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ON A FAMILY OF SPECIAL NUMBERS AND POLYNOMIALS ASSOCIATED WITH APOSTOL-TYPE NUMBERS AND POYNOMIALS AND COMBINATORIAL NUMBERS

Irem Kucukoglu and Yilmaz Simsek *

In this article, we examine a family of some special numbers and polynomials not only with their generating functions, but also with computation algorithms for these numbers and polynomials. By using these algorithms, we provide several values of these numbers and polynomials. Furthermore, some new identities, formulas and combinatorial sums are obtained by using relations derived from the functional equations of these generating functions. These identities and formulas include the Apostol-type numbers and polynomials, and also the Stirling numbers. Finally, we give further remarks and observations on the generating function including λ-Apostol-Daehee numbers, special numbers, and finite sums.

1. INTRODUCTION

In this paper, by using a computation formula including the Apostol-Bernoulli numbers and the Stirling numbers of the first kind, we define a new family of special numbers and polynomials associated with the Apostol-type numbers and polynomials. This new family provides more information about not only the Apostol-type numbers and polynomials, but also λ-Apostol-Daehee numbers and polynomials. Moreover, with the help of this aforementioned computation formula, computation algorithms are presented for calculating values of the numbers and the polynomials belonging to this new family.

*Corresponding author. Yilmaz Simsek
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In this paper, we first need to specify that $\mathbb{Z}$ and $\mathbb{C}$ corresponds to the set of integers and the set of complex numbers, respectively, and also $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Since the generating functions and their functional equations are used effectively in order to obtain results of this paper, we recall some definitions associated with the well-known special numbers and polynomials with their generating functions as follows:

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ are defined as follows

$$F_{\mathcal{B}}(t,x;\lambda) = \frac{te^{tx}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{t^n}{n!} \tag{1}$$

where $\lambda \in \mathbb{C}$; $|t| < 2\pi$ when $\lambda = 1$; $|t| < |\log \lambda|$ when $\lambda \neq 1$. For $x = 0$, yields the Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$ given by

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda)\frac{t^n}{n!} \tag{2}$$

so that a few values of the polynomials $\mathcal{B}_n(x;\lambda)$ and the numbers $\mathcal{B}_n(\lambda)$ are given by, respectively

$$\begin{aligned}
\mathcal{B}_0(x;\lambda) &= 0, \mathcal{B}_1(x;\lambda) = \frac{1}{\lambda - 1}, \mathcal{B}_2(x;\lambda) = \frac{2}{\lambda - 1}x - \frac{2\lambda}{(\lambda - 1)^2}, \\
\mathcal{B}_3(x;\lambda) &= \frac{3}{\lambda - 1}x^2 - \frac{6\lambda}{(\lambda - 1)^2}x + \frac{3\lambda(\lambda + 1)}{(\lambda - 1)^3}, \dots
\end{aligned}$$

and

$$\mathcal{B}_0(\lambda) = 0, \mathcal{B}_1(\lambda) = \frac{1}{\lambda - 1}, \mathcal{B}_2(\lambda) = -\frac{2\lambda}{(\lambda - 1)^2}, \mathcal{B}_3(\lambda) = \frac{3\lambda(\lambda + 1)}{(\lambda - 1)^3},$$

and so on (*cf.* [**1**], [**8**], [**10**], [**17**], [**23**]; and see also the references cited therein).

An explicit formula for the numbers $\mathcal{B}_n(\lambda)$ is given by the following combinatorial sum (*cf.* [**16**, Eq-(2.22)]):

$$\mathcal{B}_n(\lambda) = \frac{n}{\lambda - 1} \sum_{m=0}^{n-1} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\frac{\lambda}{\lambda - 1}\right)^m k^{n-1}. \tag{3}$$

In [**10**], Kim et al. modified the Apostol-Bernoulli polynomials by the λ-Bernoulli polynomials $\mathfrak{B}_n(x;\lambda)$ given as follows

$$F_{\mathfrak{B}}(t,x;\lambda) = \frac{\log \lambda + t}{\lambda e^t - 1}e^{tx} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x;\lambda)\frac{t^n}{n!}, \tag{4}$$

for $\lambda \in \mathbb{C}$ with assuming that $\log \lambda$ denotes the principal branch of the many-valued function $\log \lambda$ with the imaginary part $\mathrm{Im}(\log \lambda)$ constrained by

$$-\pi < \mathrm{Im}(\log \lambda) \leq \pi.$$

The λ-Stirling numbers are given by

$$\frac{(\lambda e^t-1)^m}{m!}=\sum_{n=0}^{\infty}S_2(n,m;\lambda)\frac{t^n}{n!} \tag{5}$$

which, for $\lambda=1$, reduces to the generating function for the Stirling numbers of the second kind $S_2(n,m)$ with

$$S_2(n,m)=S_2(n,m;1)$$

which satisfy the following relations: $S_2(0,0)=1$, $S_2(n,1)=S_2(n,n)=1$, $S_2(n,0)=0$ if $n>0$, and $S_2(n,m)=0$ if $m>n$. Note that these numbers are of applications in areas related to especially partition theory (*cf.* [**3**], [**4**], [**5**], [**6**], [**13**], [**15**], [**17**], [**19**], [**21**]; and see also the references cited therein).

