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CHORDAL, INTERVAL, AND CIRCULAR-ARC PRODUCT GRAPHS

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For each of the four standard products of graphs, namely the Cartesian, the strong, the direct, and the lexicographic product, we characterize when a nontrivial product of two graphs is chordal, interval, or circular-arc, respectively.

1. INTRODUCTION

A graph G is *chordal* if every cycle of length at least 4 in G has a chord, *interval* if it is the intersection graph of a family of closed intervals on the real line, and *circular-arc* if it is the intersection graph of a set of closed arcs on a circle. The classes of chordal, interval, and circular-arc graphs are well known and well studied in the literature. Every interval graph is both a chordal and a circular-arc graph; both inclusions are proper. Chordal graphs and interval graphs are subclasses of the class of perfect graphs. For more information on these graph classes we refer the reader to [**7, 14, 5, 8, 2, 15**], for example.

In this paper we consider the four standard graph products: the Cartesian product, the strong product, the direct product, and the lexicographic product. For each of these four products, we completely characterize when a nontrivial product of two graphs G and H is chordal, interval, or circular-arc, respectively. While the characterizations for chordal and interval graphs are rather straightforward and can be proved directly, the characterizations of circular-arc product graphs are more involved and are derived using characterizations of 1-perfectly orientable product graphs (for each of the four standard products) due to Hartinger and

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Milanič [**11**]. A graph is said to be 1-*perfectly orientable* if it admits an orientation such that the out-neighborhood of every vertex induces a tournament. As shown by Urrutia and Gavril [**22**] and by Skrien [**21**], respectively, the class of 1-perfectly orientable graphs generalizes both chordal graphs and circular-arc graphs.

Product graphs within various graph classes have been considered in several papers; however, complete characterizations of graph theoretic properties within all four standard products are often difficult to obtain. Ravindra and Parthasarathy [**20**] characterized Cartesian, direct, and lexicographic perfect product graphs; the Cartesian case was also studied further by de Werra and Hertz [**4**]. There is no known characterization of perfect strong product graphs; partial characterizations and sufficient conditions were obtained by Ravindra [**19**] (see also [**1**]). Characterizations of line graphs and total graphs for various products were given by Rao [**17**] and by Rao and Vartak [**18**], of modulo m well covered lexicographic product graphs by Orlovich [**16**], and of uniquely pairable Cartesian product graphs by Che [**3**]. The results of this paper contribute to the knowledge of characterizations of graph classes within graphs decomposable with respect to one the four standard graph products, by adding chordal, interval, and circular-arc graphs to the list.

The paper is organized as follows. Section 2 includes the basic definitions, notation, and preliminaries on 1-perfectly orientable product graphs and on circular-arc graphs that will be used in some of the proofs. In Sections 3, 4, 5, and 6 we deal, respectively, with Cartesian product graphs, lexicographic product graphs, direct product graphs, and strong product graphs, and state and prove the corresponding characterizations of chordal, interval, and circular-arc graphs decomposable with respect to the considered product. The corresponding theorems are Theorems 11, 13, 15, and 18, respectively.

2. PRELIMINARIES

All graphs considered in this paper are finite, simple, and undirected. The *neighborhood* of a vertex v in a graph G is the set of all vertices adjacent to v and will be denoted by $N_G(v)$ (or simply by $N(v)$ if the graph is clear from the context). The *degree* of v is the size of its neighborhood. A *leaf* in a graph is a vertex of degree 1. The *closed neighborhood* of v in G is the set $N_G(v) \cup \{v\}$, denoted by $N_G[v]$ (or simply by $N[v]$ if the graph is clear from the context). For a set $S \subseteq V(G)$, the *subgraph of G induced by S* is the graph, denoted by $G[S]$, with vertex set S and edge set $\{uv : u \in S, v \in S, uv \in E(G)\}$.

Given two graphs G and H, their *disjoint union* is the graph $G + H$ with vertex set $V(G) \cup V(H)$ (disjoint union) and edge set $E(G) \cup E(H)$. We write $2G$ for $G + G$. The *join* of two graphs G and H is the graph denoted by $G * H$ and obtained from the disjoint union of G and H by adding to it all edges joining vertex of G with a vertex of H. Given two graphs G and H and a vertex v of G, the *substitution* of v in G for H consists in replacing v with H and making each vertex of H adjacent to every vertex in $N_G(v)$ in the new graph. Two distinct vertices u and v in a graph G are said to be *true twins* if $N_G[u] = N_G[v]$. The operation of *true twin addition* to a graph G is defined as adding a new vertex w to G and making it adjacent to some vertex v of G and all its neighbors. We say that a graph G is *true-twin-free* if no pair of vertices of G are true twins. A vertex v in a graph G is *simplicial* if its neighborhood forms a clique.

