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THE RIEMANN ZETA FUNCTION AND CLASSES OF INFINITE SERIES

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We present one-parameter series representations for the following series involving the Riemann zeta function

$$\sum_{\substack{n=3\\ n\,\mathrm{odd}}}^{\infty} \frac{\zeta(n)}{n} s^n \quad \text{and} \quad \sum_{\substack{n=2\\ n\,\mathrm{even}}}^{\infty} \frac{\zeta(n)}{n} s^n$$

and we apply our results to obtain new representations for some mathematical constants such as the Euler (or Euler-Mascheroni) constant, the Catalan constant, $\log 2$, $\zeta(3)$ and π.

1. INTRODUCTION

The classical Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1)$$

is one of the most important special functions in mathematics and physics. In view of many applications in analysis, number theory, statistics, quantum field theory and other branches, the properties of the ζ-function have been studied thoroughly by numerous researchers. For more information on this subject we refer to the monographs [**13**], [**14**], [**19**] and the references cited therein.

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The study of series involving the zeta function and its relatives has a long history. Srivastava [**18**] pointed out that in 1729 Goldbach discovered a theorem which is equivalent to

$$\sum_{n=2}^{\infty}\big(\zeta(n)-1\big)=1$$

and Euler published in 1771 the remarkable summation formula

$$\zeta(3)=\frac{\pi^2}{7}\Big(1-4\sum_{\substack{n=2\\ n\text{ even}}}^{\infty}\frac{\zeta(n)}{(n+1)(n+2)2^n}\Big). \tag{1}$$

Since then many research papers have been published providing various classes of series involving the ζ-function. A detailed collection of series involving the zeta function as well as historical comments on this subject can be found in [**19**, Chapter 3].

In this note we study series involving the ζ-function with odd-integer and even-integer arguments, namely

$$U(s)=\sum_{\substack{n=3\\ n\text{ odd}}}^{\infty}\frac{\zeta(n)}{n}s^n \quad\text{and}\quad V(s)=\sum_{\substack{n=2\\ n\text{ even}}}^{\infty}\frac{\zeta(n)}{n}s^n \quad (-1<s<1).$$

These functions can be expressed in terms of Euler's gamma function. Indeed, we have (see, e.g., [**19**, p. 270, Eqs. (11) and (12)])

$$U(s)=\frac{1}{2}\log\frac{\Gamma(1-s)}{\Gamma(1+s)}-\gamma s \quad\text{and}\quad V(s)=\frac{1}{2}\log\big(\Gamma(1-s)\Gamma(1+s)\big), \tag{2}$$

where γ denotes the Euler (or Euler-Mascheroni) constant defined by

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\cong 0.57721.... \tag{3}$$

Since Euler, numerous works on the constant γ have been published. Recently, in a tantalizing blend of history and mathematics, Havil [**12**] takes us on a journey through logarithms and the harmonic series, the two defining elements of γ as in (3), toward the first account of γ's place in mathematics. Finch [**10**, pp. 35–40] recorded a fairly comprehensive bibliography of as many as 136 references on γ. For a very recent work of Euler's constant and its modern developments, one may refer to [**16**] (see also [**19**, Section 1.2]).

In the next section, we present integral representations for U and V and we show that these formulas can be applied to deduce one-parameter series representations for both functions. As special cases we obtain new series representations for some mathematical constants such as Euler's constant, Catalan's constant, $\log 2$, $\zeta(3)$, and π.

2. THE FUNCTIONS U AND V

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}_0^-$ the sets of complex numbers, positive and non-positive integers. Moreover, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The following result plays a key role in our paper. It is a special case of a more general theorem which is proved in [**4**].

