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THE NUMBER OF DISJOINT PERFECT MATCHINGS IN SEMI-REGULAR GRAPHS

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We obtain a sharp result that for any even $n \geq 34$, every $\{D_n,\, D_n+1\}$-regular graph of order n contains $\lceil n/4 \rceil$ disjoint perfect matchings, where $D_n = 2\lceil n/4 \rceil - 1$. As a consequence, for any integer $D \geq D_n$, every $\{D,\, D+1\}$-regular graph of order n contains $(D - \lceil n/4 \rceil + 1)$ disjoint perfect matchings.

1. INTRODUCTION

Vizing's theorem [25] states that the edge-chromatic number of any graph G equals either the maximum degree $\Delta(G)$, or $\Delta(G)+1$. Despite the fact that an $(\Delta(G)+1)$-edge-colouring of any graph can be found in polynomial time, Holyer [12] showed that the problem of deciding whether G is $\Delta(G)$-edge-colorable is NP-complete, even if $\Delta(G) = 3$. For any regular graph, its edge-chromatic number equals its maximum degree if and only if the graph is 1-factorizable, i.e., its edge set can be decomposed into perfect matchings. Seymour [22] provided an algebraic viewpoint for the 1-factorization of graphs, by associating every perfect matching with its $\{0,1\}$ characteristic function on edges. Among the considerable number of conjectures involving edge-colorings of regular graphs, see Jensen and Toft's book [14], the 1-factorization Conjecture 1.1 has been being regarded as one of the most famous one.

Conjecture 1.1 (The 1-factorization conjecture)**.** *Every regular simple graph of even order and with degree at least half the order is* 1*-factorizable.*

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Conjecture 1.1 was formulated by Chetwynd and Hilton [2], who, however, pointed the origin of Conjecture 1.1 to Gabriel Dirac in the early 1950s; see [14] for more references. In fact, it is easy to see that D-regular graphs of odd order is not D-edge-colourable, and that D-regular graphs of even order with $D < n/2$ may still be not D-edge-colourable.

Chetwynd and Hilton [2] showed that every graph of even order n with minimum degree at least $6n/7$ is 1-factorizable. An alternative proof can be found in [4]. This bound was improved to $(\sqrt{7}-1)n/2$, by the same authors [3], and Niessen and Volkmann [16] independently. Plantholt and Tipnis [20] generalized the above results to multigraphs. Chetwynd and Hilton [2] also noted that Roland Häggkvist has announced that for every $\epsilon > 0$, there exists $N(\epsilon) > 0$ such that any D-regular simple graph of even order $n > N(\epsilon)$ with $D \ge (1/2+\epsilon)n$ is 1-factorable. This asymptotic result is explicitly established by Perković and Reed [19] with an algorithmic proof. The combined efforts of Hoffman and Rodger [11], De Simone and Picinin de Mello [6], and De Simone and Galluccio [7] confirmed Conjecture 1.1 for D-regular graphs that are join of two graphs, and provided a polynomial time algorithm for finding a D-edge-colouring of these graphs.

For any even integer n, define

$$D_n \;=\; 2\left\lceil \frac{n}{4} \right\rceil - 1 \;=\; \begin{cases} \dfrac{n}{2} - 1, & \text{if } n \equiv 0 \pmod 4; \\ \dfrac{n}{2}, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Recently, Csaba et al. [5] confirmed Conjecture 1.1 for regular graphs of sufficiently large order with degree at least D_n. Parallel to the study of 1-factorizations, one is interested in the least number $N(D)$ such that any D-regular graph of order n has at least $N(D)$ disjoint perfect matchings (abbreviated as DPMs). We call it the *DPM problem* for regular graphs. Focusing on D-regular graphs of even order n such that $D \ge n/2$, Hilton [10] confirmed the existence of $\lfloor D/3 \rfloor$ DPMs, which was improved remarkably by Zhang and Zhu [26] to the bound $\lfloor D/2 \rfloor$.

Theorem 1.2 (Zhang and Zhu)**.** *Any D-regular graph of even order n such that $D \ge n/2$ contains at least $\lfloor D/2 \rfloor$ DPMs.*

A graph is said to be $\{D,\, D+1\}$-regular if the degree of every vertex is either D or $(D+1)$. Following Akiyama and Kano [1, Section 5.2], a $\{D,\, D+1\}$-regular graph is said to be *semi-regular*. Semi-regular graphs, along with regular graphs, have been paid much attention on graph factor problems. For example, Thomassen [23] showed that every $\{r, r+1\}$-graph has a $\{k, k+1\}$-factor for any $1 \le k < r$. Considering the DPM problem for semi-regular graphs, Hou [13] obtained the following analogue of Theorem 1.2.

Theorem 1.3 (Hou)**.** *Every $\{D,\, D+1\}$-regular graph of even order n such that $D \ge n/2$ contains at least $(D - n/2 + \lfloor n/6 \rfloor + 1)$ DPMs.*

In this paper, we improve Theorem 1.3 to the sharp result that every $\{D,\, D+1\}$-regular graph of even order $n \ge 34$ contains $(D - \lceil n/4 \rceil + 1)$ DPMs; see Theorem 3.12. It is essentially a corollary of Theorem 3.11, whose proof occupies most of this paper. For a comparison with Csaba et al.'s results, see the last section.

2. PRELIMINARY

In this paper, we consider finite undirected simple graphs without loops or multiple edges. The number of vertices in a graph G is said to be the order of G, denoted $|G|$. As usual, we denote the neighbor set of a vertex subset W of G by $N_G(W)$, or simply $N(W)$ if there is no confusion. One of the earliest cornerstones in the matching theory is Hall's theorem [**9**].

Theorem 2.4 (Hall)**.** *Let $G = (X, Y)$ be a bipartite graph. Then G has a matching covering X if and only if $|W| \le |N(W)|$ for every subset W of X.*

The famous Tutte's theorem [**24**] states that a graph G has a perfect matching if and only if for any vertex subset S, the number of odd components of the graph $G - S$ is at most the order $|S|$. In this paper, we will use the following stronger version of Tutte's theorem, see Lovász and Plummer's book [**15**, Exercise 3.3.18 (b)]. A graph G is said to be *factor-critical* if the subgraph $G - u$ has a perfect matching for every vertex v. Obviously the order of a factor-critical graph must be odd.

Theorem 2.5. *Let G be a graph without perfect matchings. Then G has a vertex subset S such that every component of the subgraph $G - S$ is factor-critical, and*

$$o(G - S) \equiv |S| \pmod 2 \quad \textit{and} \quad o(G - S) \ \ge \ |S| + 2,$$

where $o(G - S)$ is the number of factor-critical components of the subgraph $G - S$.

We also need some known results judging the graph structure with aid of the minimum degree. A graph that contains a Hamiltonian cycle is called *Hamiltonian*. Next is a classical criterion for graph Hamiltonicity due to Dirac [**8**].

Theorem 2.6 (Dirac)**.** *Every graph with minimum degree at least half of its order is Hamiltonian.*

A graph is said to be *Hamiltonian-connected* if it contains a Hamiltonian path between every two distinct vertices. Ore [**17, 18**] discovered a criterion for this stronger property.

Theorem 2.7 (Ore)**.** *Let G be a 2-connected graph. If the degree sum of any two non-adjacent vertices of G is larger than the order $|G|$, then G is Hamiltonian-connected.*

A graph is said to be *bi-critical* if the subgraph obtained by removing any two distinct vertices has a perfect matching. Plummer [**21**] established a criterion for the bi-criticality of graphs.

Theorem 2.8 (Plummer)**.** *Let G be a graph of even order. If the degree sum of any two non-adjacent vertices of G is larger than the order $|G|$, then the graph G is bi-critical.*

Let us give an overview of notion and notations that we need in the sequel. For any vertex subset S of V, we denote by $G[S]$ the subgraph of G induced by S, and write $G - S = G[V(G) - S]$. For a graph G and an edge set $\tilde{E}$, we denote by $G \cup \tilde{E}$ the graph with vertex set $V(G) \cup V(\tilde{E})$ and edge set $E(G) \cup \tilde{E}$.

For any vertex subsets X and Y of a graph G, we denote by $E_G(X, Y)$ the set of edges with one end in X and the other end in Y. It is clear that $E_G(X, Y) = E_G(Y, X)$. Denote $e_G(X, Y) = |E_G(X, Y)|$. As usual, we use the notation

$$\partial_G X \;=\; E_G(X,\, V(G) - X).$$

The degree of a vertex v in a graph G is denoted by $\deg_G(v)$. The minimum degree of vertices of a vertex set X in a graph G is denoted by $\delta_G(X)$. As usual, we denote $\delta(G) = \delta_G(V(G))$. When the symbol X or Y denotes a subgraph of G, we use the same notation $E_G(X, Y)$ to denote the edge set $E_G(V(X),\, V(Y))$, and use the similar convention $\delta_G(X) = \delta_G(V(X))$.

3. MAIN RESULT

Lemma 3.9 will be of considerable help in the proof of Theorem 3.11.