Let

$$\chi:(\mathbb{Z}/d\mathbb{Z})^*\rightarrow\mathbb{C}\backslash\{0\}$$

where $(\mathbb{Z}/d\mathbb{Z})^*$ denote the unit group of reduced residue class modulo $d\in\mathbb{N}$ and

$$\chi(x+d)=\chi(x).$$

The function χ is called a Dirichlet character with conductor d. This function is also a group homomorphism (*cf.* [**2**]).

By using p-adic integral equation, the second author [**16**] defined the following generating functions for the family of special numbers $Y_{n,\chi}(\lambda,q)$ and polynomials $Y_{n,\chi}(z;\lambda,q)$ including the generalized Apostol-type numbers and polynomials attached to Dirichlet character χ, respectively

$$H(t;\lambda,q,\chi)=\frac{[2:q]\sum_{j=0}^{d-1}(-1)^j\chi(j)\,(\lambda q)^j\,(1+\lambda t)^j}{(\lambda q)^d\,(1+\lambda t)^d-1}=\sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}, \tag{6}$$

where

$$[x:q]=\begin{cases}\frac{1-q^x}{1-q}, & q\neq 1\\ x, & q=1\end{cases}$$

and

$$H(t,z;\lambda,q,\chi)=(1+\lambda t)^zH(t;\lambda,q,\chi)=\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}, \tag{7}$$

where $\lambda\in\mathbb{C}$.

It follows from (6) and (7) that

$$Y_{n,\chi}(\lambda,q)=Y_{n,\chi}(0;\lambda,q),$$

and note that the relation between the numbers $Y_{n,\chi}(\lambda,q)$ and the polynomials $Y_{n,\chi}(z;\lambda,q)$ is given as follows

$$Y_{n,\chi}(z;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(z)_{n-j}Y_{j,\chi}(\lambda,q), \tag{8}$$

where $n \in \mathbb{N}_0$ (*cf.* [**16**]). In the special case when q goes to 1 and $d = 1$, the polynomials $Y_{n,\chi}(z;\lambda,q)$ are reduced to the polynomials $Y_n(z;\lambda)$, given by the following generating functions:

$$F(t,z;\lambda) = \frac{2\,(1+\lambda t)^z}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(z;\lambda)\frac{t^n}{n!}, \tag{9}$$

(*cf.* [**16**], [**24**]).

Moreover, in the special case when q goes to 1, the numbers $Y_{n,\chi}(\lambda,q)$ are reduced to the numbers $Y_{n,\chi}(\lambda)$, given by the following generating functions

$$\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)\lambda^j(1+\lambda t)^j}{\lambda^d(1+\lambda t)^d-1} = \sum_{n=0}^{\infty} Y_{n,\chi}(\lambda)\frac{t^n}{n!},$$

which, for $d = 1$, yields the generating functions for the numbers $Y_n(\lambda)$ given as follows

$$F(t;\lambda) = \frac{2}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!} \tag{10}$$

so that a few of the numbers $Y_n(\lambda)$ are given as follows

$$\begin{aligned} Y_0(\lambda) &= \frac{2}{\lambda-1}, Y_1(\lambda) = -\frac{2\lambda^2}{(\lambda-1)^2}, \\ Y_2(\lambda) &= \frac{4\lambda^4}{(\lambda-1)^3}, Y_3(\lambda) = -\frac{12\lambda^6}{(\lambda-1)^4}, \ldots \end{aligned}$$

(*cf.* [**16**], [**24**]).

Remark 1 (Remarks and Observations)**.** *The numbers $Y_n(\lambda)$ are constructed by Simsek* [**16**]*. Recently, Srivastava et al.* [**24**] *gave various novel identities and relations including the numbers $Y_n(\lambda)$, the Apostol-Bernoulli numbers and polynomials, the Apostol-Euler numbers and polynomials, the Stirling numbers of the first kind. In* [**7**]*, Choi modified the numbers $Y_n(\lambda)$ as follows:*

$$\gamma_n := \frac{1}{2n!}Y_n(\lambda)$$

in order to give new identities related to the Apostol-type Daehee polynomials. On the other hand, by using the numbers $Y_n(\lambda)$ and the Stirling numbers of the second kind, the Apostol-Bernoulli numbers are given as follows (cf. [**16**]*):*

$$\mathcal{B}_n(\lambda) = \frac{n}{2}\sum_{m=0}^{n-1}\lambda^{-m}Y_m(\lambda)S_2\left(n-1,m\right).$$