Proposition. *Suppose that the function*

$$F(x) = \frac{f(x)}{1-x}$$

satisfies $f,\, F \in L[0,1]$. *With* $\lambda < 1/2$ *and*

$$b_k = \int_0^1 x^k f(x)\,dx \quad (k \in \mathbb{N}_0),$$

it follows that

$$\int_0^1 F(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k}\, b_k. \tag{4}$$

Remark 1. It is found that the double series in (4) reduces to a single series without the parameter λ under a constraint. Indeed, let S be the double series in (4). By using the known double series manipulation (see [**8**, Eq. (1.17)])

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{k,n} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} A_{k,n}$$

we get for $\lambda < 1/2$

$$S = \sum_{k=0}^{\infty} b_k \sum_{n=k}^{\infty} \binom{n}{n-k} \frac{(-\lambda)^{n-k}}{(1-\lambda)^{n+1}}.$$

Letting $n - k = n'$ and then dropping the prime on n', we have

$$\begin{aligned} S &= \sum_{k=0}^{\infty} b_k \frac{1}{(1-\lambda)^{k+1}} \sum_{n=0}^{\infty} \binom{n+k}{n} \left(\frac{-\lambda}{1-\lambda}\right)^n \\ &= \sum_{k=0}^{\infty} b_k \frac{1}{(1-\lambda)^{k+1}} \sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} \left(\frac{-\lambda}{1-\lambda}\right)^n, \end{aligned}$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ $(\alpha \in \mathbb{C};\, n \in \mathbb{N}_0)$ denotes the Pochhammer symbol (see [**19**, p. 2]). Using the binomial theorem

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \quad (|z| < 1;\, a \in \mathbb{C})$$

we obtain

$$\sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} \left(\frac{-\lambda}{1-\lambda}\right)^n = (1-\lambda)^{k+1}.$$

Hence we find

$$\sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k}\, b_k = \sum_{k=0}^{\infty} b_k,$$

whose right side is independent of the parameter λ.

Our first theorem offers an integral representation for $U(s)$.

Theorem 1. *Let $s \in (-1,1)$. Then we have*

$$U(s) = \int_0^1 \Big(\frac{x^s - x^{-s}}{2\log x} - s\Big)\frac{dx}{1-x}. \tag{5}$$

Proof. Using Malmstén's formula (see [**19**, p. 27, Eq. (25)])

$$\log\Gamma(1+s) = \int_0^\infty \Big(s - \frac{1-e^{-st}}{1-e^{-t}}\Big)\frac{e^{-t}}{t}dt, \tag{6}$$

and the integral representation (see [**19**, p. 15, Eq. (9)])

$$\gamma = \int_0^\infty \Big(\frac{1}{1-e^{-t}} - \frac{1}{t}\Big)e^{-t}dt,$$

we conclude from (2) that

$$\begin{aligned} U(s) &= \int_0^\infty \Big(\frac{e^{st}-e^{-st}}{2(1-e^{-t})} - s\Big)\frac{e^{-t}}{t}dt - \int_0^\infty \Big(\frac{s}{1-e^{-t}} - \frac{s}{t}\Big)e^{-t}dt \\ &= \int_0^\infty \Big(\frac{\sinh(st)}{t} - s\Big)\frac{dt}{e^t-1}. \end{aligned}$$

The substitution $t = -\log x$ in the last integral gives (5). □

The following one-parameter series representation for $U(s)$ is valid.

Theorem 2. *Let $s \in (-1,1)$ and $\lambda < 1/2$. Then we have*

$$U(s) = \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k}(-\lambda)^{n-k}\Big[\frac{1}{2}\log\Big(1+\frac{2s}{k+1-s}\Big) - \frac{s}{k+1}\Big]. \tag{7}$$

Proof. Let

$$u_s(x) = \frac{x^s - x^{-s}}{2\log x} - s$$

and

$$u_k^*(s) = \int_0^1 x^k u_s(x)\,dx \quad (k \in \mathbb{N}_0).$$

Frullani's integral formula (see [**9**, Section 495]) yields

$$\int_0^1 x^k \frac{x^s - x^{-s}}{\log x}\,dx = \int_0^\infty \frac{e^{-(k+1-s)t} - e^{-(k+1+s)t}}{t} dt = \log\frac{k+1+s}{k+1-s}.$$

