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VARIATIONAL ANALYSIS FOR DIRICHLET IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS INVOLVING THE p-LAPLACIAN

Samad Mohseni Kolagar, Ghasem A. Afrouzi, John R. Graef[*]

By using variational methods and critical point theory, the authors establish the existence of infinitely many weak solutions for impulsive differential inclusions involving two parameters and the p-Laplacian and having Dirichlet boundary conditions.

1. INTRODUCTION

Because of its wide applicability in the modeling of many phenomena in various fields of physics, chemistry, biology, engineering and economics, the theory of fractional differential equations has recently been attracting increasing interest; see for instance the monographs of Hilfer [**19**], Kilbas *et al.* [**24**], Miller and Ross [**29**], Podlubny [**31**], Samko *et al.* [**33**], the papers [**1, 2, 4, 5, 6, 9, 25, 27, 35, 37**] and references therein. Particular applications of fractional differential equations to problems in fluid flow, electrical networks, control of dynamical systems, elasticity, and signal processing, for example, have appeared in the literature.

Impulsive boundary value problems for differential equations and inclusions have been intensively studied in recent years. Such problems appear in mathematical models with sudden changes in their states such as in population dynamics, pharmacology, optimal control, etc. [**26**]. Impulsive problems for fractional equations are often treated by topological methods such as in [**3, 7, 8, 23**]. The existence

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of solutions of impulsive problems is studied using variational methods and critical point theorems in [**10, 36, 39**]. The recent monograph [**17**] gives background and applications of differential inclusions.

In the present paper, motivated by the results in [**14, 18, 34, 38**] and employing an abstract critical point result (see Theorem 1 below), we obtain sufficient conditions to ensure the existence of infinitely many weak solutions to the problem (1); see Theorem 2 below. We refer to [**12**], in which related variational methods are used for non-homogeneous problems.

The aim of this article is to show the existence of infinitely many weak solutions to the two parameter impulsive fractional differential inclusion

$$(1)\quad \begin{cases} {}_xD_T^{\alpha}\phi_p\Big({}_0D_x^{\alpha}u(x)\Big) + \phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)), \text{ a.e. } x \in [0,T], \\ x \neq x_j, \\ \Delta({}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha}u))(x_j) = I_j(u(x_j)), \quad j = 1,2,\dots,m, \\ u(0) = u(T) = 0, \end{cases}$$

where $1/p < \alpha \leq 1$, $\lambda > 0$, $\mu \geq 0$, $T > 0$, $p > 1$, $\phi_p(x) = |x|^{p-2}x$, ${}_0D_x^{\alpha}$ and ${}_xD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives respectively, and $0 = x_0 < x_1 < x_2 < \ldots < x_m < x_{m+1} = T$. Here,

$$\Delta({}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha}u))(x_j) = {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^+) - {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^-),$$

where

$${}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^+) = \lim_{x\to x_j^+} {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x),$$

$${}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^-) = \lim_{x\to x_j^-} {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x),$$

and ${}_0^cD_x^{\alpha}$ is the left Caputo fractional derivative of order α. The multifunction F defined on $\mathbb{R}$ is assumed to satisfy

(F1) $F : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values;

(F2) $\min F$, $\max F : \mathbb{R} \to \mathbb{R}$ are Borel measurable;

(F3) $|\xi| \leq a(1 + |s|^{r-1})$ for all $s \in \mathbb{R}$, $\xi \in F(s)$, $r > 1$ $(a > 0)$.

Also, G is a multifunction defined on $[0,T] \times \mathbb{R}$, satisfying

(G1) $G(x,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values for a.e. $x \in [0,T] \setminus Q$, where $Q = \{x_i : i = 1,2,\dots,m\}$;

(G2) $\min G$, $\max G : ([0,T] \setminus Q) \times \mathbb{R} \to \mathbb{R}$ are Borel measurable;

(G3) $|\xi| \leq a(1 + |s|^{r-1})$ for a.e. $x \in [0,T]$, $s \in \mathbb{R}$, $\xi \in G(x,s)$, $r > 1$ $(a > 0)$.

2. BASIC DEFINITIONS AND PRELIMINARY RESULTS

Definition 1. ([**24, 31**]) Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ of u are defined by

$$_aD_t^\alpha u(t) := \frac{d^n}{dt^n}\,{}_aD_t^{\alpha-n}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)ds$$

and

$$_tD_b^\alpha u(t) := (-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{\alpha-n}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (t-s)^{n-\alpha-1}u(s)ds$$

for every $t \in [a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1 \le \alpha < n$ and $n \in \mathbb{N}$.
Here, $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1}e^{-z}dz.$$

Set $AC^n([a,b],\mathbb{R})$ to be the space of functions $u : [a,b] \to \mathbb{R}$ such that $u \in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)} \in AC([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings being $(n-1)$ times continuously differentiable on $[a,b]$. In particular, we denote $AC([a,b],\mathbb{R}) := AC^1([a,b],\mathbb{R})$.

Definition 2. ([**13, 16, 33**]) Let $\gamma \ge 0$ and $n \in \mathbb{N}$.

(i) If $\gamma \in (n-1,n)$ and $u \in AC^n([a,b],\mathbb{R})$, then the left and right Caputo fractional derivatives of order γ for the function u exist almost everywhere on $[a,b]$, and are given by

$$^c_aD_t^\gamma u(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-\gamma-1}u^{(n)}(s)ds$$

and

$$^c_tD_b^\gamma u(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b (s-t)^{n-\gamma-1}u^{(n)}(s)ds$$

for every $t \in [a,b]$, respectively.