Lemma 3.9. *Let d, k, s be integers such that $d \ge (s + k)/2 + 1$ and $s \ge k + 1$. Let $G' = (S, U)$ be a bipartite graph with part orders $|S| = s$ and $|U| = s + 1$. Suppose that the minimum degree $\delta_{G'}(U)$ is at least d, and that every vertex in the part S has degree at most $(d+2)$, with at most one vertex in S having degree $(d+2)$. Then for any vertex subset $S' \subset S$ of order k and for any vertex subset $U' \subset U$ of order $(k + 1)$, the graph $G' - S' - U'$ has a perfect matching.*

Proof. By contradiction, suppose that there exist subsets $S' \subset S$ and $U' \subset U$ such that the subgraph $H = G' - S' - U'$ has no perfect matchings. By Hall's Theorem 2.4, there exists a vertex set $T \subseteq U - U'$ such that

$$|N_H(T)| \;\le\; |T| - 1. \tag{3.1}$$

Denote $p = |N_H(T)|$. By using the hand-shaking theorem, we have

$$\sum_{u\in U} \deg_{G'}(u) \;=\; \sum_{v\in S} \deg_{G'}(v) \;=\; \sum_{v\in N_H(T)\cup S'} \deg_{G'}(v) \;+\; \sum_{v\in S-N_H(T)-S'} \deg_{G'}(v). \tag{3.2}$$

We shall estimate the three summations on both sides of Eq. (3.2) individually.

From the premise that every vertex in the part U has degree at least d, we infer that

$$\sum_{u\in U}\deg_{G'}(u) \;\ge\; d\cdot|U| \;=\; d\,(s+1).$$

From the premise that every vertex in the part S has degree at most $(d+2)$, with at most one vertex having degree $(d+2)$, we deduce that

$$\sum_{v\in N_H(T)\cup S'}\deg_{G'}(v) \;\le\; (d+2)+(d+1)\cdot(|N_H(T)\cup S'|-1) \;=\; 1+(d+1)(p+k).$$

Note that the neighbors of all vertices in the set $S-N_H(T)-S'$ are in the set $U-T$. Therefore, with the aid of Ineq. (3.1), we derive that

$$\begin{aligned}\sum_{v\in S-N_H(T)-S'}\deg_{G'}(v) \;&\le\; |S-N_H(T)-S'|\cdot|U-T|\\ &=\; (s-p-k)(s+1-|T|) \;\le\; (s-p-k)(s-p).\end{aligned}$$

Combining the above three inequalities with Eq. (3.2), we obtain that

$$d\,(s+1) \;\le\; 1+(d+1)(p+k)+(s-p-k)(s-p). \tag{3.3}$$

To deal with Ineq. (3.3), we first figure out the domain of p. On the one hand, we have $T\ne\emptyset$ in virtue of Ineq. (3.1). From the premise, every vertex in the set T has at least d neighbors. Thus $|N_{G'}(T)|\ge d$ and thereby

$$|N_H(T)| \;\ge\; |N_{G'}(T)|-|S'| \;\ge\; d-k.$$

On the other hand, from definition, we have $T\subseteq U-U'$. Together with Ineq. (3.1), we obtain

$$p \;\le\; |T|-1 \;\le\; |U-U'|-1 \;=\; (s+1)-(k+1)-1 \;=\; s-k-1.$$

Combining the above two inequalities, we find the domain

$$d-k \;\le\; p \;\le\; s-k-1.$$

In view of the premises $d\ge(s+k)/2+1$ and $s\ge k+1$, and the above domain of p, it is elementary to derive that the right hand side of Ineq. (3.3), considered as a quadratic function in the variable p, attains its maximum at the value $p=s-k-1$. Therefore, we can substitute $p=s-k-1$ into Ineq. (3.3), which gives

$$d\,(s+1) \;\le\; 1+(d+1)(s-1)+(k+1),$$

contradicting the premise $d\ge(s+k)/2+1$. This completes the proof. □

Lemma 3.10. *Let H be a graph with minimum degree at least $\lceil n/4\rceil$, consisting of factor-critical components C_1 and C_2 with $|C_1|\le|C_2|$. Let M be a perfect matching* *of the complementary graph of H. Let M' be a perfect matching of the graph $H \cup M$ such that the graph $(H \cup M) - M'$ consists of factor-critical components C_1' and C_2' with $|C_1'| \le |C_2'|$. Suppose that*

$$E_M(C_1, C_2) - M' \neq \emptyset. \tag{3.4}$$

Then we have $V(C_1') \subset V(C_2)$. In other words, we have

$$V(C_1) \cap V(C_1') \;=\; \emptyset \quad \text{and} \quad V(C_2) \cap V(C_2') \;\neq\; \emptyset.$$

Proof. Denote $H' = (H \cup M) - M'$. Since the minimum degree $\delta(H) \ge \lceil n/4 \rceil$, we find

$$|C_1| \;\ge\; \frac{n}{4} + 1. \tag{3.5}$$

For $i, j \in [2]$, we denote

$$V_{ij} \;=\; V(C_i) \cap V(C_j').$$

Then the desired results are $V_{11} = \emptyset$ and $V_{22} \neq \emptyset$. See Fig. 3.1. In the colorful version, one may see that the component C_1' is in red, while the component C_2' is in blue.

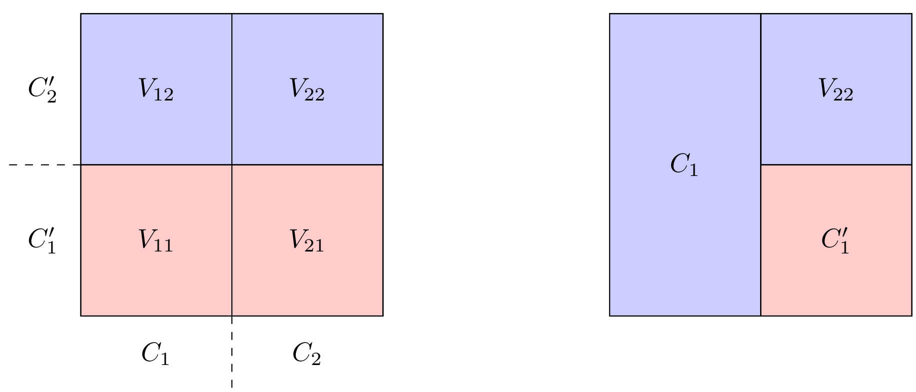

Figure 3.1: The decomposition of components of the graph H.

The vertex set $V(C_i)$ which is connected in the graph H, is decomposed into the subsets V_{i1} and V_{i2} in the graph H', one of which might be empty. Therefore, we infer that

$$E_{C_i}(V_{i1}, V_{i2}) \;\subseteq\; E(H) - E(H') \;\subseteq\; M'. \tag{3.6}$$

Let $i, j \in [2]$. From (3.6), we deduce that in the component C_i, every vertex (if it exists) in the set V_{ij} has at most one neighbor in the set $V_{ij'}$, where $j' \neq j$. Therefore, we have

$$\delta_H(V_{ij}) \;\ge\; \delta(H) - 1 \;\ge\; \left\lceil \frac{n}{4} \right\rceil - 1, \qquad \text{if } V_{ij} \neq \emptyset. \tag{3.7}$$

It follows that

$$(3.8) \qquad |V_{ij}| \;\ge\; \left\lceil \frac{n}{4} \right\rceil, \qquad \text{if } V_{ij} \ne \emptyset.$$

By way of contradiction, assume that $V_{11} \ne \emptyset$. First, we claim that

$$V(C_1') \;=\; V(C_1) \quad \text{and} \quad V(C_2') \;=\; V(C_2),$$

that is, $V_{12} = V_{21} = \emptyset$. In fact, if $V_{12} \ne \emptyset$, then Ineq. (3.8) implies that

$$|C_1| \;=\; |V_{11}| + |V_{12}| \;\ge\; \frac{n}{4} + \frac{n}{4} \;=\; \frac{n}{2}.$$

Since $|C_1| \le |C_2| = n - |C_1| \le n/2$, we infer that $|C_1| = n/2$, i.e., the equality in the above inequality holds. In particular, the odd component C_1 is composed of two vertex sets V_{11} and V_{12} of the same order, which is absurd! This proves $V_{12} = \emptyset$, i.e., $V(C_1) = V_{11}$. Now, if $V_{21} \ne \emptyset$, then Ineqs. (3.5) and (3.8) imply that

$$|C_1'| \;=\; |V_{11} \cup V_{21}| \;=\; |C_1| + |V_{21}| \;\ge\; \left(\frac{n}{4} + 1\right) + \frac{n}{4} \;>\; \frac{n}{2}.$$

It follows that $|C_2'| < |C_1'|$, contradicting the premise $|C_1'| \le |C_2'|$. This proves the claim.

From Condition (3.4), there exists an edge

$$e' \;\in\; E_M(C_1, C_2) - M' \;\subseteq\; E(H').$$

From the claim, we see that

$$e' \;\in\; E_M(C_1, C_2) \;=\; E_M(C_1', C_2').$$

Combining the above two relations, we obtain

$$(3.9) \qquad e' \;\in\; E(H') \cap E_M(C_1', C_2') \;\subseteq\; E_{H'}(C_1', C_2').$$

This is impossible since the components C_1' and C_2' are disconnected in the graph H'. This proves $V_{11} = \emptyset$.

It remains to show that $V_{22} \ne \emptyset$. In fact, the opposite relation $V_{22} = \emptyset$ implies that $V(C_2') = V(C_1)$ and $V(C_1') = V(C_2)$, resulting in the same contradiction (3.9). This proves Lemma 3.10. □

The main result of this paper is Corollary 3.12. Here out comes its essential part.

Theorem 3.11. *Let $n \ge 34$. Then every $\{D_n,\, D_n + 1\}$-regular graph of order n has at least $\lceil n/4 \rceil$ disjoint perfect matchings.*

Proof. Let $n \geq 34$. For short, we denote $D = D_n$ throughout this proof. Let G be an $\{D, D+1\}$-regular graph with a family $\mathcal{M}$ of the maximum number l of perfect matchings.