A *clique* (resp., *independent set*) in a graph G is a set of pairwise adjacent (resp., non-adjacent) vertices of G. The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set as G in which two distinct vertices are adjacent if and only if they are not adjacent in G. The fact that two graphs G and H are isomorphic to each other will be denoted by $G \cong H$. Given a family $\mathcal{F}$ of graphs, we say that a graph is *$\mathcal{F}$-free* if it has no induced subgraph isomorphic to a graph of $\mathcal{F}$. K_n, C_n and P_n denote the n-vertex complete graph, cycle, and path, respectively. The *claw* is the complete bipartite graph $K_{1,3}$. A graph G is *bipartite* if its vertex set can be partitioned into two independent sets, and *co-bipartite* if it is the complement of a bipartite graph. For graph theoretic notions not defined here, see, e.g. [**23**].

We will recall the definitions of the four graph products studied in the respective sections (Sec. 3–6). For each of the four considered products, we say that the product of two graphs is *nontrivial* if both factors have at least 2 vertices. For further details regarding product graphs and their properties, we refer to [**9, 13**].

2.1. Characterizations of 1-perfectly orientable product graphs

We now state the characterizations of 1-perfectly orientable graphs for each of the four standard products due to Hartinger and Milanič. For a positive integer k, we say that a *k-linear forest* is a disjoint union of paths each having at most k vertices. In particular, 1-linear forests are exactly the edgeless graphs, and 2-linear forests are exactly the graphs consisting only of isolated vertices and isolated edges.

Theorem 1 ([**11**])**.** *A nontrivial Cartesian product, $G\square H$, of two graphs G and H is 1-perfectly orientable if and only if one of the following conditions holds:*

(i) *G is edgeless and H is 1-perfectly orientable, or vice versa.*

(ii) *G and H are 2-linear forests.*

Theorem 2 ([**11**])**.** *A nontrivial lexicographic product, $G[H]$, of two graphs G and H is 1-perfectly orientable if and only if one of the following conditions holds:*

(i) *G is edgeless and H is 1-perfectly orientable.*

(ii) *G is 1-perfectly orientable and H is complete.*

(iii) *Every component of G is complete and H is co-bipartite.*

A *pseudoforest* is a graph each component of which contains at most one cycle.

Theorem 3 ([**11**])**.** *A nontrivial direct product,* $G \times H$*, of two graphs* G *and* H *is* 1*-perfectly orientable if and only if one of the following conditions holds*:

(i) *G is a* 1*-linear forest and H is any graph, or vice versa.*

(ii) *G is a* 2*-linear forest and H is a pseudoforest, or vice versa.*

(iii) *G is a* 3*-linear forest and H is a* 4*-linear forest, or vice versa.*

A graph G is a *co-chain graph* if its vertex set can be partitioned into two cliques, say X and Y, such that the vertices in X can be ordered as $X = \{x_1, \dots, x_{|X|}\}$ so that for all $1 \leq i < j \leq |X|$, we have $N[x_i] \subseteq N[x_j]$ (or, equivalently, $N(x_i) \cap Y \subseteq N(x_j) \cap Y$). Such a pair (X, Y) will be referred to as a *co-chain partition* of G. We say that a connected graph is 2*-complete* if it is not complete and it is the union of two complete graphs. (Equivalently, if it can be obtained from P_3 by applying a sequence of true twin additions.)

Theorem 4 ([**11**])**.** *A nontrivial strong product,* $G \boxtimes H$*, of two graphs* G *and* H *is* 1*-perfectly orientable if and only if one of the following conditions holds*:

(i) *Every component of G is complete and H is* 1*-perfectly orientable, or vice versa.*

(ii) *Every component of G is* 2*-complete and every component of H is co-chain, or vice versa.*

Since every chordal, interval, or circular-arc graph is 1-perfectly orientable, Theorems 1–4 give necessary conditions that every chordal, interval, resp. circular-arc product graph must satisfy. We will also need the following characterization of 1-perfectly orientable nontrivial strong products of two connected true-twin-free graphs. Given a non-negative integer $n \geq 0$, the *raft of order n* is the graph R_n consisting of two disjoint cliques on $n + 1$ vertices each, say $X = \{x_0, x_1, \dots, x_n\}$ and $Y = \{y_0, y_1, \dots, y_n\}$ together with additional edges between X and Y such that for every $0 \leq i, j \leq n$, vertex x_i is adjacent to vertex y_j if and only if $i + j \geq n + 1$. The cliques X and Y will be referred to as the *parts* of the raft.

