1)^k \binom{j-k-1}{k} C_{\ell-k-1}$$

$$= (-1)^j \sum_{\substack{i/2\le \ell\le i\\ \ell\ge j}} \sum_{k=0}^{\lfloor (j-1)/2\rfloor} (-1)^{\ell-k} \binom{\ell}{i-\ell}\binom{j-k-1}{k} C_{\ell-k-1}$$

and

$$0 = \sum_{\ell=j}^{i} b_{i,\ell} a_{\ell,j} = \sum_{\substack{i\ge \ell\ge j\\ \ell\le 2j}} b_{i,\ell} a_{\ell,j}$$

$$= \sum_{\substack{i\ge \ell\ge j\\ \ell\le 2j}} (-1)^{i-\ell} \sum_{k=0}^{\lfloor (\ell-1)/2\rfloor} (-1)^k \binom{\ell-k-1}{k} C_{i-k-1} \binom{j}{\ell-j}$$

$$= (-1)^i \sum_{\substack{i\ge \ell\ge j\\ \ell\le 2j}} \sum_{k=0}^{\lfloor (\ell-1)/2\rfloor} (-1)^{\ell-k} \binom{j}{\ell-j}\binom{\ell-k-1}{k} C_{i-k-1}.$$

The identities (5.23) and (5.24) are thus derived. The proof of Theorem 5.6 is complete. □

**Theorem 5.7.** *Let $m,n\in\mathbb{N}$. If $n\ge 2m\ge 2$, then*

$$\frac{\displaystyle\sum_{\ell=0}^{m-1}(-1)^\ell \binom{2m-\ell-1}{\ell}\frac{n+2\ell+1}{n-\ell+1} C_{n-\ell-1}}{\displaystyle\sum_{\ell=0}^{m-1}(-1)^\ell \binom{2m-\ell-2}{\ell}\frac{1}{2m-2\ell-1} C_{n-\ell-1}} = m(2m-1). \tag{5.26}$$

*Proof.* Employing the expression (5.21) and making use of Theorem 5.5, we can write the recursive equation (3.17) as

$$\begin{aligned}
&2(k-2n)(-1)^{n-k} \sum_{\ell=0}^{\lfloor (k-1)/2\rfloor} (-1)^\ell \binom{k-\ell-1}{\ell} C_{n-\ell-1}\\
&\quad + k(-1)^{n-k+1} \sum_{\ell=0}^{\lfloor (k-2)/2\rfloor} (-1)^\ell \binom{k-\ell-2}{\ell} C_{n-\ell-1}\\
&= (-1)^{n-k+1}\Bigg\{ k \sum_{\ell=0}^{\lfloor (k-2)/2\rfloor} (-1)^\ell \binom{k-\ell-2}{\ell} C_{n-\ell-1}\\
&\quad - 2(k-2n) \sum_{\ell=0}^{\lfloor (k-1)/2\rfloor} (-1)^\ell \binom{k-\ell-1}{\ell} C_{n-\ell-1}\Bigg\}
\end{aligned}$$

$$= (-1)^{n-k+1}(n+1) \sum_{\ell=0}^{\lfloor (k-1)/2 \rfloor} (-1)^\ell \binom{k-\ell-1}{\ell} C_{n-\ell}$$

for $n \ge 2$, that is,

$$k \sum_{\ell=0}^{\lfloor (k-2)/2 \rfloor} (-1)^\ell \binom{k-\ell-2}{\ell} C_{n-\ell-1} - 2(k-2n) \sum_{\ell=0}^{\lfloor (k-1)/2 \rfloor} (-1)^\ell \binom{k-\ell-1}{\ell} C_{n-\ell-1} = (n+1) \sum_{\ell=0}^{\lfloor (k-1)/2 \rfloor} (-1)^\ell \binom{k-\ell-1}{\ell} C_{n-\ell}$$

for $n \ge 2$. When $k = 2m$ and $m \in \mathbb{N}$, the above equality is equivalent to

$$2m \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-2}{\ell} C_{n-\ell-1} - 4(m-n) \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-1}{\ell} C_{n-\ell-1} = (n+1) \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-1}{\ell} C_{n-\ell},$$