We thus have

$$u_k^*(s) = \frac{1}{2}\log\frac{k+1+s}{k+1-s} - \frac{s}{k+1}.$$

Applying (5) and the Proposition leads to (7). □

Remark 2. The function U can be expressed in terms of the hypergeometric function ${}_2F_1$. Using (7) with $\lambda = 0$ gives

$$U(s) = \sum_{n=0}^\infty \Big[\frac{1}{2}\log\Big(1 + \frac{2s}{n+1-s}\Big) - \frac{s}{n+1}\Big] = \frac{1}{2}\sum_{n=0}^\infty \Big[\log\frac{1+z_n(s)}{1-z_n(s)} - 2z_n(s)\Big]$$

with $z_n(s) = s/(n+1)$. Recalling the known formula (see [**19**, p. 67, Eq. (24)])

$$\log\frac{1+z}{1-z} = 2z\,{}_2F_1\Big(\frac{1}{2}, 1; \frac{3}{2}; z^2\Big)$$

we obtain

$$U(s) = s\sum_{n=1}^\infty \frac{1}{n}\Big[{}_2F_1\Big(\frac{1}{2}, 1; \frac{3}{2}; \frac{s^2}{n^2}\Big) - 1\Big].$$

We investigate the derivative and higher order derivatives of U. We recall the psi (or digamma) function ψ defined by

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,dt \tag{8}$$

and the polygamma functions $\psi^{(N)}$ ($N \in \mathbb{N}$) defined by

$$\psi^{(N)}(z) = \frac{d^{N+1}}{dz^{N+1}}\log\Gamma(z) = \frac{d^N}{dz^N}\psi(z). \tag{9}$$

Applying (2), (5), (8), (9), the Proposition and the integral formula (see [**11**, Formula 4.272(6)]

$$\int_0^1 x^{a-1}(-\log(x))^{b-1}dx = \frac{\Gamma(b)}{a^b} \quad (a > 0,\, b > 0) \tag{10}$$

we obtain

$$\begin{aligned}
U'(s) = \sum_{\substack{n=3\\ n\,\mathrm{odd}}}^{\infty} \zeta(n) s^{n-1} &= -\frac{1}{2}\bigl(\psi(1+s)+\psi(1-s)\bigr)-\gamma \\
&= \int_0^1 \Bigl(\frac{x^s+x^{-s}}{2}-1\Bigr)\frac{dx}{1-x} \\
&= s^2 \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(k+1)\bigl[(k+1)^2-s^2\bigr]}
\end{aligned}$$

and, for $N \in \mathbb{N}\setminus\{1\}$,

(11)

$$\begin{aligned}
U^{(N)}(s) &= \sum_{\substack{n=3\\ n\,\mathrm{odd}}}^{\infty} (n-1)(n-2)\cdots(n-N+1)\,\zeta(n)\,s^{n-N} \\
&= -\frac{1}{2}\Bigl(\psi^{(N-1)}(1+s)+(-1)^{N-1}\,\psi^{(N-1)}(1-s)\Bigr) \\
&= \int_0^1 \frac{(-1)^{N-1}x^s+x^{-s}}{2}(-\log x)^{N-1}\frac{dx}{1-x} \\
&= \frac{(N-1)!}{2}\sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-\lambda)^{n-k}\Bigl[\frac{(-1)^{N-1}}{(k+1+s)^N}+\frac{1}{(k+1-s)^N}\Bigr].
\end{aligned}$$

Remark 3. The generalized (or Hurwitz) zeta function is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad (\Re(s)>1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-).$$

Applying a useful relation between the n-th polygamma function and the generalized zeta function

$$\psi^{(N)}(z) = (-1)^{N+1}\,N!\,\zeta(N+1,z)$$

to the second representation in (11) we obtain

$$U^{(N)}(s) = -\frac{(N-1)!}{2}\Bigl((-1)^N\,\zeta(N,1+s)-\zeta(N,1-s)\Bigr) \quad (N\in\mathbb{N}\setminus\{1\}).$$