(ii) If $\gamma = n-1$ and $u \in AC^{n-1}([a,b],\mathbb{R})$, then

$$^c_aD_t^{n-1}u(t) = u^{(n-1)}(t) \quad \text{and} \quad {}^c_tD_b^{n-1}u(t) = (-1)^{(n-1)}u^{(n-1)}(t)$$

for every $t \in [a,b]$.

With these definitions, we have the following formulas for fractional integration by parts and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator as proved in [**24**] and [**33**].

Proposition 1. *The following property of fractional integration*

$$\int_a^b [{}_aD_t^{-\gamma}u(t)]v(t)dt = \int_a^b [{}_tD_b^{-\gamma}v(t)]u(t)dt, \quad \gamma > 0$$

holds provided that $u \in L^p([a,b],\mathbb{R})$, $v \in L^q([a,b],\mathbb{R})$, and $p \geq 1$, $q \geq 1$, and $1/p + 1/q \leq 1 + \gamma$, or $p \neq 1$, $q \neq 1$, and $1/p + 1/q = 1 + \gamma$.

Proposition 2. *Let $n \in \mathbb{N}$ and $n-1 < \gamma \leq n$. If $u \in AC^n([a,b],\mathbb{R})$ or $u \in C^n([a,b],\mathbb{R})$, then*

$${}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}u(t)) = u(t) - \sum_{j=0}^{n-1} \frac{u^{(j)}(a)}{j!}(t-a)^j$$

and

$${}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}u(t)) = u(t) - \sum_{j=0}^{n-1} \frac{(-1)^j u^{(j)}(b)}{j!}(b-t)^j$$

for every $t \in [a,b]$. In particular, if $0 < \gamma \leq 1$ and $u \in AC([a,b],\mathbb{R})$ or $u \in C^1([a,b],\mathbb{R})$, then

$$(2) \qquad {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}u(t)) = u(t) - u(a) \quad \textit{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}u(t)) = u(t) - u(b).$$

Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $\varphi : X \to \mathbb{R}$ is called *locally Lipschitz* if, for all $u \in X$, there exist a neighborhood U of u and a real number $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq L\|v - w\|_X \quad \text{for all } v, w \in U.$$

If φ is locally Lipschitz and $u \in X$, the *generalized directional derivative* of φ at u along the direction $v \in X$ is defined by

$$\varphi^\circ(u; v) := \limsup_{w \to u,\, \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

The *generalized gradient* of φ at u is the set

$$\partial\varphi(u) := \{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

Thus, $\partial\varphi : X \to 2^{X^*}$ is a multifunction. We say that φ has a *compact gradient* if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

Lemma 1 ([**30**, Proposition 1.1])**.** *Let $\varphi \in C^1(X)$ be a functional. Then φ is locally Lipschitz and*

$$\varphi^\circ(u; v) = \langle\varphi'(u), v\rangle \quad \textit{for all } u, v \in X;$$
$$\partial\varphi(u) = \{\varphi'(u)\} \quad \textit{for all } u \in X.$$

Lemma 2 ([**30**, Proposition 1.3])**.** *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. Then $\varphi^\circ(u;\cdot)$ is subadditive and positively homogeneous for all $u \in X$, and*

$$\varphi^\circ(u;v) \le L\|v\| \quad \text{for all } u, v \in X,$$

with $L > 0$ being a Lipschitz constant for φ around u.

Lemma 3 ([**15**])**.** *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. Then $\varphi^\circ : X \times X \to \mathbb{R}$ is upper semicontinuous, and for all $\lambda \ge 0$ and u, $v \in X$,*

$$(\lambda\varphi)^\circ(u;v) = \lambda\varphi^\circ(u;v).$$

Moreover, if φ, $\psi : X \to \mathbb{R}$ are locally Lipschitz functionals, then

$$(\varphi+\psi)^\circ(u;v) \le \varphi^\circ(u;v) + \psi^\circ(u;v) \quad \text{for all } u,\ v \in X.$$

Lemma 4 ([**30**, Proposition 1.6])**.** *Let φ, $\psi : X \to \mathbb{R}$ be locally Lipschitz functionals. Then*

$$\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u) \quad \text{for all } u \in X,\ \lambda \in \mathbb{R},$$
$$\partial(\varphi+\psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u) \quad \text{for all } u \in X.$$

Lemma 5 ([**20**, Proposition 1.6])**.** *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional with a compact gradient. Then φ is sequentially weakly continuous.*

We say that $u \in X$ is a *(generalized) critical point* of a locally Lipschitz functional φ if $0 \in \partial\varphi(u)$; i.e.,

$$\varphi^\circ(u;v) \ge 0 \quad \text{for all } v \in X.$$

If a non-smooth functional $g : X \to (-\infty,+\infty)$ is expressed as a sum of a locally Lipschitz function, $\varphi : X \to \mathbb{R}$, and a convex, proper, and lower semicontinuous function, $j : X \to (-\infty,+\infty)$, that is, $g := \varphi + j$, a (generalized) critical point of g is every $u \in X$ such that

$$\varphi^\circ(u;v-u) + j(v) - j(u) \ge 0$$

for all $v \in X$ (see [**30**, Chapter 3]).