$$2m \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-2}{\ell} C_{n-\ell-1} - 4m \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-1}{\ell} C_{n-\ell-1} = (n+1) \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-1}{\ell} C_{n-\ell} - 4n \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-1}{\ell} C_{n-\ell-1},$$

$$2m \sum_{\ell=0}^{m-1} (-1)^\ell \left[\binom{2m-\ell-2}{\ell} - 2\binom{2m-\ell-1}{\ell}\right] C_{n-\ell-1} = \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-1}{\ell} [(n+1)C_{n-\ell} - 4nC_{n-\ell-1}],$$

$$m(2m-1) \sum_{\ell=0}^{m-1} (-1)^\ell \frac{(2m-\ell-2)!}{\ell!(2m-2\ell-1)!} C_{n-\ell-1} = \sum_{\ell=0}^{m-1} (-1)^\ell \binom{2m-\ell-1}{\ell} \frac{n+2\ell+1}{n-\ell+1} C_{n-\ell-1}$$

which can be rearranged as

$$\sum_{\ell=0}^{m-1} (-1)^\ell \left[m(2m-1) - \frac{(2m-\ell-1)(n+2\ell+1)}{n-\ell+1}\right] \frac{(2m-\ell-2)!}{\ell!(2m-2\ell-1)!} C_{n-\ell-1} = 0$$

for $n \ge 2m \ge 2$. This can be further rewritten as (5.26). The proof of Theorem 5.7 is complete. $\square$

## 6. REMARKS

Finally, we give some remarks on the closely related results stated in previous sections.

*Remark* 6.1. The identity (5.22) recovers [**50**, p. 2187, Theorem 2, Eq. (15b)]. It can also be verified alternatively and directly by the same method used in the proof of the identity (5.20).

Actually, the identity (5.22) is a special case $i = j \in \mathbb{N}$ of the identity (5.20). In other words, the identity (5.20) generalizes, or say, extends (5.22).

It is clear that the proof of the identity (5.22) in this paper is simpler than the one adopted in [**50**] and the related references therein.

In [**7**, p. 322, Theorem 12.1], it was given that

$$C_n = \sum_{r=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r}, \quad n \ge 1 \tag{6.27}$$

which can be rearranged as

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} C_{n-k-1} = 0, \quad n \ge 1. \tag{6.28}$$

This identity is a special case $j = 1$ of the equality (5.23). Indeed, when $j = 1$, the identity (5.23) becomes

$$\sum_{\ell=\lceil i/2 \rceil}^{i} (-1)^{\ell} \binom{\ell}{i-\ell} C_{\ell-1} = 0.$$

Further letting $k = i - \ell$ leads to (6.28).

The identity (6.27) was also generalized by the third identity (7) in [**9**, Theorem 1].

*Remark* 6.2. Let $A_n = I_n + M_n$ and $I_n$ be the identity matrix of order $n$. By linear algebra, it is easy to see that $M_n^n = 0$ and

$$(I_n + M_n)\bigl(I_n - M_n + M_n^2 - M_n^3 + \dots + (-1)^{n-1} M_n^{n-1}\bigr) = I_n - M_n^n = I_n.$$

This means that

$$A_n^{-1} = (I_n + M_n)^{-1} = I_n + \sum_{k=1}^{n-1} (-1)^k M_n^k.$$

In theory, this formula is useful for computing the inverse $A_n^{-1}$. But, in practice, it is too difficult to acquire the simple forms in (4.18) and (5.21).

Can one conclude a general and concrete formula for computing $M_n^k$? If yes, then one can find an alternative method to compute $A_n^{-1}$.

*Remark* 6.3. In [**11**, p. 387, 15.4.18], it was listed that the formula

$$ {}_2F_1\left(a, a+\frac{1}{2}; 2a; z\right) = \frac{1}{\sqrt{1-z}}\left(\frac{1}{2}+\frac{\sqrt{1-z}}{2}\right)^{1-2a}, \quad |z|<1 \tag{6.29} $$

holds for $a, a+\frac{1}{2} \notin \{0, -1, -2, \dots\}$ and for the principal branch. Replacing $z$ by $\frac{1}{t^2}$ leads to the equality

$$ {}_2F_1\left(a, a+\frac{1}{2}; 2a; \frac{1}{t^2}\right) = \frac{1}{2^{1-2a}}\frac{|t|}{\sqrt{t^2-1}}\left(1+\frac{\sqrt{t^2-1}}{|t|}\right)^{1-2a} $$

for $a, a+\frac{1}{2} \notin \{0, -1, -2, \dots\}$ and $|t|>1$.