Lemma 5 ([**11**])**.** *A nontrivial strong product,* $G \boxtimes H$*, of two true-twin-free connected graphs G and H is* 1*-perfectly orientable if and only if one of them is isomorphic to P_3 and the other one belongs to* $\{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$.

2.2. Preliminaries on circular-arc graphs

To prove our results, we also need some properties of circular-arc graphs, which we now summarize. Given a circular-arc graph G and a representation of G with arcs around a circle, a set of arcs whose union equals the entire circle is said to *cover the circle.* Notice that if the set of arcs in the representation does not cover the circle, the corresponding circular-arc graph G is an interval graph. The following characterization of when a disjoint union of two graphs is circular-arc is easy to see and well known.

Lemma 6. *The disjoint union $G + H$ of two graphs G and H is a circular-arc graph if and only if both G and H are interval graphs.*

The following fact is well known, see, e.g., [**5**].

Fact 7. *For every $n \geq 4$, every circular-arc graph is $C_n + K_1$-free.*

While a characterization of the class of circular-arc graphs by forbidden induced subgraphs remains an open problem, in a recent study the first forbidden structure characterization of circular-arc graphs was obtained [**6**]. The class of co-bipartite circular-arc graphs, however, has been characterized in many ways (see, e.g., [**15**, Section 7] and [**5**]). In particular, we now state a characterization of co-bipartite circular-arc graphs due to Hell and Huang and its consequence, which we will use in the characterization of the circular-arc nontrivial lexicographic product graphs in Section 4.

Let G be a co-bipartite graph with a bipartition $\{U, U'\}$ of its vertex set into two cliques. An edge of G connecting a vertex from U with a vertex of U' is said to be a *crossing edge* of G. A coloring of the crossing edges of G with colors red and blue is said to be *good* (with respect to $\{U, U'\}$) if for every induced C_4 in G, the two crossing edges in it are of the opposite color. The following characterization of co-bipartite circular-arc graphs is a reformulation of [**12**, Corollary 2.3].

Theorem 8 ([**12**])**.** *Let G be a co-bipartite graph with a bipartition $\{U, U'\}$ of its vertex set into two cliques. Then G is a circular-arc graph if and only if it has a good coloring.*

Lemma 9. *The class of co-bipartite circular-arc graphs is closed under join.*

Proof. Let G and H be co-bipartite circular-arc graphs, with bipartitions of their vertex sets into two cliques U_1 and U_2, and V_1 and V_2, respectively. Then $F = G * H$ is co-bipartite as well, with bipartition into two cliques $W_1 = U_1 \cup V_1$ and $W_2 = U_2 \cup V_2$. We will now show that F admits a good coloring. By Theorem 8 this will imply that the join of G and H is circular-arc.

By Theorem 8 there exists a good coloring of G and a good coloring of H. Every crossing edge of F is exactly of one of the following four types: a crossing edge of G, a crossing edge of H, a U_1, V_2-edge, or a U_2, V_1-edge. We construct a good coloring of F as follows: the crossing edges of G or of H are colored as in (some fixed) good colorings of G, resp. H, every U_1, V_2-edge is colored red, and every U_2, V_1-edge is colored blue. Since every induced C_4 in F either lies entirely in one of G and H, or it is formed by two non-adjacent vertices in G and two non-adjacent vertices in H, the so obtained coloring is indeed a good coloring of F. □

In the following lemma we summarize the known characterizations of interval (resp., circular-arc) forests. A *caterpillar* is a tree T such that the removal of all degree-one vertices yields a path. A *caterpillar forest* is a disjoint union of caterpillars. A *bipartite claw* is the graph obtained from the claw by subdividing each of its edges exactly once.

Lemma 10 ([**14, 10**])**.** *Let F be a forest. Then, the following are equivalent*:

1. *F is an interval graph,*
2. *F is a circular-arc graph,*
3. *F is a caterpillar forest,*
4. *F contains no induced bipartite claw.*

Proof. Let F be a forest. Clearly, if F is interval then it is circular-arc. Now, assume F is circular-arc. Since F contains no cycle, the set of arcs in any circular-arc representation of F cannot cover the circle, which implies that F is interval. The fact that F is interval if and only if it contains no induced bipartite claw follows from the characterization of interval graphs from [**14**]. The fact that F is a caterpillar forest if and only if F contains no induced bipartite claw was proved in [**10**].