Hereafter, we assume that X is a reflexive real Banach space, $\mathcal{N} : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive parameter, $j : X \to (-\infty,+\infty)$ is a convex, proper, and lower semicontinuous functional, and $D(j)$ is the effective domain of j. Let

$$\mathcal{M} := \Upsilon - j, \quad I_\lambda := \mathcal{N} - \lambda\mathcal{M} = (\mathcal{N} - \lambda\Upsilon) + \lambda j.$$

We also assume that $\mathcal{N}$ is coercive and

$$D(j) \cap \mathcal{N}^{-1}((-\infty,r)) \neq \emptyset \tag{3}$$

for all $r > \inf_X \mathcal{N}$. Moreover, owing to (3) and provided $r > \inf_X \mathcal{N}$, we can define

$$\varphi(r) := \inf_{u\in\mathcal{N}^{-1}((-\infty,r))} \frac{\left(\sup_{v\in\mathcal{N}^{-1}((-\infty,r))} \mathcal{M}(v)\right) - \mathcal{M}(u)}{r - \mathcal{N}(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \mathcal{N})^+} \varphi(r).$$

If $\mathcal{N}$ and Υ are locally Lipschitz functionals, the following result is proved in [**11**, Theorem 2.1]; it is a more precise version of [**28**, Theorem 1.1] (also see [**32**]).

Theorem 1. *Under the above assumptions on X, $\mathcal{N}$ and $\mathcal{M}$, we have:*

(a) For every $r > \inf_X \mathcal{N}$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda = \mathcal{N} - \lambda\mathcal{M}$ to $\mathcal{N}^{-1}((-\infty, r))$ admits a global minimum that is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b1) I_λ possesses a global minimum, or

(b2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \mathcal{N}(u_n) = +\infty$.

(c) If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either

(c1) there is a global minimum of $\mathcal{N}$ that is a local minimum of I_λ, or

(c2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n\to+\infty} \mathcal{N}(u_n) = \inf_X \mathcal{N}$, that converges weakly to a global minimum of $\mathcal{N}$.

To establish a variational structure for the main problem, it is necessary to construct appropriate function spaces. Following [**22**], denote by $C_0^\infty([0,T],\mathbb{R})$ the set of all functions $g \in C^\infty([0,T],\mathbb{R})$ with $g(0) = g(T) = 0$.

Definition 3. Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure with respect to the weighted norm

$$\|u\|_{\alpha,p} := \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt + \int_0^T |u(t)|^p dt \right)^{1/p}, \text{ for all } u \in E_0^{\alpha,p}. \tag{4}$$

Clearly, the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p[0,T]$ having an α-order fractional derivative ${}_0D_t^\alpha u \in L^p[0,T]$ and satisfying $u(0) = u(T) = 0$. From [**22**, Proposition 3.1], we know for $0 < \alpha \le 1$, the space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

For every $u \in E_0^\alpha$, set

$$\|u\|_{L^s} := \left(\int_0^T |u_i(t)|^s dt \right)^{1/s}, \qquad s \ge 1,$$

and

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)|.$$

Remark 1. For any $u \in E_0^{\alpha,p}$ according to (2), and in view of the fact that $u(0) = u(T) = 0$, we have ${}_0D_t^\alpha u(t) = {}_0^cD_t^\alpha u(t)$ and ${}_tD_T^\alpha u(t) = {}_t^cD_T^\alpha u(t)$ for $t \in [0,T]$.

Lemma 6 ([**22**])**.** *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\Big)^{1/p}.$$

Moreover, If $\alpha > \frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}\|u\|_\infty &\le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\Big)^{1/p}\\ &= \frac{1}{\Gamma(\alpha)}\frac{T^{\alpha-1/p}}{(\frac{(\alpha-1)p}{p-1}+1)^{\frac{p-1}{p}}}\Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\Big)^{1/p}.\end{aligned} \tag{5}$$

As a consequence of this lemma, we can consider $E_0^{\alpha,p}$ with the norm

$$\|u\|_{\alpha,p} := \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{1/p}, \quad u \in E_0^{\alpha,p},$$

which is equivalent to (4).

Lemma 7 ([**22**])**.** *Assume that $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_n\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_n \rightharpoonup u$. Then $\{u_n\}$ converges strongly to u in $C([0,T]),\mathbb{R})$, i.e., $\|u_n - u\|_\infty \to 0$, as $n \to \infty$.*

For $v \in E_0^{\alpha,p}$, by Remark 1 and Definition 1 we have

$$\begin{aligned}\int_0^T [{}_xD_T^\alpha \phi_p({}_0D_x^\alpha u(x))]v(x)dx &= \int_0^T [{}_xD_T^\alpha \phi_p({}_0^cD_x^\alpha u(x))]v(x)dx\\ &= -\int_0^T v(x)d[{}_xD_T^{\alpha-1}\phi_p({}_0^cD_x^\alpha u(x))]\\ &= \int_0^T [{}_xD_T^{\alpha-1}\phi_p({}_0^cD_x^\alpha u(x))]v'(x)dx.\end{aligned}$$

Thus, from Proposition 1 and Definition 2 we have

$$\begin{aligned}\int_0^T [{}_xD_T^\alpha \phi_p({}_0D_x^\alpha u(x))]v(x)dx &= \int_0^T \phi_p({}_0^cD_x^\alpha u(x)){}_0D_x^{\alpha-1}v'(x)dx\\ &= \int_0^T \phi_p({}_0^cD_x^\alpha u(x)){}_0^cD_x^\alpha v(x)dx.\end{aligned}$$

So we can define the weak solutions of FBVP (1) as follows.