By the way, the formula (6.29) can also be derived from the facts that

$$ {}_2F_1(a, b; b; z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!} = (1-z)^{-a}, \quad |z|<1, $$
$$ \frac{\mathrm{d}^n}{\mathrm{d}z^n}(1-z)^{-a} = (a)_n(1-z)^{-a-n}, \quad \frac{\mathrm{d}^n}{\mathrm{d}z^n}(1-z)^{-a}\bigg|_{z=0} = (a)_n, $$
$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad \Gamma(2z) = \frac{2^{2z-1/2}}{\sqrt{2\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), $$

where the first formula can be found in [**11**, p. 1015, Item 9.121(1)] and the last formula is the duplication formula [**1**, p. 256, Item 6.1.18] for the classical gamma function $\Gamma(z)$.

*Remark* 6.4. Comparing main results of this paper with those in [**20, 21, 32**], we can see that there exist some close connections among the Chebyshev polynomials of the second kind $U_n$, the Catalan numbers $C_n$, the central Delannoy numbers $D_n$, the Fibonacci polynomials $F_n(x)$, and triangular and tridiagonal matrices.

Comparing Theorem 3.1 with Theorem 5.5 reveals that the equality (3.11) can be reformulated in terms of the Catalan numbers $C_n$ as

$$ \sum_{k=1}^{n}\left[\sum_{\ell=0}^{\lfloor (k-1)/2\rfloor}(-1)^{\ell}\binom{k-\ell-1}{\ell}C_{n-\ell-1}\right](2x)^k U_k(x) = (2x)^{2n}. \tag{6.30} $$

Taking $x=3$ in (6.30) and considering results in [**20**, Section 9] disclose that

$$ \sum_{k=1}^{n}6^k\left[\sum_{\ell=0}^{\lfloor (k-1)/2\rfloor}(-1)^{\ell}\binom{k-\ell-1}{\ell}C_{n-\ell-1}\right]\left[\sum_{\ell=0}^{k}D(\ell)D(k-\ell)\right] = 6^{2n}, $$

where $D(k)$ denotes the central Delannoy numbers which are combinatorially the numbers of "king walks" from the $(0,0)$ corner of an $n\times n$ square to the upper right corner $(n,n)$ and can be generated analytically by

$$ \frac{1}{\sqrt{1-6x+x^2}} = \sum_{k=0}^{\infty}D(k)x^k = 1+3x+13x^2+63x^3+\cdots $$

Taking $x = \frac{s}{2}\sqrt{-1}$ in (6.30) and utilizing results in [**20**, Section 7] expose that

$$\sum_{k=1}^{n}(-1)^k\left[\sum_{\ell=0}^{\lfloor (k-1)/2\rfloor}(-1)^\ell\binom{k-\ell-1}{\ell}C_{n-\ell-1}\right]s^kF_{k+1}(s) = (-1)^n s^{2n},$$

where the Fibonacci polynomials

$$F_n(s) = \frac{1}{2^n}\frac{\left(s+\sqrt{4+s^2}\,\right)^n - \left(s-\sqrt{4+s^2}\,\right)^n}{\sqrt{4+s^2}}$$

can be generated [**24**] by

$$\frac{t}{1-ts-t^2} = \sum_{n=1}^{\infty}F_n(s)t^n = t + st^2 + \left(s^2+1\right)t^3 + \left(s^3+2s\right)t^4 + \cdots .$$

*Remark* 6.5. Recently Theorem 4.4 has been cited and applied in the papers [**13, 34**] and closely related references therein.

*Remark* 6.6. We conjecture that the range of $k \in \mathbb{N}$ in Lemma 2.6 can be extended to $k \in \mathbb{R}$.

*Remark* 6.7. This paper is an extended and revised version of two preprints [**25, 41**].

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