3. THE CARTESIAN PRODUCT

The *Cartesian product* $G\square H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ in which two distinct vertices (u,v) and (u',v') are adjacent if and only if

(a) $u=u'$ and v is adjacent to v' in H, or

(b) $v=v'$ and u is adjacent to u' in G.

The Cartesian product of two graphs is commutative, in the sense that $G\square H\cong H\square G$. In the following theorem we characterize when a nontrivial Cartesian product of two graphs G and H is chordal, interval, or circular-arc, respectively.

Theorem 11. *A nontrivial Cartesian product, $G\square H$, of two graphs G and H is*:

- *chordal if and only if G is edgeless and H is chordal, or vice versa*,
- *interval if and only if G is edgeless and H is interval, or vice versa*,
- *circular-arc if and only if one of the following conditions holds*:
  - (i) *G is edgeless and H is an interval graph, or vice versa*,
  - (ii) $G\cong H\cong K_2$.

Proof. First we characterize the chordal case. If G is edgeless and H is chordal, then $G\square H$ is isomorphic to a disjoint union of $|V(G)|$ copies of H. Thus, since chordal graphs are closed under disjoint union, the stated condition is sufficient. To show necessity, assume now that $G\square H$ is chordal. Both graphs G and H must be chordal since they are induced subgraphs of $G\square H$. Suppose that none of G

and H is edgeless. In that case, $G\square H$ contains an induced $K_2\square K_2 \cong C_4$ and is therefore not chordal, a contradiction. Thus, at least one of G and H is edgeless.

Suppose now that $G\square H$ is interval. Both graphs, G and H, must be interval since they are induced subgraphs of $G\square H$. Since $G\square H$ is interval, it is chordal, and thus, by the above, one of G and H, say G, must be edgeless. Conversely, if G is edgeless and H is interval, the Cartesian product $G\square H$ is isomorphic to a disjoint union of copies of H, and therefore interval.

For the circular-arc case, it is clear that any of the conditions (*i*) and (*ii*) is sufficient for $G\square H$ to be a circular-arc graph. To prove necessity, suppose that $G\square H$ is circular-arc. If one of G and H, say G, is edgeless, then, since the product is nontrivial, it is isomorphic to the disjoint union of $|V(G)| \geq 2$ copies of H. By Lemma 6 and an inductive argument on the number of components of G, we infer that H is interval. Suppose now that both G and H have an edge. Since $G\square H$ is circular-arc, it is also 1-perfectly orientable. By Theorem 1, G and H are 2-linear forests. If one of G and H contains at least 2 edges, then $G\square H$ contains $2C_4$ as an induced subgraph. This would imply the existence of an induced $C_4 + K_1$, contrary to Fact 7. A similar reasoning shows that each of G and H has a unique component, and thus each of them is isomorphic to K_2. ☐

Since the Cartesian product of a graph G with an n-vertex edgeless graph is isomorphic to the disjoint union of n copies of G, we obtain the following.

Corollary 12. *Let $\mathcal{C}_\square$, $\mathcal{I}_\square$, resp. $\mathcal{CA}_\square$ denote the sets of (isomorphism classes of) nontrivial Cartesian product graphs that are chordal, interval, resp. circular-arc. Then:*

$$\begin{aligned}
\mathcal{C}_\square &= \{nG : G \text{ chordal}, n \geq 2, |V(G)| \geq 2\}\,,\\
\mathcal{I}_\square &= \{nG : G \text{ interval}, n \geq 2, |V(G)| \geq 2\}\,,\\
\mathcal{CA}_\square &= \{nG : G \text{ interval}, n \geq 2, |V(G)| \geq 2\} \cup \{C_4\}\,.
\end{aligned}$$

4. THE LEXICOGRAPHIC PRODUCT

Given two graphs G and H, the *lexicographic product* of G and H, denoted by $G[H]$ (sometimes also by $G \circ H$) is the graph with vertex set $V(G) \times V(H)$, in which two distinct vertices (u, v) and (u', v') are adjacent if and only if

(a) u is adjacent to u' in G, or

(b) $u = u'$ and v is adjacent to v' in H.

Note that contrary to the other three products considered in this paper, the lexicographic product is not commutative, that is, $G[H] \ncong H[G]$ in general.