Definition 4. A function $u \in X$ is a weak solution of the problem (1) if there exists $u^* \in L^p([0,T])$ (for some $p > 1$) such that

$$\int_0^T [\phi_p({}_0^c D_x^\alpha u(x)){}_0^c D_x^\alpha v(x) + \phi_p(u(x))v(x) - u^*(x)v(x)]\,dx + \sum_{i=1}^m I_i(u(x_i))v(x_i) = 0$$

for all $v \in X$ and $u^* \in \lambda F(u(x)) + \mu G(x, u(x))$ for a.e. $x \in [0,T]$.

For a.e. $x \in [0,T]$ and all $s \in \mathbb{R}$, we introduce the Aumann-type set-valued integral

$$\int_0^s F(t)dt = \Big\{ \int_0^s f(t)dt \,:\, f : \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } F \Big\},$$

and we set $\mathcal{F}(u) = \int_0^T \min \int_0^u F(s)\,ds\,dx$ for all $u \in L^p([0,T])$. We also have the Aumann-type set-valued integral

$$\int_0^s G(x,t)dt = \Big\{ \int_0^s g(x,t)dt \,:\, g : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } G \Big\}$$

and set $\mathcal{G}(u) = \int_0^T \min \int_0^u G(x,s)\,ds\,dx$ for all $u \in L^p([0,T])$.

Lemma 8 ([**21**, Lemma 3.1])**.** *The functionals $\mathcal{F}, \mathcal{G} : L^p([0,T]) \to \mathbb{R}$ are well defined and Lipschitz on any bounded subset of $L^p([0,T])$. Moreover, for all $u \in L^p([0,T])$ and all $u^* \in \partial(\mathcal{F}(u) + \mathcal{G}(u))$,*

$$u^*(x) \in F(u(x)) + G(x, u(x)) \quad \textit{for a.e. } x \in [0,T].$$

We define an energy functional for the problem (1) by setting

$$I_\lambda(u) = \frac{1}{p}\|u\|_\alpha^p - \lambda\mathcal{F}(u) - \mu\mathcal{G}(u) + \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds$$

for all $u \in X$.

Lemma 9 ([**34**, Lemma 4.4])**.** *The functional $I_\lambda : X \to \mathbb{R}$ is locally Lipschitz. Moreover, for each critical point $u \in X$ of I_λ, u is a weak solution of* (1).

3. MAIN RESULTS

We are now ready to formulate our main result using the following assumptions:

(F4)

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}$$
$$< \frac{1}{pc}\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x}F(s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^m\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)}ds};$$

(I1) $I_i(0)=0$, $I_i(s)s>0$, $s\in\mathbb{R}$, $i=1,2,\dots,m$.

(A1) $\frac{1}{p}<\alpha\le 1$.

We define

$$c:=\frac{1}{(\Gamma(\alpha))^p}\Big(\frac{T^{\alpha p-1}}{(\frac{(\alpha-1)p}{p-1}+1)^{p-1}}\Big)$$

and

$$w_\alpha:=(\frac{2}{T})^p\Big(\int_0^{T/2}x^{p(1-\alpha)}dx+\int_{T/2}^{T}(x^{1-\alpha}-2(x-T/2)^{1-\alpha})^p dx\Big).$$

Theorem 2. *Assume that* (F1)–(F4), (I1) and (A1) *hold. Let*

$$\lambda_1:=1\Big/\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x}F(s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^m\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)}I_i(s)ds},$$

$$\lambda_2:=1\Big/\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\frac{1}{pc}\xi^p}.$$

Then, for every $\lambda\in(\lambda_1,\lambda_2)$, and every multifunction G satisfying (G1)–(G3), *and*

(G4) $\int_0^T\min\int_0^t G(x,s)\,ds\,dx\ge 0$ for all $t\in\mathbb{R}$, and

(G5) $G_\infty:=\lim_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t G(x,s)ds}{\xi^p}<+\infty$,

if we put

$$\mu_{G,\lambda}:=\frac{1}{pcG_\infty}\big(1-\lambda pc\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}\big),$$

where $\mu_{G,\lambda}=+\infty$ when $G_\infty=0$, problem (1) *admits an unbounded sequence of weak solutions for every $\mu\in[0,\mu_{G,\lambda})$ in X.*

Proof. Our aim is to apply Theorem 1(b) to (1). To this end, we fix $\overline{\lambda} \in (\lambda_1, \lambda_2)$ and let G be a multifunction satisfying (G1)–(G5). Since $\overline{\lambda} < \lambda_2$, we have

$$\mu_{G,\lambda} := \frac{1}{pcG_\infty}\big(1 - \lambda pc \liminf_{\xi \to +\infty} \frac{\sup_{|t| \le \xi} \min \int_0^t F(s)ds}{\xi^p}\big) > 0.$$

Now fix $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$, set $\nu_1 := \lambda_1$, and

$$\nu_2 := \frac{\lambda_2}{1 + pc\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2\, G_\infty}.$$

If $G_\infty = 0$, then $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$ and $\overline{\lambda} \in (\nu_1, \nu_2)$. If $G_\infty \neq 0$, since $\overline{\mu} < \mu_{G,\overline{\lambda}}$, we have

$$\frac{\overline{\lambda}}{\lambda_2} + pc\overline{\mu}\, G_\infty < 1,$$

and so

$$\frac{\lambda_2}{1 + pc\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2\, G_\infty} > \overline{\lambda},$$

namely, $\overline{\lambda} < \nu_2$. Hence, taking into account that $\overline{\lambda} > \lambda_1 = \nu_1$, we have $\overline{\lambda} \in (\nu_1, \nu_2)$. Now, set

$$J(x, s) := F(s) + \frac{\overline{\mu}}{\overline{\lambda}} G(x, s)$$

for all $(x, s) \in [0, T] \times \mathbb{R}$. Assume j is identically zero in X, and for each $u \in X$, take

$$\mathcal{N}(u) := \frac{1}{p}\|u\|_\alpha^p + \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds, \quad \Upsilon(u) := \int_0^T \min \int_0^u J(x, s)\, ds\, dx,$$
$$\mathcal{M}(u) := \Upsilon(u) - j(u) = \Upsilon(u),$$

and

$$I_{\overline{\lambda}}(u) := \mathcal{N}(u) - \overline{\lambda}\mathcal{M}(u) = \mathcal{N}(u) - \overline{\lambda}\Upsilon(u).$$