By way of contradiction, we assume $l \leq \lceil n/4 \rceil - 1$. It follows that

$$D - l \;\geq\; \left\lceil \frac{n}{4} \right\rceil. \tag{3.10}$$

Since $n \geq 34$, by Ineq. (3.10), we have

$$D - l \;\geq\; 9. \tag{3.11}$$

Let $H = G - \mathcal{M}$ denote the graph obtained by removing all edges constituting the matchings in the family $\mathcal{M}$. Then the graph H is $\{D - l,\, D - l + 1\}$-regular. Thus for any vertex v, we have

$$D - l \;\leq\; \deg_H(v) \;\leq\; D - l + 1. \tag{3.12}$$

By the choice of the family $\mathcal{M}$, the graph H has no perfect matchings. By Theorem 2.5, there is a vertex subset S such that the graph $H - S$ consists of factor-critical components $C_1, C_2, \ldots, C_q$ with

$$q \;\geq\; s + 2, \tag{3.13}$$
$$q \equiv s \pmod 2, \tag{3.14}$$
$$c_i \equiv 1 \pmod 2, \quad \text{and} \tag{3.15}$$
$$1 \;\leq\; c_1 \;\leq\; c_2 \;\leq\; \cdots \;\leq\; c_q, \tag{3.16}$$

where $s = |S|$ and $c_i = |C_i|$. By using Ineq. (3.12), we infer that

$$\sum_{i=1}^{q} |\partial_H C_i| \;=\; |\partial_H S| \;\leq\; (D - l + 1) \cdot s. \tag{3.17}$$

On the other hand, by counting the vertices in H, we find

$$n \;=\; s + \sum_{i=1}^{q} c_i, \tag{3.18}$$

Together with Ineqs. (3.16) and (3.13), we infer that $n \geq s + q \geq 2s + 2$, that is,

$$s \;\leq\; \frac{n}{2} - 1. \tag{3.19}$$

Let $i \in [q]$. Since every vertex in the component C_i has at most $(c_i - 1)$ neighbors inside itself, it has at least $(D - c_i + 1)$ neighbors outside. Thus we have

$$|\partial_G C_i| \;\geq\; c_i \cdot (D - c_i + 1). \tag{3.20}$$

Along the same line, we can deduce

$$|\partial_H C_i| \;\geq\; c_i \cdot (D - l + 1 - c_i).$$

Regarding the right hand side of the above inequality as a quadratic function in the variable c_i, we obtain

$$|\partial_H C_i| \;\geq\; D - l, \qquad \text{if } 1 \leq c_i \leq D - l; \tag{3.21}$$

$$|\partial_H C_i| \;\geq\; 2(D - l - 1), \qquad \text{if } 3 \leq c_i \leq D - l - 1; \quad \text{and} \tag{3.22}$$

$$|\partial_H C_i| \;\geq\; 3(D - l - 2), \qquad \text{if } 3 \leq c_i \leq D - l - 2. \tag{3.23}$$

In this proof, we often make effort to find the range of some order c_i so as to use the corresponding lower bound of the number $|\partial_H C_i|$ given by one of Ineqs. (3.21), (3.22), and (3.23).

Assume that $c_q \leq D - l$, then Ineq. (3.16) implies that $1 \leq c_i \leq D - l$ for all $i \in [q]$. Thus, Ineqs. (3.17), (3.21), and (3.13) imply that

$$(D - l) \cdot (s + 2) \;\leq\; (D - l) \cdot q \;\leq\; \sum_{i=1}^{q} |\partial_H C_i| \;\leq\; (D - l + 1) \cdot s.$$

Simplifying it, and by using Ineq. (3.10), we find $s \geq 2(D - l) \geq n/2$, contradicting Ineq. (3.19). Therefore, we have $c_q \geq D - l + 1$. By using Ineq. (3.10) again, we can deduce

$$c_q \;\geq\; D - l + 1 \;\geq\; \frac{n}{4} + 1. \tag{3.24}$$

Together with Eq. (3.18) and Ineq. (3.13), we infer that

$$n \;=\; s + \sum_{i=1}^{q-1} c_i + c_q \;\geq\; s + (q - 1) + \left(\frac{n}{4} + 1\right) \;\geq\; 2s + \frac{n}{4} + 2,$$

that is,

$$s \;\leq\; \frac{3n}{8} - 1. \tag{3.25}$$

Below we will handle the cases $s = 1$, $s \geq 2$, and $s = 0$, individually. As will be seen, the case $s = 1$ is relatively easy, the case $s = 2$ implies that $s \geq \lceil n/4 \rceil$, and the case $s = 0$ is proved to be reducible to the previous cases.

Case 1. $s \geq 2$.

First, we show that $s \geq \lceil n/4 \rceil$ in this case, and figure out some basic relation among the parameters.

Claim 1.1. Suppose that $s \;\geq\; 2$. Then we have

(i) $s \geq D - l \geq \lceil n/4 \rceil$;

(ii) $q = s + 2$;

(iii) $c_i = 1$ for $i \in [q-1]$;

(iv) $c_q = n - 2s - 1 \in [\, n/4 + 1,\, n/2 - 1\,]$.

(v) $|\partial_H C_q| \leq s + l - D$, and the subgraph C_q is Hamiltonian-connected.

We shall show the above results one by one.

(i). In order to show the desired lower bound $D-l$ of the number s, we suppose, to the contrary, that $s < D-l$. If the component C_1 consists of a single vertex, then all neighbors of this vertex lie in the set S. As a consequence, by Ineq. (3.12), the set S contains at least $D-l$ vertices, a contradiction. Note that all the components C_i are of odd order. Therefore, we have

$$c_1 \;\geq\; 3. \tag{3.26}$$

It will be used to judge the condition when we apply Ineqs. (3.22) and (3.23).

From Ineq. (3.13), we see that $q \geq 4$. Thus the notation C_{q-3} is well defined. Assume that $c_{q-3} \geq D - l$. By Eq. (3.18) and Ineqs. (3.16) and (3.24), we have

$$n \;\geq\; c_{q-3} + c_{q-2} + c_{q-1} + c_q \;\geq\; 3(D-l) + (D-l+1),$$

contradicting Ineq. (3.10). Thus, we have $C_{q-3} \leq D - l - 1$. Together with Ineq. (3.26), we find

$$3 \;\leq\; c_i \;\leq\; D - l - 1, \qquad \text{for all } i \in [q-3]. \tag{3.27}$$

Therefore, by using Ineq. (3.22), we can deduce from Ineq. (3.17) that

$$(D-l+1)s \;\geq\; \sum_{i=1}^{q} |\partial_H C_i| \;\geq\; \sum_{i=1}^{q-3} |\partial_H C_i| \;\geq\; 2(D-l-1)(q-3). \tag{3.28}$$

Assume that $q \geq s + 3$. Then Ineq. (3.28) implies $D - l + 1 \;\geq\; 2(D-l-1)$, contradicting Ineq. (3.11). This proves that $q \leq s + 2$. In view of Ineq. (3.13), we derive that $q = s + 2$. Consequently, Ineq. (3.28) implies that

$$s \;\leq\; 2\left(1 + \frac{2}{D-l-3}\right) \;\leq\; \frac{8}{3}.$$

Therefore, we find $s = 2$ and $q = 4$.

Assume that $c_1 \leq D - l - 2$. By using Ineqs. (3.22) and (3.23), we can deduce from Ineq. (3.17) that

$$2(D-l+1) \;\geq\; |\partial_H C_1| \;\geq\; 3(D-l-2),$$

contradicting Ineq. (3.11). From Ineq. (3.27), we deduce that

$$c_1 \;=\; D-l-1 \;\geq\; \left\lceil \frac{n}{4} \right\rceil - 1.$$

In view of Eq. (3.18) that $n-2=\sum_{i=1}^{4} c_i$, we find

$$c_1 \;=\; c_2 \;=\; c_3 \;=\; c_4 \;=\; \frac{n-2}{4},$$

contradicting Ineq. (3.24). This completes the proof of the lower bound part $s \geq D-l$ in Claim 1.1 (i). By Ineq. (3.10) again, we obtain $s \geq \lceil n/4 \rceil$ immediately.

(ii). Note that Eq. (3.18) and Ineqs. (3.13) and (3.16) give that

$$n \;=\; s+\sum_{i=1}^{q-2} c_i + (c_{q-1}+c_q) \;\geq\; s+(q-2)+2c_{q-1} \;\geq\; 2(s+c_{q-1}). \tag{3.29}$$

Together with the inequality $s \geq D-l$ confirmed in Claim 1.1 (i), and Ineq. (3.10), we find that

$$c_{q-1} \;\leq\; \frac{n}{2}-D+l \;\leq\; D-l.$$

Therefore, Ineqs. (3.17) and (3.21) give

$$(D-l+1)s \;\geq\; \sum_{i=1}^{q} |\partial_H C_i| \;\geq\; \sum_{i=1}^{q-1} |\partial_H C_i| \;\geq\; (D-l)(q-1),$$

which can be recast as $(D-l)(q-s-1) \leq s$. By using Ineq. (3.19), we infer that

$$q-s-1 \;\leq\; \frac{s}{D-l} \;\leq\; \frac{n/2-1}{n/4} \;<\; 2.$$

It follows that $q \leq s+2$. In view of Ineq. (3.13), we derive that $q = s+2$.

(iii). Suppose to the contrary that $c_{q-1} \geq 3$.