The following theorem characterizes when a nontrivial lexicographic product of two graphs G and H is chordal, interval, or circular-arc, respectively.

Theorem 13. *A nontrivial lexicographic product*, $G[H]$, *of two graphs* G *and* H *is:*

- *chordal if and only if one of the following conditions holds*:
  - (i) G *is edgeless and* H *is chordal*,
  - (ii) G *is chordal and* H *is complete*,
- *interval if and only if one of the following conditions holds*:
  - (i) G *is edgeless and* H *is interval*,
  - (ii) G *is interval and* H *is complete*,
- *circular-arc if and only if one of the following conditions holds*:
  - (i) G *is edgeless and* H *is interval*,
  - (ii) G *is circular-arc and* H *is complete*,
  - (iii) G *is complete and* H *is co-bipartite circular-arc.*

Proof. First, we characterize the chordal case. Suppose first that $G[H]$ is chordal. Then, both G and H are chordal since they are induced subgraphs of $G[H]$. If neither of conditions (i) or (ii) above holds, then G has an edge and H is not complete. This implies that the product $G[H]$ contains an induced subgraph isomorphic to $K_2[2K_1] \cong C_4$, contrary to the fact that it is chordal. For the converse direction, we will show that in both cases (i) and (ii), the product graph $G[H]$ is chordal. If G is edgeless and H is chordal, then the product $G[H]$ is isomorphic to the disjoint union of $|V(G)|$ copies of H, and therefore chordal. If G is chordal and H is complete, then the product $G[H]$ is isomorphic to the graph obtained by repeatedly substituting a vertex of G with a complete graph, and this operation is easily seen to preserve chordality.

Now we analyze the interval case. Assume that $G[H]$ is interval. Then, G and H are interval. Since $G[H]$ is interval, in particular $G[H]$ is chordal, and thus we obtain the desired result. Conversely, if G is edgeless and H interval, $G[H]$ is isomorphic to a disjoint union of copies of H, and if G is interval and H is complete, $G[H]$ can be obtained from a sequence of true twin additions to H. In both cases the lexicographic product $G[H]$ is interval.

Finally, we characterize the circular-arc case. Suppose first that $G[H]$ is a circular-arc graph. Then, both G and H are circular-arc graphs, since they are induced subgraphs of $G[H]$. If G is edgeless, then the lexicographic product $G[H]$ is isomorphic to the Cartesian product $G \square H$ and by Theorem 11, condition (i) holds. So we may assume that G has an edge. If H is complete then condition (ii) holds. Suppose now that G is not edgeless and that H is not complete. Since $G[H]$ is 1-perfectly orientable, one of conditions (i)–(iii) from Theorem 2 holds, and so we infer that every component of G is complete and H is co-bipartite. Therefore, the product $G[H]$ contains an induced subgraph isomorphic to $K_2[2K_1] \cong C_4$, from which we infer that G is connected (that is, complete), since by Fact 7 $G[H]$ is

$C_4 + K_1$-free. Therefore, condition (iii) holds. This completes the proof of the forward direction.

For the converse direction, we will show that in any of the three cases, the product graph $G[H]$ is circular-arc. If G is edgeless and H interval, then the lexicographic product $G[H]$ is isomorphic to the disjoint union of $|V(G)|$ copies of H, and therefore circular-arc. If G is circular-arc and H is complete, then the product $G[H]$ is isomorphic to the graph obtained by repeatedly substituting a vertex of G with a complete graph. Substituting a vertex v with a complete graph is the same as adding a sequence of true twins to vertex v, an operation easily seen to preserve the property of being a circular-arc graph. Finally, suppose that G is complete and H is a co-bipartite circular-arc graph. In this case, an inductive argument on the order of G together with the fact that the class of co-bipartite circular-arc graphs is closed under join (by Lemma 9) shows that $G[H]$ is a circular-arc graph. □

Since the lexicographic product of an n-vertex edgeless graph with a graph G is isomorphic to the disjoint union of n copies of G, Theorem 13 has the following consequence.