We set

$$U_0(s) = s\,U'(s) = \sum_{\substack{n=3\\ n\,\mathrm{odd}}}^{\infty}\zeta(n)s^n.$$

Then we get

$$\begin{aligned} U_0'(s) &= \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} n\,\zeta(n)s^{n-1} \\ &= -\frac{1}{2}\big(\psi(1+s)+\psi(1-s)\big) - \frac{s}{2}\big(\psi'(1+s)-\psi'(1-s)\big) - \gamma \\ &= s^2 \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{3(k+1)^2 - s^2}{(k+1)\big[(k+1)^2 - s^2\big]^2}. \end{aligned}$$

Using some known special values (see [**5**, Section 6], [**15**], [**19**, Section 1.3]) we obtain

$$U(1/2) = \frac{1}{2}(\log 2 - \gamma); \tag{12}$$

(13)

$$U'(1/2) = 2\log 2 - 1;\ U'(1/3) = \frac{3}{2}(\log 3 - 1);\ U'(1/5) = \frac{5}{2}\big(\log\sqrt{5} - 1\big) + \frac{\sqrt{5}}{2}\log\frac{1+\sqrt{5}}{2};$$

$$U^{(2)}(1/4) = 8 - 8G; \quad U^{(2)}(3/4) = \frac{8}{9} + 8G; \tag{14}$$

$$U^{(3)}(1/2) = 14\,\zeta(3) - 8; \tag{15}$$

$$U_0'(1/4) = 3\,\log 2 - 2\,G. \tag{16}$$

Here G denotes Catalan's constant defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cong 0.91596....$$

A large number of various representations for G are known in the literature, for example,

$$G = -\int_0^1 \frac{\log x}{1+x^2}\,dx$$

and

$$G = \frac{1}{16}\Big(\zeta(2, 1/4) - \zeta(2, 3/4)\Big).$$

Euler's formula (1) expresses $\zeta(3)$ in terms of a series involving the zeta function with even-integer arguments. Applying (11) with $N = 3$, $s = 1/2$ and

(15) yields a counterpart of this result. We find a series representation for $\zeta(3)$ involving the ζ-function with odd-integer arguments

$$\zeta(3) = \frac{1}{3}\Big(2 + \sum_{\substack{n=4 \\ n \text{ even}}}^{\infty} \frac{(n-1)\, n\, \zeta(n+1)}{2^n}\Big).$$

Using the special values (12)-(16) and the one-parameter series representations for $U^{(N)}$ $(N = 0, 1, 2, 3)$ and U_0' we obtain series representations for various mathematical constants such as

$$\log 2 - \gamma = \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \Big[\log\Big(1 + \frac{2}{2k+1}\Big) - \frac{1}{k+1}\Big];$$

$$\log 2 = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(2k+1)(2k+2)(2k+3)}; \tag{17}$$

$$\log 3 = 1 + 2\sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(3k+2)(3k+3)(3k+4)}; \tag{18}$$

$$\begin{aligned} \log \frac{1+\sqrt{5}}{2} =& \sqrt{5}\big(1 - \log\sqrt{5}\big) \\ & + 2\sqrt{5} \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(5k+4)(5k+5)(5k+6)}; \end{aligned} \tag{19}$$

$$G = 1 - 16\sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{k+1}{(4k+3)^2(4k+5)^2}; \tag{20}$$

$$G = -\frac{1}{9} + 48\sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{k+1}{(4k+1)^2(4k+7)^2}; \tag{21}$$

$$\zeta(3) = \frac{4}{7} + \frac{16}{7}\sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{(k+1)(4k^2+8k+7)}{(2k+1)^3(2k+3)^3}; \tag{22}$$

$$3\log 2 - 2G = \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{48k^2+96k+47}{(k+1)(4k+3)^2(4k+5)^2}. \tag{23}$$