If $c_{q-1} \leq D-l-1$, then Ineqs. (3.17), (3.21) and (3.22) yield that

$$(D-l+1)s \;\geq\; \sum_{i=1}^{q-2} |\partial_H C_i| + |\partial_H C_{q-1}| \;\geq\; (D-l)s + 2(D-l-1),$$

that is, $s \geq 2(D-l-1) \geq n/2-2$. Therefore, Ineq. (3.29) implies $n \geq 2(s+3) \geq n+2$, a contradiction. Therefore, we have $c_{q-1} \geq D-l$. Together with Claim 1.1 (i) that $s \geq D-l$, we see that all the equalities in Ineq. (3.29) hold true. In particular, one has $c_q = n/4$, contradicting Ineq. (3.24). This confirms Claim 1.1 (iii).

(iv). Now, by Claim 1.1 (ii) and (iii), Eq. (3.18) reduces to

$$n \;=\; s+(q-1)+c_q \;=\; 2s+1+c_q,$$

which gives the desired formula for c_q. By using Ineq. (3.10) and using $s \ge D - l$ from Claim 1.1 (i), we find the desired upper bound $n/2-1$ of c_q. The lower bound has been shown in Ineq. (3.24). This proves Claim 1.1 (iv).

(v). From Claim 1.1 (iii) and Ineq. (3.21), we infer that $|\partial_H C_i| \ge D - l$ for all $i \in [q-1]$. Together with Claim 1.1 (i), (ii), and Ineq. (3.17), we deduce that

$$|\partial_H C_q| \ \le\ (D-l+1)s - (q-1)(D-l) \ =\ s + l - D.$$

Together with Ineq. (3.12) and Claim 1.1 (i) and (iv), we infer that

$$\delta(C_q) \ \ge\ \delta_H(C_q) - |\partial_H C_q| \ \ge\ (D-l) - (s+l-D) \quad \frac{c_q}{2}.$$

By Theorem 2.7, the subgraph C_q is Hamiltonian-connected.

This completes the proof of Claim 1.1. □

Claim 1.2. There exists a matching $M_0 \in \mathcal{M}$ such that

$$|\partial_{M_0} C_q| \ \ge\ 3. \tag{3.30}$$

By Claim 1.1 (iv), we see that $n/4 + 1 \le c_q \le D$. Therefore, Ineq. (3.20) implies that

$$|\partial_G C_q| \ \ge\ c_q(D - c_q + 1) \ \ge\ D.$$

Assume that $|\partial_M C_q| \le 1$ for all $M \in \mathcal{M}$ (if $\mathcal{M} \ne \emptyset$). By using Claim 1.1 (v), we deduce

$$s - D + l \ \ge\ |\partial_H C_q| \ =\ |\partial_G C_q| - \sum_{M \in \mathcal{M}} |\partial_M C_q| \ \ge\ D - l,$$

which implies that $s \ge n/2$ by Ineq. (3.10), contradicting Ineq. (3.19). Hence, there exists a matching $M_0 \in \mathcal{M}$ such that $|\partial_{M_0} C_q| \ge 2$. Since the component C_q is of odd order, the cardinality $|\partial_M C_q|$ is odd for all matchings M. Thus $|\partial_{M_0} C_q| \ge 3$. This proves Claim 1.2. □

Denote $U = \cup_{i=1}^{q-1} V(C_i)$. From Claim 1.1 (iii), we see that the set U consists of $(s+1)$ isolated vertices in the graph H. Now the graph H has three parts S, U, and C_q. Denote by F the bipartite graph with vertex parts S and U, and with edge set $E_H(S, U)$. It can be obtained alternatively from the graph $H - C_q$ by removing the edges among vertices in the set S.

By Claim 1.2, we can take a matching $M_0 \in \mathcal{M}$ subject to Ineq. (3.30). Since the perfect matching M_0 covers the vertices of the set U, we have

$$s + 1 \ =\ |U| \ =\ e_{M_0}(U, S) + e_{M_0}(U, C_q) + 2e_{M_0}(U, U). \tag{3.31}$$

For the same reason, we have

$$s \ =\ e_{M_0}(S, U) + e_{M_0}(S, C_q) + 2e_{M_0}(S, S) \ \ge\ e_{M_0}(S, U) + e_{M_0}(S, C_q). \tag{3.32}$$

Subtracting Eq. (3.31) from Ineq. (3.32), and by using Ineq. (3.30), we obtain

$$-1 \;\ge\; e_{M_0}(S, C_q) - e_{M_0}(U, C_q) - 2e_{M_0}(U, U) \;\ge\; 3 - 2e_{M_0}(U, C_q) - 2e_{M_0}(U, U).$$

It follows that

$$e_{M_0}(U, U) \;\ge\; 2 - e_{M_0}(U, C_q). \tag{3.33}$$

Below we have three subcases to treat. In each of them, we will apply Lemma 3.9 twice, taking $k \in \{0, 1\}$ and $d \in \{D - l,\, D - l - 1\}$. Here we verify the condition $d \ge (s + k)/2 + 1$ and $s \ge k + 1$, as

$$D - l - 1 \;\ge\; \frac{s+1}{2} + 1 \quad \text{and} \quad s \;\ge\; 2 \;\ge\; k + 1, \tag{3.34}$$

whose truth can be seen from Ineqs. (3.10), (3.25), and (3.11) directly. In this way, we obtain two disjoint perfect matchings in the graph $H \cup M_0$, contradicting the choice the family $\mathcal{M}$.

Subcase 1.1. Suppose that $e_{M_0}(U, C_q) \ge 2$.

Let $e_{21}, e_{22} \in E_{M_0}(U, C_q)$. Note that we use the first subscript 2 to indicate we are in the subcase with the assumption $e_{M_0}(U, C_q) \ge 2$. See Fig. 3.2.

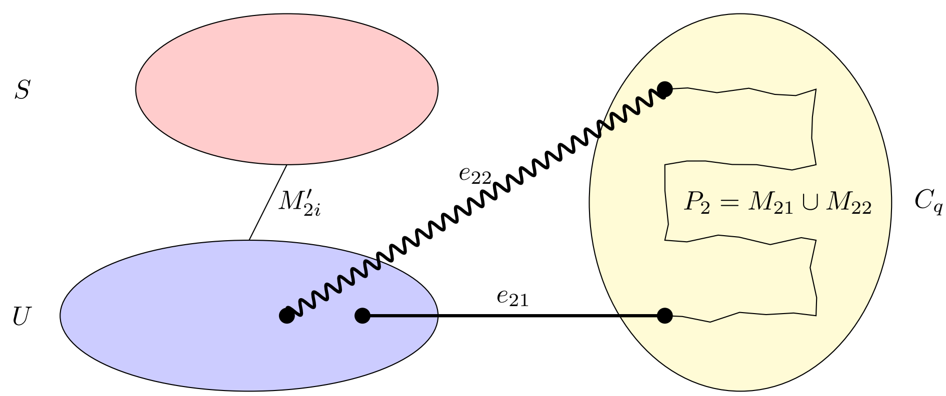

Figure 3.2: The perfect matchings $M_{21} \cup M'_{21} \cup \{e_{21}\}$ and $M_{22} \cup M'_{22} \cup \{e_{22}\}$.

By Claim 1.1 (v), the component C_q has a Hamiltonian path, say, P_2, from the vertex $V(e_{21}) \cap V(C_q)$ to the vertex $V(e_{22}) \cap V(C_q)$. For $i = 1, 2$, since the path $P_2 - V(e_{2i})$ has an even number of vertices, it has a unique perfect matching, say, M_{2i}.

In Lemma 3.9, we take

$$d = D - l, \quad k = 0, \quad G' = F, \quad S' = \emptyset, \quad \text{and} \quad U' = V(e_{21}) \cap U.$$

In the graph F, by Ineq. (3.12), every vertex in the set S has degree at most $(D - l + 1)$, and the minimum degree $\delta_F(U)$ is at least $(D - l)$. In view of Ineq. (3.34), we infer from Lemma 3.9 that the graph $F-V(e_{21})$ has a perfect matching, say, M'_{21}. Now, we take

$$d=D-l-1,\quad k=0,\quad G'=F-M'_{21},\quad S'=\emptyset,\quad \text{and}\quad U'=V(e_{22})\cap U.$$

Consider the graph $F-M'_{21}$. Since the matching M'_{21} is perfect, by Ineq. (3.12), every vertex in the set S has degree at most $(D-l)$, and that the minimum degree $\delta_{F-M'_{21}}(U)$ is at least $(D-l-1)$. Again, Lemma 3.9 provides a perfect matching M'_{22} of the graph $F-V(e_{22})-M'_{21}$.

From definition, we obtain two disjoint perfect matchings

$$M''_{2i} \;=\; M_{2i}\cup M'_{2i}\cup\{e_{2i}\}\qquad (i=1,2),$$

of the graph $H\cup M_0$. As a consequence, the family $(\mathcal{M}-M_0)\cup\{M''_{21},\,M''_{22}\}$ consists of $(l+1)$ disjoint perfect matchings, contradicting the choice of the family $\mathcal{M}$.

Subcase 1.2. Suppose that $e_{M_0}(U,C_q)=0$.

In this case, by Ineq. (3.30), we have $e_{M_0}(S,C_q)\geq 3$. Thus we can choose two edges $e_{01},e_{02}\in E_{M_0}(S,C_q)$. See Fig. 3.3.