Other relation between the numbers $Y_n(\lambda)$ and the Apostol-Bernoulli numbers is given as follows (cf. [**24**, Theorem 9], [**16**, Corollary 4]*):*

$$Y_n(\lambda) = 2\lambda^n \sum_{m=0}^{n} \frac{S_1(n,m)\,\mathcal{B}_{m+1}(\lambda)}{m+1}, \tag{11}$$

where $S_1(n,m)$ denotes the Stirling numbers of the first kind given by

$$\frac{(\log(1+t))^m}{m!} = \sum_{n=0}^{\infty} S_1(n,m)\frac{t^n}{n!} \tag{12}$$

so that these numbers satisfy the following relations: $S_1(0,0)=1$, $S_1(0,m)=0$ if $m>0$, $S_1(n,0)=0$ if $n>0$, and $S_1(n,m)=0$ if $m>n$. Note that the numbers $S_1(n,m)$ are of applications in areas related to permutations (cf. [**3**], [**5**], [**6**], [**23**]*; and see also the references cited therein). An explicit formula for the numbers $S_1(n,m)$ is given by the following combinatorial sum:*

$$S_1(n,m) = \sum_{r=0}^{n-m}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{n+r-1}{m-1}\binom{2n-m}{n-m-r}\frac{j^{n-m+r}}{r!}, \tag{13}$$

(cf. [**3**, p. 233, Theorem 4.66], [**5**, p. 291, Eq-(8.21)]*).*

On the other hand, replacing λ by $-\lambda$ in (11)*, another relation between the numbers $Y_n(\lambda)$ and the other special numbers such as the Apostol-Euler numbers are given as follows (cf.* [**24**, Theorem 10]*):*

$$Y_n(-\lambda) = (-1)^{n+1}\lambda^n \sum_{m=0}^{n} \mathcal{E}_m(\lambda) S_1(n,m) \tag{14}$$

where $\mathcal{E}_m(\lambda)$ denotes the Apostol-Euler numbers given by the following generating function

$$F_{\mathcal{E}}(t;\lambda) = \frac{2}{\lambda e^t+1} = \sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}$$

where $\lambda \in \mathbb{C}$; $|t|<\pi$ when $\lambda=1$; $|t|<|\log(-\lambda)|$ when $\lambda \neq 1$ (cf. [**11**], [**12**], [**9**], [**14**], [**22**], [**23**]*; and see also the references cited therein).*

The motivation of this paper, in the light of the above knowledge regarding to the numbers $Y_n(\lambda)$, is to not only define a new family of special numbers and polynomials associated with the Apostol-type numbers and polynomials with their computation algorithms, but also derive miscellaneous novel identities, relations and combinatorial sums including the Stirling numbers and Apostol-type numbers and polynomials.

A summary of this article is given in the following manner:

In Section 2, we define a new family of special numbers and polynomials associated with the Apostol-type numbers and polynomials. In Section 3, we give computation algorithm for these numbers and polynomials. By using these algorithms, we provide several values of these numbers and polynomials. In Section 4, by combining functional equation and generating function techniques, we give some results including not only these numbers and polynomials, but also the Stirling numbers, and Apostol-type numbers and polynomials. In Section 5, we observe generating function for the λ-Apostol-Daehee numbers and we give some remarks on this observation. Moreover, we provide an explicit formula for the λ-Apostol-Daehee numbers with their several values.

2. GENERATING FUNCTIONS FOR A NEW FAMILY OF SPECIAL NUMBERS AND POLYNOMIALS

In this section, we define a new family of special numbers and polynomials associated with the Apostol-type numbers and polynomials.

Let $\lambda q \neq 1$ with $\lambda, q \in \mathbb{C}$ and $d \in \mathbb{N}$. We set the following generating functions for a new family of special numbers denoted by $I_{m,d}(\lambda, q)$:

$$F_d(t; \lambda, q) = \frac{\log(1+\lambda t)}{(\lambda q)^d (1+\lambda t)^d - 1} = \sum_{m=0}^{\infty} I_{m,d}(\lambda, q) \frac{t^m}{m!}. \tag{15}$$

We also set a new family of special polynomials $I_{m,d}(x; \lambda, q)$ by the following generating function:

$$G_d(t, x; \lambda, q) = (1+\lambda t)^x F_d(t; \lambda, q) = \sum_{m=0}^{\infty} I_{m,d}(x; \lambda, q) \frac{t^m}{m!} \tag{16}$$

so that, obviously,

$$I_{m,d}(\lambda, q) = I_{m,d}(0; \lambda, q).$$

We now give computation formula for the numbers $I_{m,d}(\lambda, q)$ associated with the Apostol-type numbers and the Stirling numbers of the first kind. By using Eq-(2.2) in [**16**], we set

$$\frac{1}{(\lambda q)^d (1+\lambda t)^d - 1} = \frac{d\log(1+\lambda t)}{d\log(1+\lambda t)\left((\lambda q)^d e^{d\log(1+\lambda t)} - 1\right)}.$$