It is a simple matter to verify that $\mathcal{N}$ is sequentially weakly lower semicontinuous on X. Clearly, $\mathcal{N} \in C^1(X)$. By Lemma 1, $\mathcal{N}$ is locally Lipschitz on X. By Lemma 8, $\mathcal{F}$ and $\mathcal{G}$ are locally Lipschitz on $L^p([0, T])$. So, Υ is locally Lipschitz on $L^p([0, T])$. Moreover, X is compactly embedded into $L^p([0, T])$, so Υ is locally Lipschitz on X. Furthermore, Υ is sequentially weakly upper semicontinuous. For all $u \in X$, by (I1),

$$\int_0^{u(x_i)} I_i(s)ds > 0, \quad i = 1, 2, \ldots, m.$$

Hence, we have

$$\mathcal{N}(u) = \frac{1}{p}\|u\|_{\alpha,p}^p + \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds > \frac{1}{p}\|u\|_{\alpha,p}^p$$

for all $u \in X$. Hence, $\mathcal{N}$ is coercive and $\inf_X \mathcal{N} = \mathcal{N}(0) = 0$.

We want to show that, under our hypotheses, there exists a sequence $\{\overline{u}_n\} \subset X$ of critical points for the functional $I_{\overline{\lambda}}$, that is, every element $\overline{u}_n$ satisfies

$$I^{\circ}_{\overline{\lambda}}(\overline{u}_n, v - \overline{u}_n) \geq 0, \quad \text{for every } v \in X.$$

Now, we claim that $\gamma < +\infty$. To see this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\sup_{|t|\leq\xi_n} \min \int_0^t J(x,s)ds}{\xi_n^p} = \liminf_{\xi\to+\infty} \frac{\sup_{|t|\leq\xi} \min \int_0^t J(x,s)ds}{\xi^p}.$$

Take

$$r_n := \frac{1}{pc}\xi_n^p, \quad \text{for all } n \in \mathbb{N}.$$

Then, for all $v \in X$ with $\mathcal{N}(v) < r_n$, taking into account that $\|v\|^p_{\alpha,p} < pr_n$ and (5), one has $|v(x)| \leq \xi_n$ for every $x \in [0,T]$. Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\mathcal{N}^{-1}((-\infty,r))} \frac{\big(\sup_{v\in\mathcal{N}^{-1}((-\infty,r))} \mathcal{M}(v)\big) - \mathcal{M}(u)}{r - \mathcal{N}(u)} \\
&\leq \frac{\sup_{\|v\|^p_{\alpha,p}<pr_n} \big(\mathcal{F}(v) + \frac{\overline{\mu}}{\overline{\lambda}}\mathcal{G}(v)\big)}{r_n} \\
&\leq \frac{\sup_{|t|\leq\xi_n} \big(\int_0^T \min \int_0^t F(s)\,ds\,dx + \frac{\overline{\mu}}{\overline{\lambda}} \int_0^T \min \int_0^t G(x,s)\,ds\,dx\big)}{r_n} \\
&\leq pc\Big[\frac{\sup_{|t|\leq\xi_n} \min \int_0^t F(s)ds}{\xi_n^p} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\sup_{|t|\leq\xi_n} \min \int_0^t G(x,s)ds}{\xi_n^p}\Big].
\end{aligned}$$

Moreover, from conditions (F4) and (G5), we have

$$\lim_{n\to+\infty} \frac{\sup_{|t|\leq\xi_n} \min \int_0^t F(s)ds}{\xi_n^p} + \lim_{n\to+\infty} \frac{\overline{\mu}}{\overline{\lambda}} \frac{\sup_{|t|\leq\xi_n} \min \int_0^t G(x,s)ds}{\xi_n^p} < +\infty,$$

which implies

$$\lim_{n\to+\infty} \frac{\sup_{|t|\leq\xi_n} \min \int_0^t J(x,s)ds}{\xi_n^p} < +\infty.$$

Therefore,

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq pc \liminf_{\xi\to+\infty} \frac{\sup_{|t|\leq\xi} \min \int_0^t J(x,s)ds}{\xi^p} < +\infty. \tag{6}$$

Since

$$\frac{\sup_{|t|\leq\xi} \min \int_0^t J(x,s)ds}{\xi^p} \leq \frac{\sup_{|t|\leq\xi} \min \int_0^t F(s)ds}{\xi^p} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\sup_{|t|\leq\xi} \min \int_0^t G(x,s)ds}{\xi^p},$$

and taking (G5) into account, we get

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t J(x,s)ds}{\xi^p}\le\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}+\frac{\overline{\mu}}{\overline{\lambda}}G_\infty. \tag{7}$$

Moreover, from Assumption (G4) we obtain

$$\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x}J(x,s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)}J(x,s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^m\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)}I_i(s)ds}$$

$$\ge\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x}F(s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^m\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)}I_i(s)ds}. \tag{8}$$

Therefore, from (7) and (8), we observe that

$$\begin{aligned}\overline{\lambda}\in(\nu_1,\nu_2)\subseteq&\left(\frac{1}{\limsup\limits_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x}F(s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^m\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)}I_i(s)ds}},\right.\\&\left.\frac{1}{\liminf\limits_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t J(x,s)ds}{\frac{1}{pc}\xi^p}}\right)\\&\subseteq(0,1/\gamma).\end{aligned}$$

For the fixed $\overline{\lambda}$, the inequality (6) ensures that the condition (b) of Theorem 1 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1) such that $\lim_{n\to\infty}\|u_n\|=+\infty$.