Corollary 14. *Let $\mathcal{C}_{\ell ex}$, $\mathcal{I}_{\ell ex}$, resp. $\mathcal{CA}_{\ell ex}$, denote the sets of (isomorphism classes of) nontrivial lexicographic product graphs that are chordal, interval, resp. circular-arc. Then:*

$$\begin{aligned}
\mathcal{C}_{\ell ex} &= \{nG : G \textit{ chordal}, n \geq 2, |V(G)| \geq 2\} \\
&\quad \cup \{G[K_n] : G \textit{ chordal }, n \geq 2, |V(G)| \geq 2\}\,, \\
\mathcal{I}_{\ell ex} &= \{nG : G \textit{ interval},\ n \geq 2,\ |V(G)| \geq 2\} \\
&\quad \cup \{G[K_n] : G \textit{ interval},\ n \geq 2,\ |V(G)| \geq 2\}\,, \\
\mathcal{CA}_{\ell ex} &= \{nG : G \textit{ interval},\ n \geq 2,\ |V(G)| \geq 2\} \\
&\quad \cup \{G[K_n] : G \textit{ circular-arc},\ n \geq 2,\ |V(G)| \geq 2\} \\
&\quad \cup\ \{K_n[G] : n \geq 2,\ G \textit{ co-bipartite circular-arc},\ |V(G)| \geq 2\}\,.
\end{aligned}$$

5. THE DIRECT PRODUCT

The *direct product* $G \times H$ of two graphs G and H (sometimes also called *tensor product*, *categorical product*, or *Kronecker product*) is the graph with vertex set $V(G) \times V(H)$ in which two distinct vertices (u, v) and (u', v') are adjacent if and only if

(a) u is adjacent to u' in G, and

(b) v is adjacent to v' in H.

The direct product of two graphs is commutative, in the sense that $G \times H \cong H \times G$.

In the next theorem we characterize when a nontrivial direct product of two graphs G and H is chordal, interval, or circular-arc, respectively. A *circular caterpillar* (resp. *odd circular caterpillar*) is a connected graph such that the removal of all degree-one vertices yields a cycle (resp. an odd cycle).

Theorem 15. *A nontrivial direct product, $G \times H$, of two graphs G and H is:*

- *chordal if and only if one of the following conditions holds:*
  - (i) *at least one of G and H is edgeless,*
  - (ii) *G is a 2-linear forest and H is a forest, or vice versa,*
- *interval if and only if one of the following conditions holds:*
  - (i) *at least one of G and H is edgeless,*
  - (ii) *G is a 2-linear forest and H is a caterpillar forest, or vice versa,*
- *circular-arc graph if and only if one of the following conditions holds:*
  - (i) *at least one of G and H is edgeless,*
  - (ii) *G is a 2-linear forest and H is a caterpillar forest, or vice versa,*
  - (iii) *$G \cong K_2$ and H is an odd circular caterpillar, or vice versa.*

Proof. We prove the three equivalences in the order as stated in the theorem.

First suppose that $G \times H$ is chordal, and that both G and H contain an edge. We claim that G (and then, by symmetry, also H) is a forest. Indeed, if G contained a cycle, then $G \times H$ would contain an induced subgraph isomorphic to the direct product of K_2 with a cycle, which contains an induced cycle of length at least 4, contrary to the fact that $G \times H$ is chordal. It remains to show that at least one of G and H is a 2-linear forest. If this were not the case, then $G \times H$ would contain an induced copy of $P_3 \times P_3$, which contains an induced C_4 and therefore is not chordal, a contradiction.

For the converse direction, suppose that one of conditions (i) and (ii) holds. If condition (i) holds, then $G \times H$ is edgeless and hence chordal. Assume now that condition (ii) holds, say G is a 2-linear forest and H is a forest. In this case, for each component T of H, the graphs $K_1 \times T$ and $K_2 \times T$ are acyclic, and hence so is $G \times H$, which is the disjoint union of such graphs. It follows that $G \times H$ is chordal.

Assume now that $G \times H$ is interval. Since $G \times H$ is interval, it is chordal, and therefore one of the conditions for the chordal case holds. Therefore, necessity of the stated conditions is achieved, unless (without loss of generality) G is a 2-linear forest containing an edge and H is a forest that is not a caterpillar forest. By Lemma 10, H contains an induced bipartite claw, and consequently $G \times H$ contains an induced subgraph, say F, isomorphic to the direct product of K_2 with the bipartite claw. A direct inspection shows that F is isomorphic to the disjoint union of two copies of the bipartite claw, therefore by Lemma 10, F is not interval, and hence neither is $G \times H$. This stablishes necessity. Let us now prove sufficiency.

If one of G and H, say G, is edgeless, then $G \times H$ is edgeless and therefore interval. Now, if G is a 2-linear forest and H is a caterpillar forest, then each component of $G \times H$ is interval. This is because for each component K of H, the components of $G \times H$ are either $K_1 \times K$ or $K_2 \times K$, both caterpillar forests, and in particular interval graphs (Lemma 10). The result now follows from the fact that interval graphs are closed under disjoint union.