Combining the above equation with (2) yields

$$\frac{1}{(\lambda q)^d (1+\lambda t)^d - 1} = \frac{1}{\log(1+\lambda t)} \sum_{n=0}^{\infty} \mathcal{B}_n((\lambda q)^d) \frac{d^{n-1}(\log(1+\lambda t))^n}{n!}.$$

Therefore, combining (12) with the above equation yields

$$\frac{1}{(\lambda q)^d (1+\lambda t)^d - 1} = \frac{1}{\log(1+\lambda t)} \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} d^{n-1} \lambda^m \mathcal{B}_n((\lambda q)^d) S_1(m,n) \right) \frac{t^m}{m!}.$$

Combining the above equation with (15) yields

$$\sum_{m=0}^{\infty} I_{m,d}\left(\lambda, q\right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} d^{n-1} \lambda^m \mathcal{B}_n((\lambda q)^d) S_1\left(m, n\right) \right) \frac{t^m}{m!}.$$

Therefore, comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation yields a computation formula for the numbers $I_{m,d}\left(\lambda, q\right)$ by the following theorem:

Theorem 1. *Let $m \in \mathbb{N}_0$. Then we have*

$$I_{m,d}\left(\lambda, q\right) = \lambda^m \sum_{n=0}^{m} d^{n-1} \mathcal{B}_n((\lambda q)^d) S_1\left(m, n\right). \tag{17}$$

Remark 2. *By using the special case when q goes to 1 and $d = 1$ of (17), we set*

$$I_{m,1}\left(\lambda\right) = \lim_{q \to 1} I_{m,1}\left(\lambda, q\right).$$

Thus, for $\lambda \neq 1$, we have

$$I_{m,1}\left(\lambda\right) = \lambda^m \sum_{n=0}^{m} \mathcal{B}_n\left(\lambda\right) S_1\left(m, n\right). \tag{18}$$

Let $\left(x\right)_m$ denote the well-known falling factorial polynomial given by

$$\left(x\right)_m = x\left(x-1\right)\cdots\left(x-m+1\right) \qquad \left(x \in \mathbb{C}; m \in \mathbb{N}\right)$$

with $\left(x\right)_0 = 1$. Using (15) and (16) yields

$$\sum_{m=0}^{\infty} I_{m,d}\left(x; \lambda, q\right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(x\right)_m \lambda^m \frac{t^m}{m!} \sum_{m=0}^{\infty} I_{m,d}\left(\lambda, q\right) \frac{t^m}{m!}.$$

Using the Cauchy product in the above equation yields

$$\sum_{m=0}^{\infty} I_{m,d}\left(x; \lambda, q\right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} \left(x\right)_{m-k} \lambda^{m-k} I_{k,d}\left(\lambda, q\right) \right) \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation yields the following theorem:

Theorem 2. *Let $m \in \mathbb{N}_0$. Then we have*

$$I_{m,d}\left(x; \lambda, q\right) = \sum_{k=0}^{m} \binom{m}{k} \left(x\right)_{m-k} \lambda^{m-k} I_{k,d}\left(\lambda, q\right). \tag{19}$$

3. COMPUTATION ALGORITHMS FOR THE NUMBER $I_{m,d}(\lambda, q)$ AND THE POLYNOMIALS $I_{m,d}(x; \lambda, q)$

In this section, in order to compute values of the numbers $I_{m,d}(\lambda, q)$ and the polynomials $I_{m,d}(x; \lambda, q)$, we give some computation algorithms. For the purpose of computing these numbers and polynomials, equations (17) and (19) are used. Since equation (17) contains the Apostol-Bernoulli numbers and the Stirling numbers of the second kind, we firstly give computation algorithms including procedures for these numbers. Then, by using these procedures, we also give computation algorithms for the numbers $I_{m,d}(\lambda, q)$ and the polynomials $I_{m,d}(x; \lambda, q)$.

By Algorithm 1, we give a computation algorithm including `APOST_BERN_NUM` procudure for calculating values of the Apostol-Bernoulli numbers.

Algorithm 1 Let n be a nonnegative integer and $\lambda \in \mathbb{C} \setminus \{1\}$. By using the equation (3), this algorithm will return the n-th Apostol-Bernoulli number $\mathcal{B}_n(\lambda)$.

```
procedure APOST_BERN_NUM(n, λ)
    Global variable B ← 0
    Local variable m, k : integer
    if n = 0 then
        return 0
    else
        if n > 0 then
            for m = 0; m ≤ n − 1; m = m + 1 do
                for k = 0; k ≤ m; k = k + 1 do
                    B ← B + power (−1, k) ∗Binomial_Coef(m, k)
                        ↪ ∗power (λ/ (λ − 1) , m) ∗ power (k, n − 1)
                end for
            end for
            return (n/ (λ − 1)) ∗ B
        end if
    end if
end procedure
```

By Algorithm 2, we give a computation algorithm including `STIRLING_FIRST_NUM` procedure as a computation algorithm for calculating values of the Stirling numbers of the first kind.