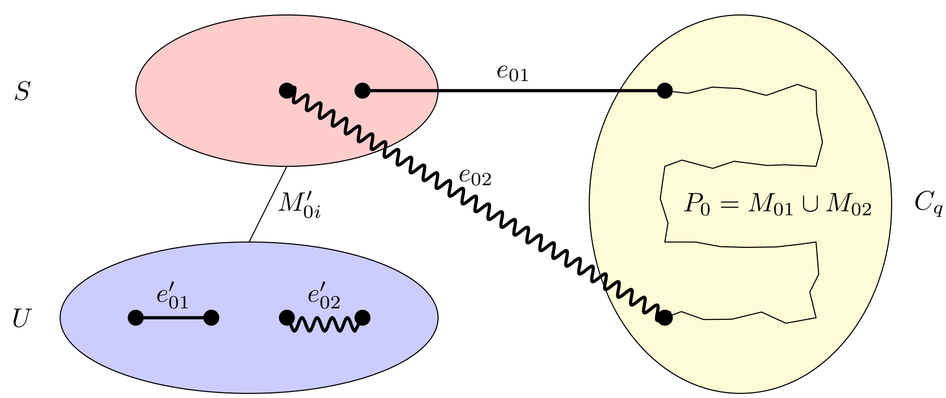

Figure 3.3: The perfect matchings $M_{0i}\cup M'_{0i}\cup\{e_{0i},\,e'_{0i}\}$ $(i=1,2)$.

By Claim 1.1 (v), the component C_q has a Hamiltonian path, say, P_0, from the vertex $V(e_{01})\cap V(C_q)$ to the vertex $V(e_{02})\cap V(C_q)$. Same to Subcase 1.1, for $i=1,2$, we denote by M_{0i} the unique perfect matching of the path $P_0-V(e_{0i})$. From Ineq. (3.33), we infer that $e_{M_0}(U,U)\;\geq\;2$. Thus, we can pick edges $e'_{01},e'_{02}\in E_{M_0}(U,U)$. In Lemma 3.9, we take

$$d=D-l,\quad k=1,\quad G'=F,\quad S'=V(e_{01})\cap S,\quad \text{and}\quad U'=V(e'_{01}).$$

Same to Subcase 1.1, the graph $F-V(e_{01})-V(e'_{01})$ has a perfect matching, say, M'_{01}. Then, we take

$$d=D-l-1,\quad k=1,\quad G'=F-M'_{01},\quad S'=V(e_{02})\cap S,\quad \text{and}\quad U'=V(e'_{02}).$$

Note that in the graph $F - M'_{01}$, the vertex in the set $V(e_{01}) \cap S$ has degree at most $(D - l + 1)$, every other vertex in the set S has degree at most $(D - l)$, and that the minimum degree $\delta_{F-M'_{01}}(U)$ is at least $(D - l - 1)$. Again, Lemma 3.9 offers a perfect matching M'_{02} of the graph $F - V(e_{02}) - V(e'_{02})$. From definition, we obtain two disjoint perfect matchings $M_{0i} \cup M'_{0i} \cup \{e_{0i}, e'_{0i}\}$ $(i = 1, 2)$ of the graph $H \cup M_0$, the same contradiction as in Subcase 1.1.

Subcase 1.3. Suppose that $e_{M_0}(U, C_q) = 1$.

In this case, we can choose an edge $e_{11} \in E_{M_0}(U, C_q)$. See Fig. 3.4.

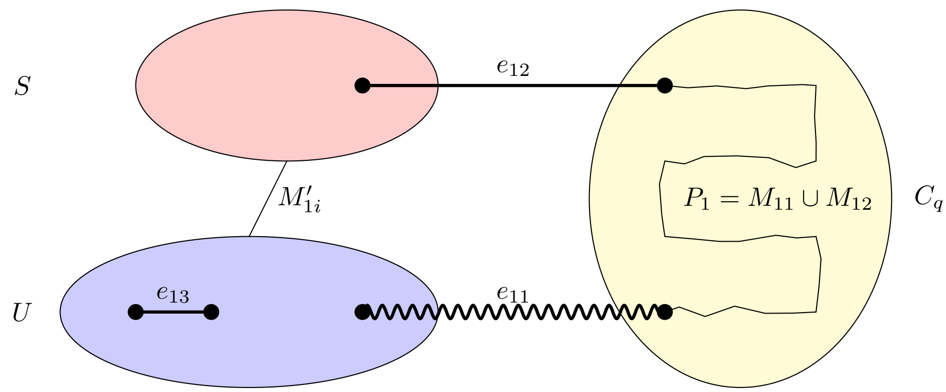

Figure 3.4: The perfect matchings $M_{11} \cup M'_{11} \cup \{e_{11}\}$ and $M_{12} \cup M'_{12} \cup \{e_{12}, e_{13}\}$.

From Ineq. (3.30), we infer that $e_{M_0}(C_q, S) \geq 2$, which allows us to pick an edge $e_{12} \in E_{M_0}(C_q, S)$ such that $V(e_{11}) \cap V(e_{12}) = \emptyset$. Same to Subcase 1.1, let P_1 be a Hamiltonian path from the vertex $V(e_{11}) \cap V(C_q)$ to the vertex $V(e_{12}) \cap V(C_q)$. Denote by M_{1i} the perfect matching of the path $P_1 - V(e_{1i})$ for $i = 1, 2$. Taking

$$d = D - l, \quad k = 0, \quad G' = F, \quad S' = \emptyset, \quad \text{and} \quad U' = V(e_{11}) \cap U,$$

we infer from Lemma 3.9 that the graph $F - V(e_{11})$ has a perfect matching, say, M'_{11}. By Ineq. (3.33), we have $e_{M_0}(U, U) \geq 1$. Let $e_{13} \in E_{M_0}(U, U)$. Then, we put

$$d = D - l - 1, \quad k = 1, \quad G' = F - M'_{11}, \quad S' = V(e_{12}) \cap S, \quad \text{and} \quad U' = V(e_{13}).$$

Again, Lemma 3.9 results in a perfect matching M'_{12} of the graph $F - V(e_{11}) - V(e_{12}) - V(e_{13})$. From definition, we obtain two disjoint perfect matchings

$$M_{11} \cup M'_{11} \cup \{e_{11}\} \quad \text{and} \quad M_{12} \cup M'_{12} \cup \{e_{12}, e_{13}\}$$

are disjoint perfect matchings of the graph $H \cup M_0$, the same contradiction.

This completes the proof for Case 1.

Case 2. $s = 1$.

Before dealing with the other cases $s = 1$ and $s = 0$, we give some common properties for these two cases. Let $j \in [q]$. Every vertex in the subgraph $H[C_j]$ has at most s neighbors outside C_j. Therefore, by Ineq. (3.12), every vertex in $H[C_j]$ has at least $(D - l - s)$ neighbors inside C_j. In other words,

$$\delta(C_j) \;\geq\; D - l - s \;\geq\; \left\lceil \frac{n}{4} \right\rceil - s. \tag{3.35}$$

It follows that

$$c_j \;\geq\; \delta(C_j) + 1 \;\geq\; D - l - s + 1 \;\geq\; \left\lceil \frac{n}{4} \right\rceil - s + 1. \tag{3.36}$$

From Eq. (3.18) and that $s \in \{0, 1\}$, we have

$$n \;=\; s + \sum_{j=1}^{q} c_j \;\geq\; s + q \cdot \left(\frac{n}{4} - s + 1 \right) \quad q \cdot \frac{n}{4}.$$

It follows that $q \leq 3$. From Ineq. (3.13) and Eq. (3.14), we infer that

$$q \;=\; s + 2. \tag{3.37}$$

From Claim 1.2, we see that the graph G has a perfect matching if $s \geq 2$. In fact, this is also true for $s \in \{0, 1\}$.

Claim 2.1. Let $s \in \{0, 1\}$. Then the graph G has a perfect matching, i.e., we have $l \geq 1$.

By Eq. (3.18) and Ineqs. (3.16), (3.35), and (3.37), we find

$$n \;=\; s + \sum_{i=1}^{q} c_i \;\geq\; s + (s + 2) \cdot c_1 \;\geq\; s + (s + 2) \cdot (D - l - s + 1). \tag{3.38}$$

Assume that $l = 0$. For $s = 1$, Ineq. (3.38) implies $n \geq 1 + 3D \geq 1 + 3(n/2 - 1)$, contradicting $n \geq 34$. For $s = 0$, Ineq. (3.38) implies $n \geq 2(D + 1) = 4\lceil n/4 \rceil \geq n$. Thus the equality in Ineq. (3.38) holds. In particular, we have $n \equiv 0 \pmod 4$ and $c_1 = D + 1 = n/2$ is even, contradicting Eq. (3.15). This proves Claim 2.1. □

From Eq. (3.37), we have $q = 3$. We rename the components C_1, C_2, and C_3 by T_1, T_2, and T_3, so that

$$e_H(S, T_3) \;=\; \max_{1 \leq i \leq 3} e_H(S, C_i). \tag{3.39}$$

Denote $|T_i| = t_i$. This case $s = 1$ will be handled by presenting a family of disjoint perfect matchings larger than $\mathcal{M}$. To do this, we will discover a matching $M \in \mathcal{M}$ such that the graph $H \cup M$ has two disjoint perfect matchings. Claims 2.2 and 2.3 will be of use.

Claim 2.2. We have

$$\left\lceil \frac{n}{4} \right\rceil + 1 \;\le\; t_i \;\le\; \frac{n}{2} - 3, \qquad \text{for } i = 1, 2, \qquad \text{and}$$
$$\left\lceil \frac{n}{4} \right\rceil \;\le\; t_3 \;\le\; \frac{n}{2} - 3.$$

As a consequence, every component T_j $(j = 1, 2, 3)$ is Hamiltonian-connected.