Finally, we consider the circular-arc case. Suppose first that $G \times H$ is a circular-arc graph. Then it is 1-perfectly orientable, in particular, one of the conditions (i)–(iii) from Theorem 3 holds. Condition (i) from that theorem coincides with condition (i) in Theorem 15, so we may assume that both G and H contain an edge.

Suppose that condition (ii) from Theorem 3 holds, say G is a 2-linear forest and H is a pseudoforest (the other case is symmetric). We consider two cases depending on whether H is acyclic or not.

Case 1 : *H is acyclic.* We claim that in this case H is a caterpillar forest (and hence condition (ii) holds in this case). If this is not the case, then, by Lemma 10, H would contain an induced subgraph, say K, isomorphic to the bipartite claw, but then $G \times H$ would contain $K_2 \times K \cong 2K$ as induced subgraph, contradicting the fact that $2K$ is not is a circular-arc graph (by Lemma 10). Hence, condition (ii) of the proposition holds in this case.

Case 2 : *H contains a component, say K, with a cycle.* If K contains an even cycle (say of length $2k \geq 4$), then $G \times H$ contains $2C_{2k}$ as induced subgraph, contrary to the fact that it is a circular-arc graph (by Fact 7, $C_{2k} + K_1$ is not circular-arc and therefore neither is $2C_{2k}$). Hence, K contains a (unique) odd cycle, say C. If H has a vertex with no neighbors on C, then $G \times H$ contains an induced subgraph isomorphic to $C_{2k} + K_1$ where $k \geq 3$ is the length of C, contrary to the fact that $G \times H$ is a circular-arc graph. It follows that every vertex not in C has a neighbor in C, and in particular, since H is a pseudoforest, that every vertex not in C has a unique neighbor in C and that $V(H) \setminus C$ is an independent set in H. Consequently, H is an odd circular caterpillar. If G were not isomorphic to K_2, the product $G \times H$ would contain an induced $C_{2k} + K_1$, contradicting Fact 7. We conclude that $G \cong K_2$ and hence condition (iii) applies in this case.

Finally, suppose that condition (iii) from Theorem 3 holds, say G is a 3-linear forest and H is a 4-linear forest. To avoid the already considered condition (ii) (from Theorem 3), we may assume that neither of G and H is a 2-linear forest. But then $G \times H$ contains an induced copy of $P_3 \times P_3$, which is not a circular-arc graph (since it contains an induced $C_4 + K_1$), a contradiction.

For the converse direction, suppose that one of the conditions (i)–(iii) holds. If condition (i) holds, then $G \times H$ is edgeless and hence circular-arc. Assume now that both G and H contain an edge and that condition (ii) holds, say G is a 2-linear forest and H is a caterpillar forest. In this case, for each component K of H, the graphs $K_1 \times K$ and $K_2 \times K$ are caterpillar forests, in particular, by Lemma 10, they are interval graphs. It follows that $G \times H$, which is the disjoint union of such graphs, is also interval, and hence circular-arc. Finally, if condition (iii) holds, say

$G \cong K_2$ and H is an odd circular caterpillar, then $G \times H$ is a circular caterpillar. It is easy to see that every circular caterpillar is circular-arc: we can obtain a circular-arc representation of it by covering the circle with arcs corresponding to vertices of the cycle, and placing a new arc corresponding to each leaf within the arc corresponding to its unique neighbor in the cycle without intersecting any other arc. □

Theorem 15 implies the following.

Corollary 16. *Let* $\mathcal{C}_\times$, $\mathcal{I}_\times$, *resp.* $\mathcal{CA}_\times$, *denote the sets of (isomorphism classes of) nontrivial direct product graphs that are chordal, interval, resp. circular-arc. Then:*

$$\begin{aligned}
\mathcal{C}_\times &= \{mnK_1 : m \geq 2,\ n \geq 2\}\\
&\quad \cup \{2mF + n|V(F)|K_1 : F \text{ is a forest},\ m \geq 1,\ n \geq 0,\ |V(F)| \geq 2\},\\
\mathcal{I}_\times &= \{mnK_1 : m \geq 2,\ n \geq 2\}\\
&\quad \cup \{2mF + n|V(F)|K_1 : F \text{ is a caterpillar forest},\ m \geq 1,\ n \geq 0,\ |V(F)| \geq 2\},\\
\mathcal{CA}_\times &= \{mnK_1 : m \geq 2,\ n \geq 2\}\\
&\quad \cup \{2mF + n|V(F)|K_1 : F \text{ is a caterpillar forest},\ m \geq 1,\ n \geq 0,\ |V(F)| \geq 2\}\\
&\quad \cup\ \{G : G \text{ is a circular caterpillar satisfying conditions } (*)\},
\end{aligned}$$

where conditions $(*)$ *are the following:*

- *the unique cycle* C *of* G *is of length* $4k+2$ *for some* $k \geq 1$, *and*
- *every two vertices at distance* $2k+1$ *on* C *are of the same degree in* G.