Since

$$\gamma = \frac{1}{2} - 2U(1/2) + \frac{1}{2}U'(1/2), \tag{24}$$

we get

$$\gamma = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \Big[\frac{8k^2+16k+7}{(2k+1)(2k+2)(2k+3)} - \log\frac{2k+3}{2k+1}\Big].$$

In particular, if we set $\lambda = 0$ in (19), (20), (21), (23) and $\lambda = -1$ in (17), (18), (22), then we obtain the following representations:

$$\log\frac{1+\sqrt{5}}{2} = \sqrt{5}\big(1-\log\sqrt{5}\big) + \frac{2}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{n(25n^2-1)}; \tag{25}$$

$$G = 1 - 16\sum_{n=1}^{\infty} \frac{n}{(16n^2-1)^2};$$

$$G = -\frac{1}{9} + 48\sum_{n=1}^{\infty} \frac{n}{(16n^2-9)^2};$$

$$3\log 2 - 2G = \sum_{n=1}^{\infty} \frac{48n^2-1}{n(16n^2-1)^2};$$

$$\log 2 = \frac{1}{2} + \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)(2k+1)(2k+3)};$$

$$\log 3 = 1 + \frac{1}{3}\sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)(3k+2)(3k+4)};$$

$$\zeta(3) = \frac{4}{7} + \frac{8}{7}\sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(k+1)(4k^2+8k+7)}{(2k+1)^3(2k+3)^3}.$$

Remark 4. The number $(1+\sqrt{5})/2 \cong 1.61803\ldots$ appearing in (19) and (25) is known as the golden ratio which was defined by Euclid more than two thousand years ago. Since then it has shown a propensity to appear in the most astonishing variety of places - from mollusk shells, sunflower florets, and the crystals of some materials, to the shapes of galaxies containing billions of stars (see [**17**]). From the last formula in (13) we conclude that the logarithm of the golden ratio can be expressed as series involving the Riemann zeta function with odd-integer arguments

$$\log\frac{1+\sqrt{5}}{2} = \sqrt{5}\big(1-\log\sqrt{5}\big) + 2\sqrt{5}\sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \frac{\zeta(n)}{5^n}.$$

Remark 5. Using the second formula in (14) and (24) we obtain the following representations for the Catalan and Euler constants in terms of the ζ-function:

$$G = -\frac{1}{9} + \frac{2}{9} \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \left(\frac{3}{4}\right)^n (n-1)\zeta(n) \quad \text{and} \quad \gamma = \frac{1}{2} + \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \frac{(n-2)\zeta(n)}{n\, 2^n}.$$

We begin the study of the function V with an integral representation.

Theorem 3. *Let $s \in (-1,1)$. Then we have*

$$V(s) = \int_0^1 \frac{x^s + x^{-s} - 2}{-2\log x} \frac{dx}{1-x}. \tag{26}$$

Proof. From (2) and (6) we get

$$V(s) = \int_0^\infty \Big(\frac{e^{st} + e^{-st}}{2} - 1\Big) \frac{e^{-t}}{(1-e^{-t})\, t} dt = \int_0^\infty \frac{\cosh(st) - 1}{t} \frac{dt}{e^t - 1}.$$

Substituting $t = -\log x$ gives (26). □

Next, we offer a one-parameter series representation for $V(s)$, which is a counterpart of (7) in Theorem 2.