The next step is to show that for the fixed $\overline{\lambda}$, the functional $I_{\overline{\lambda}}$ has no global minimum. To do this, we first show that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\begin{aligned}\frac{1}{\overline{\lambda}}&<\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x}F(s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^m\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)}I_i(s)ds}\\&\le\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x}J(x,s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)}J(x,s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^m\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)}I_i(s)ds},\end{aligned}$$

there exists a sequence $\{\xi_n\}$ of positive numbers and a constant τ such that $\lim_{n\to+\infty}\xi_n = +\infty$ and

(9)
$$\frac{1}{\overline{\lambda}} < \tau < \frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\xi_n}{T}x} J(x,s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\xi_n}{T}(T-x)} J(x,s)\,ds\,dx}{\frac{1}{p}\xi_n^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi_n}{T}(T-x_i)} I_i(s)ds}$$

for each sufficiently large $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, set

$$w_n(x) = \begin{cases} \frac{2\Gamma(2-\alpha)\xi_n}{T}x, & x \in [0, \frac{T}{2}), \\ \frac{2\Gamma(2-\alpha)\xi_n}{T}(T-x), & x \in (\frac{T}{2}, T]. \end{cases}$$

Clearly $w_n(0) = w_n(T) = 0$ and $w_n \in L^P([0,T])$. A direct calculation shows that

$${}_0^cD_x^\alpha w_n(x) = \begin{cases} \frac{2\xi_n}{T}x^{1-\alpha}, & x \in [0, \frac{T}{2}), \\ \frac{2\xi_n}{T}(x^{1-\alpha} - 2(x-T/2)^{1-\alpha}), & x \in (\frac{T}{2}, T] \end{cases}$$

Furthermore,

$$\begin{aligned} \int_0^T |{}_0^cD_t^\alpha w_n(x)|^p dt &= \int_0^{T/2} (|{}_0^cD_x^\alpha w_n(x)|^p dt + \int_{T/2}^T (|{}_0^cD_x^\alpha w_n(x)|^p dt \\ &= (\frac{2\xi_n}{T})^p \left\{ \int_0^{T/2} x^{p(1-\alpha)} dt + \int_{T/2}^T (x^{1-\alpha} - 2(x-T/2)^{1-\alpha})^p dt \right\}. \\ &= w_\alpha \xi_n^p. \end{aligned}$$

Thus, $w_n \in X$, in particular,

$$\|w_n\|_{\alpha,p}^p = \int_0^T |{}_0^cD_t^\alpha w_n(x)|^p dt = w_\alpha \xi_n^p,$$

and so

$$\mathcal{N}(w_n) = \frac{1}{p}\xi_n^p w_\alpha + \sum_{i=1}^m \int_0^{w_n(x_i)} I_i(s)ds. \tag{10}$$

By (9) and (10), we see that

$$\begin{aligned} I_{\overline{\lambda}}(w_n) &= \mathcal{N}(w_n) - \overline{\lambda}\mathcal{M}(w_n) \\ &= \frac{1}{p}\xi_n^p w_\alpha + \sum_{i=1}^m \int_0^{w_n(x_i)} I_i(s)ds \\ &\quad - \overline{\lambda}\Big(\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\xi_n}{T}x} J(x,s)\,ds\,dx + \int_{\frac{T}{2}}^T \min \int_0^{\frac{2\Gamma(2-\alpha)\xi_n}{T}(T-x)} J(x,s)\,ds\,dx \Big) \\ &< \Big(\frac{1}{p}\xi_n^p w_\alpha + \sum_{i=1}^m \int_0^{w_n(x_i)} I_i(s)ds \Big)(1 - \overline{\lambda}\tau) \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\overline{\lambda}\tau > 1$ and $\lim_{n\to+\infty} \xi_n = +\infty$, we have

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n) = -\infty.$$

Thus, the functional $I_{\overline{\lambda}}$ is unbounded from below, and so it has no global minimum. Therefore, from part (b2) of Theorem 1, the functional $I_{\overline{\lambda}}$ admits a sequence of critical points $\{\overline{u}_n\} \subset E_0^{\alpha,p}$ such that $\lim_{n\to+\infty} \mathcal{N}(\overline{u}_n) = +\infty$. Since $\mathcal{N}$ is bounded on bounded sets, $\{\overline{u}_n\}$ has to be unbounded, i.e.,

$$\lim_{n\to+\infty} \|\overline{u}_n\|_{\alpha,p} = +\infty.$$

Moreover, if $\overline{u}_n \in E_0^{\alpha,p}$ is a critical point of $I_{\overline{\lambda}}$, clearly, by definition,

$$I^{\circ}_{\overline{\lambda}}(\overline{u}_n, v - \overline{u}_n) \geq 0, \quad \text{for every } v \in E_0^{\alpha,p}.$$

Finally, by Lemma 9, the critical points of $I_{\overline{\lambda}}$ are weak solutions of the problem (1) and this proves the theorem. $\square$

Remark 2. Under the conditions

$$\liminf_{\xi\to+\infty} \frac{\sup_{|t|\leq\xi} \min \int_0^t F(s)ds}{\xi^p} = 0$$

and

$$\limsup_{\xi\to+\infty} \frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds} = +\infty,$$

from Theorem 2, we see that for every $\lambda > 0$ and for each $\mu \in \big[0, \frac{1}{pcG_\infty}\big)$, problem (1) admits a sequence of weak solutions that is unbounded in X. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

The following result is a special case of Theorem 2 with $\mu = 0$.