From Ineq. (3.36), we obtain the desired lower bound of t_3 directly. Assume that $t_i = \lceil n/4 \rceil$ for some $i \in \{1, 2\}$. Let $S = \{v^*\}$. By Ineq. (3.12), every vertex in the component T_i is a neighbor of the vertex v^*. Thus $e_H(S, T_i) \ge t_i$. Therefore, by Ineq. (3.39), we have

$$\deg_H(v^*) \;=\; \sum_{j=1}^{3} e_H(S,\, T_j) \;\ge\; e_H(S, T_i) + e_H(S, T_3) \;\ge\; 2t_i \;=\; 2\left\lceil \frac{n}{4} \right\rceil.$$

By Ineq. (3.12), we find $l = 0$, contradicting Claim 2.1. Hence, both integers t_1 and t_2 have the lower bound $\lceil n/4 \rceil + 1$.

By the lower bounds of t_i that just obtained, we infer that

$$t_3 \;=\; |G - S - T_1 - T_2| \;\le\; n - 1 - \Big(\frac{n}{4} + 1\Big) - \Big(\frac{n}{4} + 1\Big) \;=\; \frac{n}{2} - 3,$$

the desired upper bound of t_3. Along the same line, we have

$$t_1 \;=\; |G - S - T_2 - T_3| \;\le\; n - 1 - \Big(\frac{n}{4} + 1\Big) - \frac{n}{4} \;=\; \frac{n}{2} - 2.$$

If $t_1 = n/2 - 2$, i.e., if the equality in the above inequality holds, then $t_2 = n/4 + 1$ and $t_3 = n/4$, having different parities. But this is impossible since the order of every component T_i has odd parity. This confirms the desired upper bound of t_1. The desired upper bound of t_2 can be shown in the same fashion.

Let $j \in [3]$. By Ineq. (3.35), we have

$$2\delta(T_j) \;\ge\; 2\Big(\frac{n}{4} - 1\Big) \;\ge\; t_j + 1.$$

By Theorem 2.7, every component T_j is Hamiltonian-connected. This proves Claim 2.2. □

Claim 2.3. There is a matching $M \in \mathcal{M}$ such that $e_M(T_1, T_2) \ge 2$.

We estimate the number of edges between the sets $T_1 \cup T_2$ and $S \cup T_3$. On the one side, from Ineqs. (3.12) and (3.39), we infer that

$$|\partial_H(S \cup T_3)| \;=\; \sum_{i=1}^{2} e_H(S, T_i) \;\le\; \frac{2}{3} \sum_{i=1}^{3} e_H(S, T_i) \;=\; \frac{2}{3} \deg_H(v^*) \;\le\; \frac{2}{3}(D + 1 - l).$$

Therefore, we have

$$|\partial_G(S\cup T_3)| \;=\; |\partial_H(S\cup T_3)|+|\partial_{G-E(H)}(S\cup T_3)| \;\le\; |\partial_H(S\cup T_3)|+|S\cup T_3|\cdot|\mathcal{M}|$$
$$\le\; \frac{2}{3}(D+1-l)+(n-t_1-t_2)\cdot l. \tag{3.40}$$

On the other hand, assume that Claim 2.3 is false. Then $e_M(T_1,T_2)\le 1$ for every matching $M\in\mathcal{M}$. It follows that

$$e_G(T_1,T_2) \;=\; e_H(T_1,T_2)+e_{G-E(H)}(T_1,T_2) \;=\; 0+\sum_{M\in\mathcal{M}} e_M(T_1,T_2) \;\le\; |\mathcal{M}| \;=\; l.$$

Therefore, we have

$$|\partial_G(T_1\cup T_2)| \;=\; \sum_{v\in T_1\cup T_2}\deg_G(v)-\sum_{i=1}^{2}\sum_{v\in T_i}\deg_{T_i}(v)-2e_G(T_1,T_2)$$
$$\ge\; D\cdot(t_1+t_2)-\sum_{i=1}^{2}t_i(t_i-1)-2l. \tag{3.41}$$

Combining Ineqs. (3.41) and (3.40) with the identity $\partial(T_1\cup T_2)=\partial(S\cup T_3)$, we infer that

$$\frac{2}{3}(D+1-l)+(n-t_1-t_2)\cdot l-\Bigl(D\cdot(t_1+t_2)-\sum_{i=1}^{2}t_i(t_i-1)-2l\Bigr) \;\ge\; 0. \tag{3.42}$$

Since the coefficient of l in the left hand side of (3.42) is $-2/3+(n-t_1-t_2)+2>0$, and since the coefficient of D in the left hand side is $2/3-(t_1+t_2)<0$, we can substitute l by its upper bound $(n-2)/4$, and substitute D by its lower bound $n/2-1$ into Ineq. (3.42), which gives

$$f(t_1)+f(t_2)+\Bigl(\frac{n^2}{4}+\frac{n}{6}-\frac{2}{3}\Bigr) \;\ge\; 0, \tag{3.43}$$

where

$$f(t) \;=\; t^2+\Bigl(-\frac{3n}{4}+\frac{1}{2}\Bigr)t.$$

From the domain of t_i $(i=1,2)$ obtained in Claim 2.2, and since $n\ge 34$, it is elementary to derive that the quadratic function $f(t_i)$ has upper bound $f(n/4+1)$.

From Ineq. (3.43), we obtain

$$2f\Bigl(\frac{n}{4}+1\Bigr)+\Bigl(\frac{n^2}{4}+\frac{n}{6}-\frac{2}{3}\Bigr) \;\ge\; 0,$$

which reduces to $n\le 28$, a contradiction to the premise $n\ge 34$. This proves Claim 2.3. □

By Claim 2.3, we can suppose that $e_1, e_2 \in E_M(T_1, T_2)$. By Claim 2.2, the component T_i has a Hamiltonian path P_i from the vertex $V(T_i) \cap V(e_1)$ to the vertex $V(T_i) \cap V(e_2)$. Thus we obtain a Hamiltonian cycle $\mathcal{C}_1 = (P_1,\, e_2,\, P_2,\, e_1)$ of the subgraph $T_1 \cup T_2 \cup \{e_1, e_2\}$. Since both the orders t_1 and t_2 are odd, the length $(t_1 + t_2)$ of the cycle $\mathcal{C}_1$ is even. See Fig. 3.5.

On the other hand, from Ineqs. (3.39) and (3.12), we have

$$e_H(S, T_3) \;\geq\; \frac{1}{3}\deg_H(v^*) \;\geq\; \frac{1}{3}\left\lceil \frac{n}{4} \right\rceil.$$

Since $n \geq 34$, we have $e_H(S, T_3) \geq 3$. Let v_{31} and v_{32} be two neighbors of the vertex v^* in the component T_3. By Claim 2.2 again, the component T_3 has a Hamiltonian path P_3 from the vertex v_{31} to the vertex v_{32}. This gives a Hamiltonian cycle $\mathcal{C}_2 = (P_3,\, v_{32}v^*v_{31})$ of the subgraph $H[S \cup V(T_3)]$. Since the order t_3 is odd, the length $t_3 + 1$ of the cycle $\mathcal{C}_2$ is even.

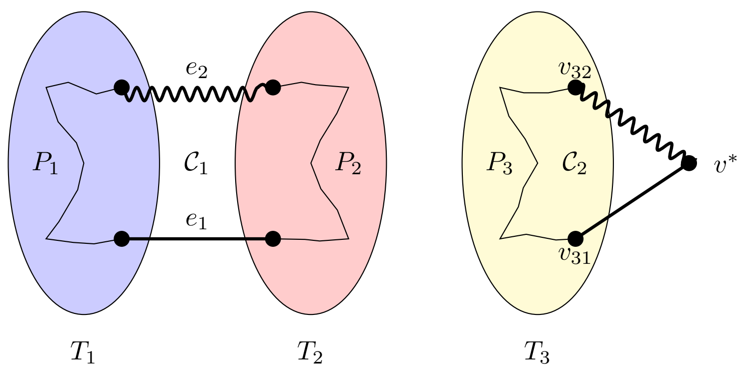

Figure 3.5: The perfect matching union $M_1 \cup M_2$.

Note that the union of the even cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ can be decomposed into two disjoint perfect matchings, say, M_1 and M_2, of the graph $H \cup M$. Then the family $(\mathcal{M} \cup \{M_1, M_2\}) - M$ consists of $(l + 1)$ disjoint perfect matchings, contradicting the choice of $\mathcal{M}$. This completes the proof for Case 2.

Case 3. $s = 0$.

From Eq. (3.37), we infer that $q = 2$. In other words, the graph H consists of factor-critical components C_1 and C_2. Claim 3.1 will be used several times for solving Case 3.

Claim 3.1. For any matching $M \in \mathcal{M}$ and for any perfect matching M' of the graph $H \cup M$, the graph $(H \cup M) - M'$ consists of two factor-critical components of orders at least $\lceil n/4 \rceil + 1$.

Let $M \in \mathcal{M}$, and let M' be a perfect matching of the graph $H \cup M$. From the choice of the family $\mathcal{M}$, we infer that the subgraph $(H \cup M) - M'$ has no perfect matchings. By Theorem 2.5, there is a vertex set S' such that the graph $H' - S'$ consists of q' factor-critical components. If $S' \neq \emptyset$, then one may consider the family $(\mathcal{M} - M) \cup \{M'\}$ of disjoint perfect matchings instead of the family $\mathcal{M}$, as in the previous proofs for Cases 1 and 2. Therefore, we can suppose that $S' = \emptyset$. Along the same lines, we are led to $q' = 2$. In analog with Ineq. (3.36), we find each component has order at least $\lceil n/4 \rceil + 1$. This proves Claim 3.1.