Algorithm 2 Let n and m be a nonnegative integer. By using the equation (13), this algorithm will return the Stirling numbers of the first kind $S_1(n,m)$.

```
procedure Stirling_First_Num(n, k)
    Global variable S ← 0
    Local variable r, j : integer
    if n = 0 ∧ m = 0 then
        return 1
    else
        if m > 0 ∨ n > 0 ∨ m > n then
            return 0
        else
            for r = 0; r ≤ n − m; r = r + 1 do
                for j = 0; j ≤ r; j = j + 1 do
                    S ← S + power(−1, j) ∗ Binomial_Coef(r, j)
                        ↪ ∗Binomial_Coef(n + r − 1, m − 1)
                        ↪ ∗Binomial_Coef(2n − m, n − m − r)
                        ↪ ∗(power(j, n − m + r)/factorial(r))
                end for
            end for
            return S
        end if
    end if
end procedure
```

Remark 3. *Algorithm 1 and Algorithm 2 may have been given up to now with different programming languages or algorithms. However, these two procedures have been given in order to run the algorithm 3 which forms the basis of this section.*

Now, we are ready to give a computation algorithm including `APOST_TYPE_NUM_I` procedure as a computation algorithm for calculating values of the numbers $I_{m,d}(\lambda,q)$ by Algorithm 3.

Algorithm 3 Let $\lambda q \neq 1$ with $\lambda, q \in \mathbb{C}$ and $d \in \mathbb{N}$. By using (17), this algorithm will return the numbers $I_{m,d}(\lambda,q)$ with the help of `APOST_BERN_NUM` and `STIRLING_FIRST_NUM` procedures given by Algorithm 1 and Algorithm 2.

```
procedure Apost_Type_Num_I(m, d, λ, q)
    Global variable Inum ← 0
    Local variable n : integer
    for n = 0; n ≤ m; n = n + 1 do
        Inum ← Inum+power(d, n − 1) ∗APOSTOL_BERN_NUM(n, power(λq, d))
                    ↪ ∗STIRLING_FIRST_NUM(m, n)
    end for
    return power(λ, m) ∗ Inum
end procedure
```

Moreover, we also give a computation algorithm including `APOST_TYPE_POLY_I` procedure as a computation algorithm for calculating values of the polynomials $I_{m,d}(x;\lambda,q)$ by Algorithm 4.

Algorithm 4 Let $\lambda q \neq 1$ with $\lambda, q \in \mathbb{C}$ and $d \in \mathbb{N}$. By using (19), this algorithm will return the polynomials $I_{m,d}(x;\lambda,q)$ with the help of `APOST_TYPE_POLY_I` procedure given by Algorithm 3.

```
procedure APOST_TYPE_POLY_I(m, d, x, λ, q)
    Global variable Ipoly ← 0
    Local variable k : integer
    for k = 0;  k ≤ m;  k = k + 1 do
        Ipoly ← Ipoly + Binomial_Coef (m, k) ∗ Falling_Fact (x, m − k)
                    ↪ ∗power (λ, m − k) ∗APOST_TYPE_NUM_I(k, d, λ, q)
    end for
    return Ipoly
end procedure
```

By Algorithm 3, we compute a few values of the numbers $I_{m,d}(\lambda,q)$ as follows:

$$\begin{aligned}
I_{0,d}(\lambda,q) &= 0, I_{1,d}(\lambda,q) = \frac{\lambda}{(\lambda q)^{d}-1}, \\
I_{2,d}(\lambda,q) &= \frac{\lambda^{2}\left(1-(1+2d)(\lambda q)^{d}\right)}{\left((\lambda q)^{d}-1\right)^{2}}, \\
I_{3,d}(\lambda,q) &= \frac{\lambda^{3}\left(\left(2+6d+3d^{2}\right)(\lambda q)^{2d}+\left(-4-6d+3d^{2}\right)(\lambda q)^{d}+2\right)}{\left((\lambda q)^{d}-1\right)^{3}}.
\end{aligned}$$

Therefore, a few values of the numbers $I_{m,1}(\lambda)$ are given as follows:

$$I_{0,1}(\lambda)=0, I_{1,1}(\lambda)=\frac{\lambda}{\lambda-1}, I_{2,1}(\lambda)=\frac{\lambda^{2}(1-3\lambda)}{(\lambda-1)^{2}}, I_{3,1}(\lambda)=\frac{\lambda^{3}\left(11\lambda^{2}-7\lambda+2\right)}{(\lambda-1)^{3}}.$$