Theorem 3. *Assume that* (F1)–(F4), (I1) and (A1) *hold. Then, for each*

$$\lambda \in \left(\frac{1}{\limsup_{\xi\to+\infty} \frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds}}, \right.$$
$$\left. \frac{1}{\liminf_{\xi\to+\infty} \frac{\sup_{|t|\leq\xi} \min \int_0^t J(x,s)ds}{\frac{1}{pc}\xi^p}} \right),$$

the problem

$$\begin{cases} {}_xD_T^{\alpha}\phi_p\Big({}_0D_x^{\alpha}u(x)\Big) + \phi_p(u(x)) \in \lambda F(u(x)), & a.e.\ x \in [0,T],\\ x \neq x_j, & \\ \Delta({}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha}u))(x_j) = I_j(u(x_j)), & j = 1,2,\dots,m,\\ u(0) = u(T) = 0 & \end{cases}$$

has an unbounded sequence of weak solutions in X.

Example 1. We choose $p = 4$, $\alpha = 0.8$, $x_1 = \frac{1}{2}$, $x_2 = \frac{7}{4}$ and $I_1(s) = I_2(s) = 2s$, for all $s \in \mathbb{R}$. Consider the problem

$$\begin{cases} {}_xD_2^{0.8}\phi_4\Big({}_0D_x^{0.8}u(x)\Big) + \phi_4(u(x)) \in \lambda F(u(x)), & \text{a.e. } x \in [0,2],\\ x \neq x_j, & \\ \Delta({}_xD_2^{-0.2}({}_0^cD_x^{0.8}))(x_j) = I_j(u(x_j)), & j = 1,2,\\ u(0) = u(2) = 0 & \end{cases} \tag{11}$$

where, for $s \in \mathbb{R}$,

$$F(s) = \begin{cases} \{0\}, & \text{if } s < 2^{-1/3},\\ [0,1], & \text{if } s = 2^{-1/3},\\ \{s - 2^{-1/3} + 1\}, & \text{if } s > 2^{-1/3}. \end{cases}$$

Simple calculations show that $c \simeq 1.8779$,

$$\sup_{|t| \le 2^{-1/3}} \min \int_0^t F(s)ds = 0,$$

and

$$\begin{aligned} &\frac{\int_0^1 \min \int_0^{\Gamma(2-0.8)\xi x} F(s)\,ds\,dx + \int_1^2 \min \int_0^{\Gamma(2-0.8)\xi(2-x)} F(s)\,ds\,dx}{\frac{1}{4c}\xi^4 w_{0.8} + \sum_{i=1}^2 \int_0^{\Gamma(2-0.8)\xi(2-x_i)} I_i(s)ds}\\ &\simeq \frac{\int_0^1 \min \int_0^{0.9182\xi x} F(s)\,ds\,dx + \int_1^2 \min \int_0^{0.9182\xi(2-x)} F(s)\,ds\,dx}{0.2835\xi^4 w_{0.8} + \int_0^{0.4591\xi} 2sds + \int_0^{0.2296\xi} 2sds}\\ &\simeq \frac{1}{0.2835\xi^4 w_{0.8} + 0.2634\xi^2} 2\Big(\int_0^1 \min \int_0^{0.9182\xi x} F(s)\,ds\,dx\Big)\\ &\simeq \frac{2}{0.2835\xi^4 w_{0.8} + 0.2634\xi^2}\Big(\int_0^{2^{-1/3}} \int_0^{0.9182\xi x} \max F(s)\,ds\,dx\\ &+ \int_{2^{-1/3}}^1 \int_0^{\xi x} \max F(s)\,ds\,dx\Big) > 0 \end{aligned}$$

for some $\xi \in \mathbb{R}$. Thus,

$$\liminf_{\xi \to +\infty} \frac{\sup_{|t| \le \xi} \min \int_0^t F(s)ds}{0.2835\xi^4} = 0$$

and

$$\limsup_{\xi\to+\infty}\frac{\int_0^1 \min\int_0^{\Gamma(2-0.8)\xi x} F(s)\,ds\,dx+\int_1^2 \min\int_0^{\Gamma(2-0.8)\xi(2-x)} F(s)\,ds\,dx}{0.2835\xi^4 w_{0.8}+\sum_{i=1}^{2}\int_0^{\Gamma(2-0.8)\xi(2-x_i)} I_i(s)ds}>0.$$

Hence, by Theorem 3, problem (11), for λ lying in a convenient interval, has an unbounded sequence of weak solutions in $E_0^{0.8,4}$.