From Ineq. (3.20), we infer that

$$\sum_{M \in \mathcal{M}} e_M(C_1, C_2) \;=\; e_G(C_1, C_2) \;\geq\; c_i(D - c_i + 1) \;=\; c_i \cdot \left(2\left\lceil \frac{n}{4} \right\rceil - c_i\right). \tag{3.44}$$

Since $c_1 \leq c_2$, we have $c_1 \leq n/2$. If $c_1 = n/2$, then the integer $n/2$, as the order of the factor-critical component, is odd. Then Ineq. (3.44) becomes

$$\sum_{M \in \mathcal{M}} e_M(C_1, C_2) \;\geq\; c_i \cdot \left(\frac{n}{2} + 1 - c_i\right) \;=\; \frac{n}{2}.$$

Otherwise, by Ineq. (3.36), we have $n/4+1 \leq c_1 \leq n/2-1$. In this case, Ineq. (3.44) implies

$$\sum_{M \in \mathcal{M}} e_M(C_1, C_2) \;\geq\; c_i \cdot \left(\frac{n}{2} - c_i\right) \;\geq\; \frac{n}{2} - 1.$$

Anyway, the sum on the left hand side of Ineq. (3.44) is at least $n/2 - 1$. Consequently, by Claim (2.1) that $l \geq 1$, and by the assumption $l \leq \lceil n/4 \rceil - 1$, there exists a matching $M_0 \in \mathcal{M}$ such that

$$e_{M_0}(C_1, C_2) \;\geq\; \frac{n/2 - 1}{l} \;\geq\; 2.$$

Since the order c_1 is odd, and the matching M_0 is perfect, the integer $e_{M_0}(C_1, C_2)$ must be odd. Thus, the above lower bound can be enhanced to

$$e_{M_0}(C_1, C_2) \;\geq\; 3. \tag{3.45}$$

Let $e_0 \in e_{M_0}(C_1, C_2)$. Since each of the components C_i is factor-critical, the subgraph $C_i - V(e_0)$ has a perfect matching, say, M_{0i}. Thus, the graph $H \cup M_0$ has the perfect matching

$$M_0' \;=\; M_{01} \cup M_{02} \cup \{e_0\}.$$

We further denote

$$\begin{aligned} H' &\;=\; (H \cup M_0) - M_0', \qquad \text{and} \\ F &\;=\; H' \cup M_0' \;=\; H \cup M_0. \end{aligned}$$

By Claim 3.1, we can suppose that the graph H' consists of factor-critical components C_1' and C_2', such that

$$\frac{n}{4} + 1 \;\leq\; |C_1'| \;\leq\; |C_2'|. \tag{3.46}$$

Denote

$$V_{ij} \;=\; V(C_i) \cap V(C'_j).$$

From Ineq. (3.45) and the definition of the matching M'_0, one may verify Condition (3.4) directly. Thus, by Lemma 3.10, we infer that

$$V(C'_1) \;\subset\; V(C_2). \tag{3.47}$$

On the other hand, from Ineqs. (3.36) and (3.46), we infer that

$$|V_{22}| \;=\; n - c_1 - |C'_1| \;\le\; n - \Big(\frac{n}{4} + 1\Big) - \Big(\frac{n}{4} + 1\Big) \;=\; \frac{n}{2} - 2. \tag{3.48}$$

From Ineqs. (3.7) and (3.48), we infer that

$$\delta_H(V_{22}) \;\ge\; \frac{n}{4} - 1 \;\ge\; \frac{|V_{22}|}{2}. \tag{3.49}$$

From Relation (3.47), we see that $V_{22} \ne \emptyset$. By Ineq. (3.8) and the premise $n \ge 34$, we find $|V_{22}| \ge 9$. By Dirac's Theorem 2.6, we conclude that the subgraph $H[V_{22}]$ is Hamiltonian. Let H_{22} be a Hamiltonian cycle of the subgraph $H[V_{22}]$.

We will find another perfect matching in the graph F in Claim 3.3, based on Claim 3.2.

Claim 3.2. The graph F contains two edges

$$e_1 \in E_{M_0 - e_0}(C_1,\, V_{22}) \quad \text{and} \quad e'_1 \in E_H(C'_1,\, V_{22}),$$

such that $V(e_1) \cap V(e'_1) = \emptyset$.

Recall that every factor-critical graph is 2-edge-connected. Since the component C_2 is factor-critical, we infer that

$$e_H(C'_1,\, V_{22}) \;\ge\; 2. \tag{3.50}$$

To show Claim 3.2, it suffices to show that

$$e_{M_0 - e_0}(C_1,\, V_{22}) \;\ge\; 2. \tag{3.51}$$

From the definition $M'_0 = M_{01} \cup M_{02} \cup \{e_0\}$, we see that

$$E_{M_0}(C_1, C_2) \cap M'_0 \;=\; \{e_0\}.$$

From the definition $H' = (H \cup M_0) - M'_0$, we can deduce that

$$E_{M_0}(C_1, C_2) - e_0 \;\subset\; E(H').$$

By Relation (3.47), we can enhanced the above relation to

$$E_{M_0}(C_1, C_2) - e_0 \;\subset\; E(C'_2).$$

Consequently, we have

$$E_{M_0}(C_1, C_2) - e_0 \;\subseteq\; E(C_2') \cap E_{M_0-e_0}(C_1, C_2) \;=\; E_{M_0-e_0}(C_1,\, V_{22}).$$

Hence, the desired Ineq. (3.51) follows from Ineq. (3.45). This proves Claim 3.2.

Let e_1 and e_1' be two edges subject to Claim 3.2. The factor-criticality of the component C_1 implies that the subgraph $C_1 - V(e_1)$ has a perfect matching, say, M_{11}, in the graph H. For the same reason, the subgraph $C_1' - V(e_1')$ has a perfect matching, say, M_{11}', in the graph H'.

Claim 3.3. The graph F has a perfect matching M'' such that

$$E_{M_0}(C_1,\, C_2) - M'' \;\neq\; \emptyset \quad \text{and} \tag{3.52}$$

$$E_{M_0'}(C_1',\, C_2') - M'' \;\neq\; \emptyset. \tag{3.53}$$

We will treat two cases according to whether the equality in Ineq. (3.48) holds or not. Assume that the equality in Ineq. (3.48) does not hold. Then the strict inequality in Ineq. (3.49) holds. By Theorem 2.8, the subgraph $H[V_{22}]$ is bi-critical. In particular, the subgraph $H[V_{22}]-V(e_1)-V(e_1')$ has a perfect matching, say, M_{12}.

Therefore, the graph F has the perfect matching $M_1' = M_{11} \cup M_{11}' \cup M_{12} \cup \{e_1,\, e_1'\}$. See Fig. 3.6.

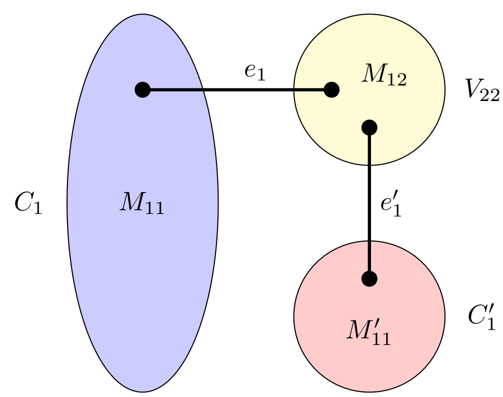

Figure 3.6: The perfect matching $M_1' = M_{11} \cup M_{11}' \cup M_{12} \cup \{e_1,\, e_1'\}$.

It follows that

$$E_{M_0}(C_1,\, C_2) \cap M_{11} \;=\; \{e_1\}, \quad \text{and} \tag{3.54}$$

$$E_{M_0'}(C_1',\, V_{22}) \cap M_{11}' \;=\; \{e_1'\}. \tag{3.55}$$

In this case, we define $M'' = M_1'$. From Ineq. (3.45) and Eq. (3.54), we obtain Ineq. (3.52). It remain to verify Ineq. (3.53). Recall from Relation (3.6) that $E_H(C_1', V_{22}) \subseteq M_0'$, we deduce that

$$E_H(C_1', V_{22}) \;\subseteq\; E_{M_0'}(C_1', V_{22}).$$

Together with Ineq. (3.50), we infer that

$$e_{M_0'}(C_1', V_{22}) \;\ge\; e_{C_2}(C_1', V_{22}) \;\ge\; 2.$$

In view of Eq. (3.55), we infer that $E_{M_0'}(C_1', V_{22}) - M_1' \neq \emptyset$. This verifies Ineq. (3.53).

Now, suppose that the equality in Ineq. (3.48) holds. Then

$$|V_{22}| \;=\; \frac{n}{2} - 2 \quad \text{and} \quad c_1 \;=\; |C_1'| \;=\; \frac{n}{4} + 1.$$

In follows that the number $n/4$ is an integer. Consider the underlying graph F. On one hand, every vertex has degree at least $n/4 + 1$. Since $\partial_F C_1 \subset M_0$, we infer that the component C_1 is isomorphic to the complete graph $K_{n/4+1}$, and that every vertex in C_1 sends an edge to the component C_2 in the matching M_0. It follows that

$$e_{M_0}(C_1,\, C_2) \;=\; \frac{n}{4} + 1. \tag{3.56}$$

Assume that $E_{M_0}(C_1, C_1') \neq \emptyset$. Then we can suppose that $e_2 \in E_{M_0}(C_1, C_1')$. Since the component C_1 is factor-critical, the subgraph $F[C_1 - V(e_2)]$ has a perfect matching, say, M_{21}. Since the component C_1' is factor-critical, the subgraph $F[C_1' - V(e_2)]$ has a perfect matching, say, M_{21}'. Let M_{22} be a perfect matching taken from the Hamiltonian cycle H_{22} of the subgraph $H[V_{22}]$. Therefore, the graph F has the perfect matching $M_2' = M_{21} \cup M_{21}' \cup M_{22} \cup \{e_2\}$. See Fig. 3.7.