Theorem 4. *Let $s \in (-1,1)$ and $\lambda < 1/2$. Then we have*

$$V(s) = \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \Big(\log(k+1) - \frac{1}{2}\log\big((k+1)^2 - s^2\big)\Big). \tag{27}$$

Proof. We define

$$v_s(x) = \frac{x^s + x^{-s} - 2}{-2\log x} \quad \text{and} \quad v_k^*(s) = \int_0^1 x^k v_s(x)\, dx \quad (k \in \mathbb{N}_0).$$

Then,

$$2v_k^*(s) = \int_0^1 \frac{x^{k+s} - x^k}{-\log x} dx + \int_0^1 \frac{x^{k-s} - x^k}{-\log x} dx = \log \frac{k+1}{k+1+s} + \log \frac{k+1}{k+1-s}.$$

Thus we find

$$v_k^*(s) = \log(k+1) - \frac{1}{2}\log\big((k+1)^2 - s^2\big).$$

Finally, we apply (26) and the Proposition. This gives (27). □

Using (2), (8), Theorem 3, the Proposition and (10) we find

$$
\begin{aligned}
V'(s) &= \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \zeta(n) s^{n-1} = \frac{1}{2}\big(\psi(1+s) - \psi(1-s)\big) \\
&= \int_0^1 \frac{x^{-s} - x^{s}}{2} \frac{dx}{1-x} \\
&= s \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(k+1)^2 - s^2},
\end{aligned} \tag{28}
$$

where the last one of which is a special case of a result given in [**2**], and for $N \in \mathbb{N} \setminus \{1\}$,

(29)

$$
\begin{aligned}
V^{(N)}(s) &= \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} (n-1)(n-2)\cdots(n-N+1)\zeta(n) s^{n-N} \\
&= \frac{1}{2}\Big(\psi^{(N-1)}(1+s) + (-1)^N \psi^{(N-1)}(1-s)\Big) \\
&= \int_0^1 \frac{(-1)^N x^{s} + x^{-s}}{2} (-\log x)^{N-1} \frac{dx}{1-x} \\
&= \frac{(N-1)!}{2} \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \Big[\frac{(-1)^N}{(k+1+s)^N} + \frac{1}{(k+1-s)^N}\Big].
\end{aligned}
$$

Remark 6. From the second representation in (29) it follows that $V^{(N)}$ can be expressed in terms of the generalized zeta function

$$
V^{(N)}(s) = \frac{(N-1)!}{2}\Big((-1)^N \zeta(N, 1+s) + \zeta(N, 1-s)\Big) \quad (N \in \mathbb{N} \setminus \{1\}).
$$

We have (see [**5**, Section 6], [**15**], [**19**, Section 1.3])

$$
V'(1/4) = 2 - \frac{\pi}{2}, \quad V^{(2)}(1/2) = \frac{1}{2}\pi^2 - 2, \quad V^{(3)}(1/4) = 64 - 2\pi^3. \tag{30}
$$

Using these values as well as (28) and (29) we obtain the following representations:

$$
\pi = 4 - 8 \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(4k+3)(4k+5)}; \tag{31}
$$

$$
\pi^2 = 4 + 8 \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{4k^2 + 8k + 5}{(2k+1)^2 (2k+3)^2}; \tag{32}
$$

$$\pi^3 = 32 - 64 \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{48k^2+96k+49}{(4k+3)^3(4k+5)^3}. \tag{33}$$

Setting $\lambda = -1$ in (31), (32) and (33) leads to

$$\frac{\pi}{4} = 1 - \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(4k+3)(4k+5)};$$

$$\frac{\pi^2}{4} = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{4k^2+8k+5}{(2k+1)^2(2k+3)^2};$$

$$\frac{\pi^3}{32} = 1 - \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{48k^2+96k+49}{(4k+3)^3(4k+5)^3}.$$

Remark 7. From (30) and the first representations in (28) and (29) we conclude that π, π^2 and π^3 can be expressed as simple series involving the ζ-function with even-integer arguments

$$\frac{\pi}{8} = \frac{1}{2} - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{\zeta(n)}{4^n};$$

$$\frac{\pi^2}{8} = \frac{1}{2} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{(n-1)\zeta(n)}{2^n};$$

$$\frac{\pi^3}{32} = 1 - \sum_{\substack{n=4 \\ n \text{ even}}}^{\infty} \frac{(n-1)(n-2)\zeta(n)}{4^n}.$$

For more series representations for mathematical constants, the interested reader may refer to [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**7**] and the references cited therein.

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