By using Algorithm 4, we also compute a few values of the polynomials $I_{m,d}(x;\lambda,q)$ as follows:

$$\begin{aligned}
I_{0,d}(x;\lambda,q) &= 0, \\
I_{1,d}(x;\lambda,q) &= \frac{\lambda}{(\lambda q)^{d}-1}, \\
I_{2,d}(x;\lambda,q) &= \frac{2\lambda^{2}}{(\lambda q)^{d}-1}x+\frac{\lambda^{2}\left(1-(1+2d)(\lambda q)^{d}\right)}{\left((\lambda q)^{d}-1\right)^{2}},
\end{aligned}$$

$$I_{3,d}\left(x;\lambda,q\right) = \frac{3\lambda^{3}}{\left(\lambda q\right)^{d}-1}x^{2}+\left(\frac{6\lambda^{3}\left(1-\left(1+d\right)\left(\lambda q\right)^{d}\right)}{\left(\left(\lambda q\right)^{d}-1\right)^{2}}\right)x$$
$$+\frac{\lambda^{3}\left(\left(2+6d+3d^{2}\right)\left(\lambda q\right)^{2d}+\left(-4-6d+3d^{2}\right)\left(\lambda q\right)^{d}+2\right)}{\left(\left(\lambda q\right)^{d}-1\right)^{3}}.$$

4. IDENTITIES AND RELATIONS

In this section, by using generating function and their functional equations technique, we give some identities, relations and combinatorial sums associated with the numbers $I_{m,d}\left(\lambda,q\right)$ and the polynomials $I_{m,d}\left(x;\lambda,q\right)$ including the Apostol-type numbers and polynomials, and the Stirling numbers.

Substituting $\lambda t=e^{z}-1$ into (16) yields

$$\frac{z}{\left(\lambda q\right)^{d}e^{zd}-1}e^{zx}=\sum_{m=0}^{\infty}I_{m,d}\left(x;\lambda,q\right)\frac{\left(e^{z}-1\right)^{m}}{\lambda^{m}m!}.$$

Combining left hand side of the above equation with (1) and right hand side of the above equation with (5) yields

$$\sum_{n=0}^{\infty}d^{n-1}\mathcal{B}_{n}\left(\frac{x}{d};\left(\lambda q\right)^{d}\right)\frac{z^{n}}{n!}=\sum_{m=0}^{\infty}\frac{I_{m,d}\left(x;\lambda,q\right)}{\lambda^{m}}\sum_{n=0}^{\infty}S_{2}\left(n,m\right)\frac{z^{n}}{n!}.$$

Since $S_{2}\left(n,m\right)=0$ when $m>n$, we have

$$\sum_{n=0}^{\infty}d^{n-1}\mathcal{B}_{n}\left(\frac{x}{d};\left(\lambda q\right)^{d}\right)\frac{z^{n}}{n!}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{I_{m,d}\left(x;\lambda,q\right)}{\lambda^{m}}S_{2}\left(n,m\right)\right)\frac{z^{n}}{n!}.$$

Comparing the coefficients of $\frac{z^{n}}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 3. *Let $n\in\mathbb{N}$. Then we have*

$$\mathcal{B}_{n}\left(\frac{x}{d};\left(\lambda q\right)^{d}\right)=\frac{1}{d^{n-1}}\sum_{m=0}^{n}\frac{I_{m,d}\left(x;\lambda,q\right)}{\lambda^{m}}S_{2}\left(n,m\right). \tag{20}$$

Substituting $t=e^{z}-\frac{1}{\lambda}$ into (16) yields the following functional equation

$$G_{d}\left(e^{z}-\frac{1}{\lambda},x;\lambda,q\right)=\lambda^{x}\left(F_{\mathfrak{B}}\left(z,x;\lambda\left(\lambda q\right)^{d}\right)-\frac{\log\left(\lambda q\right)^{d}}{z}F_{\mathcal{B}}\left(z,x;\lambda\left(\lambda q\right)^{d}\right)\right).$$

By combining the above functional equation with (16), (1), and (4), we have

$$\sum_{m=0}^{\infty} I_{m,d}\left(x;\lambda,q\right)\frac{\left(\lambda e^{z}-1\right)^{m}}{\lambda^{m}m!}=\lambda^{x}\sum_{n=0}^{\infty}\left(\mathfrak{B}_{n}(x;\lambda\left(\lambda q\right)^{d})-\frac{\log\left(\lambda q\right)^{d}}{n+1}\mathcal{B}_{n+1}(x;\lambda\left(\lambda q\right)^{d})\right)\frac{z^{n}}{n!}.$$