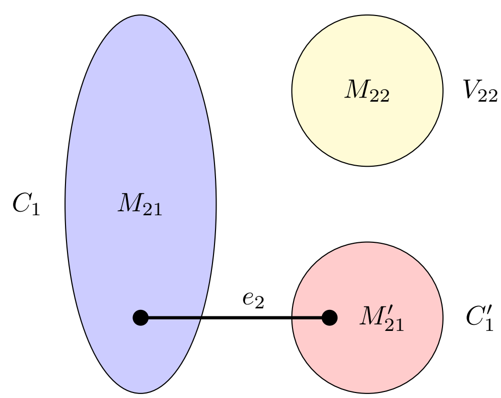

Figure 3.7: The perfect matching $M_{21} \cup M_{21}' \cup M_{22} \cup \{e_2\}$.

In this case, we define $M'' = M_2'$. By Ineq. (3.45) and the fact $M_2' \cap M_0 = \{e_2\}$, we verify Ineq. (3.52). By Ineq. (3.50) and the fact $M_2' \cap M_0' = \emptyset$, we verify Ineq. (3.53).

Otherwise, all edges with one end in the component C_1 must have the other end in the set V_{22}. By Eq. (3.56), we have $e_{M_0}(C_1, V_{22}) \ge n/4 + 1$. Recall from Claim 3.2 that $e_1' \in E_{M_0'}(C_1', V_{22})$. With the assumption $|V_{22}| = n/2 - 2$, we may choose an edge $e_3 \in E_{M_0}(C_1, V_{22})$ such that the subgraph $H_{22} - V(e_3) - V(e_1')$ consists of two paths of even orders. Consequently, the subgraph $H_{22} - V(e_3) - V(e_1')$ has a perfect matching, say, M_{32}. Since the subgraph C_1 is factor-critical, the subgraph $C_1 - V(e_3)$ has a perfect matching, say, M_{31}. Therefore, the graph F has the perfect matching $M_{31} \cup M_{11}' \cup M_{32} \cup \{e_3,\, e_1'\}$. See Fig. 3.8.

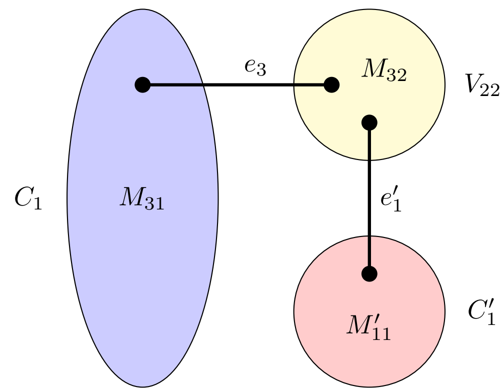

Figure 3.8: The perfect matching $M_{31} \cup M_{11}' \cup M_{32} \cup \{e_3,\, e_1'\}$.

In this case, we define $M'' = M_3$. By Ineq. (3.45) and the fact $M_3 \cap M_0 = \{e_3\}$, we verify Ineq. (3.52). By Ineq. (3.50) and the fact $M_3 \cap M_0' = \{e_1'\}$, we verify Ineq. (3.53). This proves Claim 3.3. □

Let M'' be a perfect matching of the graph F chosen subject to Ineqs. (3.52) and (3.53). By Claim 3.1, we can suppose that the graph $H'' = F - M''$ consists of the factor-critical components C_1'' and C_2'' such that

$$\left\lceil \frac{n}{4} \right\rceil + 1 \;\le\; |C_1''| \;\le\; |C_2''|. \tag{3.57}$$

Claim 3.4. We have $V(C_1'') \subseteq V_{22}$.

By Lemma 3.10 and Ineq. (3.52), we obtain

$$V(C_1'') \;\subset\; V(C_2). \tag{3.58}$$

On the other hand, we apply Lemma 3.10 by replacing the triple $(H,\, M,\, M')$ in its statement by the triple $(H',\, M_0',\, M'')$. Let us check the conditions of Lemma 3.10 one by one. First, from the definition $H' = (H \cup M_0) - M_0'$, the graph H' has minimum degree $\delta(H) \ge \lceil n/4 \rceil$, consists of factor-critical components C_1' and C_2' with $|C_1'| \le |C_2'|$, and has no intersection with the perfect matching M_0'. Second, from definition, the graph

$$(H' \cup M_0') - M'' \;=\; F - M'' \;=\; H''$$

consists of factor-critical components C_1'' and C_2'' with $|C_1''| \le |C_2''|$. Therefore, by Lemma 3.10 and Ineq. (3.53), we obtain

$$V(C_1'') \subset V(C_2'). \tag{3.59}$$

Combining Relations (3.58) and (3.59), we find

$$V(C_1'') \subseteq V(C_2) \cap V(C_2') = V_{22}.$$

This proves Claim 3.4. □

By Claim 3.4, the vertex set V_{22} is partitioned into two parts as

$$V_{22} = V(C_1'') \cup W,$$

where the vertex set W is defined by the above decomposition. Note that all the orders c_2, $|C_1'|$, and $|C_1''|$ are odd. From definition, we find the order

$$|W| = c_2 - |C_1'| - |C_1''|$$

is odd, which implies that $W \ne \emptyset$. By Relation (3.6), we have

$$E_H(W, C_1') \subseteq \partial_{C_2}(C_1') \subseteq M_0'.$$

Similarly, we have

$$E_H(W, C_1'') \subseteq \partial_{C_2}(C_1'') \subseteq M''.$$

By the above two relations, we find that every vertex in the set W has at most two neighbors outside W in the component C_2. By Ineq. (3.12), every vertex in W has degree at least $\lceil n/4 \rceil - 2$. It follows that $|W| \ge \lceil n/4 \rceil - 1$. By Ineq. (3.57), we infer that

$$|V_{22}| = |C_1''| + |W| \ge \left(\frac{n}{4} + 1\right) + \left(\frac{n}{4} - 1\right) = \frac{n}{2}, \tag{3.60}$$

contradicting Ineq. (3.48).

This completes the proof of Theorem 3.11. □

In Theorem 3.11, both the integer D_n and the bound $\lceil n/4 \rceil$ are sharp.

Sharpness of D_n. When $n/2$ is odd, consider the disjoint union of two cliques of order $n/2$. When $n/2$ is even, consider the graph obtained from the disjoint union of cliques of orders $(n/2 - 1)$ and $(n/2 + 1)$ by deleting a Hamiltonian cycle in the larger clique.

Sharpness of $\lceil n/4 \rceil$. In virtue of Theorem 3.11, it suffices to clarify the existence of a $\{D_n, D_n + 1\}$-regular graph of order n having exactly $\lceil n/4 \rceil$ disjoint perfect matchings. Let K be the complete bipartite graph with part orders $|A| = n/2 - 1$ and $|B| = n/2 + 1$. When $n/2$ is odd, such a qualified graph can be obtained from K by adding a perfect matching that covers the vertex set $V(B)$. Otherwise

$n/2$ is even. Let M be a maximal matching of the graph K. The graph obtained from the graph $K - M$ by adding a minimal edge set that covers the vertex set $V(M) - V(A)$ is qualified.

From Theorems 2.6 and 3.11, one may see the following result.

Theorem 3.12. *Let $n \geq 34$ be an even integer, and let $D \geq D_n$. Then every $\{D, D+1\}$-regular graph of order n contains $(D - \lceil n/4 \rceil + 1)$ disjoint perfect matchings.*

4. CONCLUDING REMARKS

Csaba et al.'s result is a breakthrough to the 1-factorization conjecture. Since any Hamilton cycle decomposes to a pair of edge-disjoint perfect matchings, the maximum number of edge-disjoint Hamilton cycles relates closely to the DPM problem. We remark some main relations between Csaba et al.'s results and Theorems 3.11 and 3.12.

1. The frame of Csaba et al.'s work has a global assumption, that is the sufficiently largeness of the graph order n. No clue from [5] shows how large the order n could be. In comparison, the lower bound of graph order is 34 in Theorems 3.11 and 3.12.

2. Even one is restricted to sufficiently large n only, Csaba et al.'s results do not imply Theorem 3.11. Csaba et al. showed that any graph of order n with minimum degree $\delta \geq n/2$ contains at least $N = \mathrm{reg}_{\mathrm{even}}(n, \delta)/2$ edge-disjoint Hamilton cycles, where $\mathrm{reg}_{\mathrm{even}}(n, \delta)$ denotes the degree of the largest even-regular spanning subgraph one can guarantee in a graph of order n with minimum degree δ. Such a graph has at least $2N$ DPMs. Concentrating on the DPM problem, it is still unclear that whether there is a graph with more than $2N$ DPMs.

   As to $\{D_n, D_n + 1\}$-regular graphs, this vague situation becomes completely clear by the sharpness of Theorem 3.11. In particular, the sharp bound $\lceil n/4 \rceil$ in Theorem 3.11 implies the fact that the maximum number of DPMs could be an odd integer, and thereby at least $(2N + 1)$ DPMs.

3. Yet another difference is that Csaba et al.'s result requires every vertex has degree at least $n/2$, while Theorem 3.11 allows the minimum degree to be $n/2 - 1$.

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