Next, we wish to point out the following consequences of Theorem 3 using the conditions

(F5) $\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}<\frac{1}{pc}$;

(F6) $\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} F(s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha+\sum_{i=1}^{m}\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds}>1.$

Corollary 1. *Assume that* (F1)–(F3), (F5)–(F6), (I1) and (A1) *hold. Then, the problem*

$$\begin{cases} {}_xD_T^{\alpha}\phi_p\Big({}_0D_x^{\alpha}u(x)\Big)+\phi_p(u(x))\in\lambda F(u(x)), \qquad a.e.\ x\in[0,T],\\ x\neq x_j,\\ \Delta({}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha}u))(x_j)=I_j(u(x_j)), \quad j=1,2,\dots,m,\\ u(0)=u(T)=0\end{cases}$$

has an unbounded sequence of weak solutions in X.

We conclude our paper with two more consequences of our main result. The following corollary is an immediate consequence of Corollary 1.

Corollary 2. *Assume that* (F1)–(F3) *hold,* $I_i(0)=0$, *and* $I_i(s)s>0$ *for* $s\in\mathbb{R}$, $i=1,2,\dots,m$. *If*

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}=0$$

and

$$\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} F(s)\,ds\,dx+\int_{\frac{T}{2}}^{T}\min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha-\sum_{i=1}^{m}\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds}=+\infty,$$

then problem (1) *with* $\lambda=1$ *and* $\mu=0$ *admits a sequence of pairwise distinct weak solutions.*

Corollary 3. *Let* $F_1:\mathbb{R}\to 2^{\mathbb{R}}$ *be an upper semicontinuous multifunction with compact convex values, such that* $\min F_1,\max F_1:\mathbb{R}\to\mathbb{R}$ *are Borel measurable and* $|\xi|\le a(1+|s|^{r_1-1})$ *for all* $s\in\mathbb{R}$, $\xi\in F_1(s)$, *and* $r_1>1$ $(a>0)$. *In addition, assume that:*

(C1) $\liminf\limits_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} \min\int_0^t F_1(s)ds}{\xi^p} < +\infty;$

(C2) $\limsup\limits_{\xi\to+\infty} \frac{\int_0^{\frac{T}{2}} \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} F_1(s)\,ds\,dx + \int_{\frac{T}{2}}^T \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} F_1(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds} = +\infty.$

Then, for every multifunction $F_2 : \mathbb{R} \to 2^{\mathbb{R}}$ *that is upper semicontinuous with compact convex values such that* $\min F_2$, $\max F_2 : \mathbb{R}\to\mathbb{R}$ *are Borel measurable and* $|\xi| \le b(1+|s|^{r_2-1})$ *for all* $s\in\mathbb{R}$, $\xi\in F_2(s)$, $r_2>1$ $(b>0)$, *and that satisfies the conditions*

$$\sup_{t\in\mathbb{R}} \min \int_0^t F_2(s)ds \le 0 \tag{12}$$

and

(13)
$$\liminf_{\xi\to+\infty} \frac{\int_0^{\frac{T}{2}} \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} F_2(s)\,ds\,dx + \int_{\frac{T}{2}}^T \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} F_2(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds} > -\infty,$$

for each

$$\lambda \in \left(0,\ \frac{1}{\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi}\min\int_0^t F_1(s)ds}{\frac{1}{pc}\xi^p}}\right),$$

the problem

$$\begin{cases} {}_xD_T^\alpha \phi_p\Big({}_0D_x^\alpha u(x)\Big) + \phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)), \qquad a.e.\ x\in[0,T], \\ x\ne x_j, \\ \Delta({}_xD_T^{\alpha-1}({}_0^cD_x^\alpha u))(x_j) = I_j(u(x_j)), \quad j=1,2,\ldots,m, \\ u(0)=u(T)=0 \end{cases}$$

has an unbounded sequence of weak solutions in X.

Proof. Set $F(t) = F_1(t)+F_2(t)$ for all $t\in\mathbb{R}$. Conditions (C2) and (13) yield

$$\begin{aligned} &\limsup_{\xi\to+\infty} \frac{\int_0^{\frac{T}{2}} \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^T \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds} \\ &= \limsup_{\xi\to+\infty} \frac{\int_0^{\frac{T}{2}} \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}x} (F_1(s)+F_2(s))\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds} \\ &+ \frac{\int_{\frac{T}{2}}^T \min\int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x)} (F_1(s)+F_2(s))\,ds\,dx}{\frac{1}{p}\xi^p w_\alpha + \sum_{i=1}^m \int_0^{\frac{2\Gamma(2-\alpha)\xi}{T}(T-x_i)} I_i(s)ds} = +\infty. \end{aligned}$$

Moreover, conditions (C1) and (12) ensure that

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}\le\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F_1(s)ds}{\xi^p}<+\infty.$$

Since

$$\frac{1}{\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\frac{1}{pc}\xi^p}}\ge\frac{1}{\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F_1(s)ds}{\frac{1}{pc}\xi^p}},$$

by applying Theorem 3, we have the desired conclusion. □

Remark 3. We observe that in Theorem 2 we can replace $\xi\to+\infty$ with $\xi\to 0^+$, and then by the same argument as in the proof of Theorem 2, but using conclusion (c) of Theorem 1 instead of (b), problem (1) has a sequence of weak solutions that strongly converges to 0 in X.

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John Graef
University of Tennessee at Chattanooga
Department of Mathematics 615
McCallie Ave.Chattanooga, TN 37403
USA
E-mail: *john-graef@utc.edu*

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Samad Kolagar
Department of Mathematics,
University of Mazandaran
Mazandaran Province,
Babolsar
Pasdaran Street
Iran E-mail: *mohseni.samad@gmail.com*

Ghasem Afrouzi
Department of Mathematics,
Faculty of Mathematical Sciences,
University of Mazandaran,
Babolsar, Iran
E-mail: *afrouzi@umz.ac.ir*