Combining left hand side of the above equation with (5) yields

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{I_{m,d}\left(x;\lambda,q\right)S_{2}\left(n,m;\lambda\right)}{\lambda^{m}}\frac{z^{n}}{n!}=\lambda^{x}\sum_{n=0}^{\infty}\left(\mathfrak{B}_{n}(x;\lambda\left(\lambda q\right)^{d})-\frac{\log\left(\lambda q\right)^{d}}{n+1}\mathcal{B}_{n+1}(x;\lambda\left(\lambda q\right)^{d})\right)\frac{z^{n}}{n!}.$$

By using some elementary calculations in the above equation, we arrive at the following theorem:

Theorem 4. *Let $n\in\mathbb{N}$. Then we have*

$$\sum_{m=0}^{\infty}\frac{I_{m,d}\left(x;\lambda,q\right)S_{2}\left(n,m;\lambda\right)}{\lambda^{m+x}}=\mathfrak{B}_{n}(x;\lambda\left(\lambda q\right)^{d})-\frac{\log\left(\lambda q\right)^{d}}{n+1}\mathcal{B}_{n+1}\left(x;\lambda\left(\lambda q\right)^{d}\right). \tag{21}$$

Since $S_2\left(n,m;1\right)=0$ when $m>n$, substituting $\lambda=1$ into (21) yields the following corollary:

Corollary 1. *Let $n\in\mathbb{N}$. Then we have*

$$\sum_{m=0}^{n} I_{m,d}\left(x;1,q\right)S_{2}\left(n,m\right)=\mathfrak{B}_{n}(x;q^{d})-\frac{d\log q}{n+1}\mathcal{B}_{n+1}(x;q^{d}).$$

From (6), we get the following functional equation:

$$H(t;\lambda,q,\chi)\log(1+\lambda t)=(1+q)\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}G_{d}\left(t,j;\lambda,q\right).$$

Thus,

$$\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}\left(\lambda t\right)^{m+1}}{m+1}\sum_{m=0}^{\infty}Y_{m,\chi}(\lambda,q)\frac{t^{m}}{m!}$$
$$=(1+q)\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}\sum_{m=0}^{\infty}I_{m,d}\left(j;\lambda,q\right)\frac{t^{m}}{m!}.$$

Therefore

$$\sum_{m=0}^{\infty}\left(m\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{\left(-1\right)^{k}\lambda^{k+1}k!}{(k+1)}Y_{m-k-1,\chi}(\lambda,q)\right)\frac{t^{m}}{m!}$$
$$=\sum_{m=0}^{\infty}\left(1+q\right)\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}I_{m,d}\left(j;\lambda,q\right)\frac{t^{m}}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation yields the following theorem:

Theorem 5. *Let $m \in \mathbb{N}$. Then we have*

$$m\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\frac{\lambda^{k+1}k!}{(k+1)}Y_{m-k-1,\chi}(\lambda,q) = (1+q)\sum_{j=0}^{d-1}(-1)^j\chi(j)\,(\lambda q)^j\,I_{m,d}\,(j;\lambda,q)\,. \tag{22}$$

Setting $\chi \equiv 1$ in (22) yields the following corollary:

Corollary 2. *Let $m \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{m-1}\frac{(-1)^k\,\lambda^{k+1}}{(k+1)\,(m-k-1)!}Y_{m-k-1}(\lambda,q) = \frac{1+q}{m!}I_{m,1}\,(\lambda,q)\,. \tag{23}$$

By using (10), we have

$$\sum_{m=0}^{\infty}I_{m,1}\,(\lambda)\,\frac{t^m}{m!} = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\,(\lambda t)^{m+1}}{m+1}\sum_{m=0}^{\infty}Y_m\,(\lambda)\,\frac{t^m}{m!}.$$

Using the Cauchy product in the above equation yields

$$\sum_{m=0}^{\infty}I_{m,1}\,(\lambda)\,\frac{t^m}{m!} = \frac{1}{2}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m-1}\frac{(-1)^k\,\lambda^{k+1}m!}{(k+1)\,(m-k-1)!}Y_{m-1-k}\,(\lambda)\right)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation yields the following theorem:

Theorem 6. *Let $m \in \mathbb{N}$. Then we have*

$$I_{m,1}\,(\lambda) = \frac{m!}{2}\sum_{k=0}^{m-1}\frac{(-1)^k\,\lambda^{k+1}}{(k+1)\,(m-k-1)!}Y_{m-k-1}\,(\lambda)\,. \tag{24}$$

Remark 4. *By substituting $q \to 1$ into (23), we also arrive at (24).*

Substituting (11) into (24) yields the following corollary:

Corollary 3. *Let $m \in \mathbb{N}$. Then we have*

$$I_{m,1}\,(\lambda) = m!\lambda^m\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1}\frac{(-1)^k}{(j+1)\,(k+1)\,(m-k-1)!}\mathcal{B}_{j+1}(\lambda)S_1\,(m-k-1,j)\,.$$

By replacing λ by $-\lambda$ in (14) and substituting final equation into (24) yields the following corollary:

Corollary 4. *Let $m \in \mathbb{N}$. Then we have*

$$I_{m,1}(\lambda) = \frac{m!\lambda^m}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{m-k-1} \frac{(-1)^{k+1}}{(k+1)(m-k-1)!} \mathcal{E}_j(-\lambda) S_1(m-k-1, j).$$

5. FURTHER REMARKS AND OBSERVATIONS ON THE GENERATING FUNCTION $G(t,\lambda)$ INCLUDING APOSTOL-TYPE NUMBERS

In this section, we give some survey and investigation on the generating function $G(t,\lambda)$ which has recently defined by second author [**18**], given as follows:

$$G(t,\lambda) = \frac{\log\lambda + \log(1+\lambda t)}{\lambda(1+\lambda t) - 1}.$$

What constructed function as above correspond to generating function for which type of numbers?

The main purpose of this section is to investigate the answer to this question.

In [**18**] and [**20**], the function $G(t,\lambda)$ are studied as generating functions which are related to the λ-Apostol-Daehee numbers $\mathcal{D}_n(\lambda)$. That is,

$$G(t,\lambda) = \sum_{n=0}^{\infty} \mathcal{D}_n(\lambda) \frac{t^n}{n!}.$$

Therefore, in order to give another explicit numbers for this generating function, combining (10) and (15) with the above equation, we have the following functional equation:

$$G(t,\lambda) = \frac{\log\lambda}{2} F(t;\lambda) + F_1(t;\lambda,1).$$

Thus, we have

$$\sum_{n=0}^{\infty} \mathcal{D}_n(\lambda) \frac{t^n}{n!} = \frac{\log\lambda}{2} \sum_{n=0}^{\infty} Y_n(\lambda) \frac{t^n}{n!} + \sum_{n=0}^{\infty} I_{n,1}(\lambda) \frac{t^n}{n!}.$$

Equating coefficients of the $\frac{t^n}{n!}$ on the both sides of the above equation yields the following theorem:

Theorem 7.

$$\mathcal{D}_n(\lambda) = \frac{\log\lambda}{2} Y_n(\lambda) + I_{n,1}(\lambda). \tag{25}$$

Combining the following explicit formula for the numbers $Y_n(\lambda)$:

$$Y_n(\lambda) = (-1)^n \frac{2n!}{\lambda - 1}\left(\frac{\lambda^2}{\lambda - 1}\right)^n$$

(*cf.* [**16**, Eq. (2.18)]) with (24) and (25), we obtain an explicit formula for the numbers $\mathcal{D}_n(\lambda)$ by the following theorem:

Theorem 8. *Let $n \in \mathbb{N}$. Then we have*

$$\mathcal{D}_n(\lambda) = n!\,(-1)^n \left(\frac{\lambda^2}{\lambda - 1}\right)^n \left(\frac{\log\lambda}{\lambda - 1} - \frac{1}{\lambda}\sum_{k=0}^{n-1}\frac{1}{k+1}\left(\frac{\lambda - 1}{\lambda}\right)^k\right). \tag{26}$$

By using formula in (26), we compute a few values of the number $\mathcal{D}_n(\lambda)$ as follows:

$$\begin{aligned}
\mathcal{D}_1(\lambda) &= -\frac{\lambda^2\log\lambda}{(\lambda-1)^2} + \frac{\lambda}{\lambda-1},\\
\mathcal{D}_2(\lambda) &= \frac{2\lambda^3\log\lambda}{(\lambda-1)^3} + \frac{\lambda^2(1-3\lambda)}{(\lambda-1)^2},\\
\mathcal{D}_3(\lambda) &= -\frac{6\lambda^6\log\lambda}{(\lambda-1)^4} + \frac{\lambda^3\left(11\lambda^2 - 7\lambda + 2\right)}{(\lambda-1)^3}, \ldots .
\end{aligned}$$

Remark 5. *From (26), we give the following the value of finite sum in terms of the numbers $\mathcal{D}_n(\lambda)$:*

$$\sum_{k=0}^{n-1}\frac{1}{k+1}\left(\frac{\lambda-1}{\lambda}\right)^k = \frac{(-1)^{n+1}\lambda\mathcal{D}_n(\lambda)}{n!}\left(\frac{\lambda-1}{\lambda^2}\right)^n + \frac{\lambda\log\lambda}{\lambda-1}.$$

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Irem Kucukoglu
Department of Engineering Fundamental Sciences,
Faculty of Engineering,
Alanya Alaaddin Keykubat University,
TR-07425 Antalya, Turkey.
E-mail: *irem.kucukoglu@alanya.edu.tr*

Yilmaz Simsek
Department of Mathematics,
Faculty of Science,
University of Akdeniz
TR-07058 Antalya, Turkey.
E-mail: *ysimsek@akdeniz.edu.tr*