Proof. The statement of the corollary follows immediately from the characterizations given by Theorem 15 and the following facts:

– If $G \times H$ is a nontrivial direct product such that $m = |V(G)| \geq 2$, $n = |V(H)| \geq 2$, and at least one of the two factors is edgeless, then $G \times H$ is an edgeless graph of order mn.

– If H is a bipartite graph, then $K_2 \times H \cong 2G$ (see, e.g., [**9**, Exercise 8.14]). In particular, if H is a forest (resp. caterpillar forest), then $K_2 \times H \cong 2H$.

– The direct product is distributive (up to isomorphism) with respect to the disjoint union.

– Suppose that H is an odd circular caterpillar, with its unique cycle, say C, of length $2k+1$ for some $k \geq 1$. Then, $K_2 \times H$ is isomorphic to a circular caterpillar, say G, the unique cycle of which, say C', has length $2(2k+1) = 4k+2$. Moreover, every vertex v of C corresponds to a pair of vertices v', v'' of C' at distance $2k+1$ in C', such that $d_G(v') = d_G(v'') = d_H(v)$.

6. THE STRONG PRODUCT

In this final section we consider the strong product and we characterize when a nontrivial strong product of two graphs G and H is chordal, interval, or circular-arc, respectively. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which two distinct vertices (u, v) and (u', v') are adjacent if and only if

(a) u is adjacent to u' in G and $v = v'$, or

(b) $u = u'$ and v is adjacent to v' in H, or

(c) u is adjacent to u' in G and v is adjacent to v' in H.

It is easy to see that the fact that one of the conditions (a), (b) and (c) holds is equivalent to the pair of conditions $u' \in N_G[u]$ and $v' \in N_H[v]$, that is, that $(u', v') \in N_G[u] \times N_H[v]$. Consequently, for every two vertices $u \in V(G)$ and $v \in V(H)$, we have $N_{G\boxtimes H}[(u, v)] = N_G[u] \times N_H[v]$. The strong product of two graphs is commutative, in the sense that $G \boxtimes H \cong H \boxtimes G$.

To prove the characterization of circular-arc nontrivial strong product graphs, we need one further lemma.

Lemma 17. *Let G, G', and H be graphs such that G' is obtained from G by adding a true twin. Then, $G \boxtimes H$ is circular-arc if and only if $G' \boxtimes H$ is circular-arc.*

Proof. Note that $G \boxtimes H$ is an induced subgraph of $G' \boxtimes H$, therefore if $G' \boxtimes H$ is circular-arc, then so is $G \boxtimes H$. Suppose now that $G \boxtimes H$ is circular-arc, and that G' was obtained from G by adding to it a true twin x' to a vertex x of G. Note that for every $v \in V(H)$, we have $N_{G'\boxtimes H}[(x, v)] = N_{G'}[x] \times N_H[v]$ and $N_{G'\boxtimes H}[(x', v)] = N_{G'}[x'] \times N_H[v]$. Since $N_{G'}[x] = N_{G'}[x']$, each vertex of the form (x', v) for $v \in V(H)$ is a true twin in $G' \boxtimes H$ of vertex (x, v). It follows that $G' \boxtimes H$ can be obtained from $G \boxtimes H$ by a sequence of true twin additions. Since circular-arc graphs are closed under true twin additions, $G' \boxtimes H$ is circular-arc. ☐

We now state and prove the main result of this section. Recall that a graph G is said to be *co-chain* if its vertex set can be partitioned into two cliques, say X and Y, such that the vertices in X can be ordered as $X = \{x_1, \ldots, x_{|X|}\}$ so that for all $1 \leq i < j \leq |X|$, we have $N[x_i] \subseteq N[x_j]$, and 2-*complete* if G can be obtained from P_3 by applying a sequence of true twin additions.

Theorem 18. *A nontrivial strong product, $G \boxtimes H$, of two graphs G and H is:*

- *chordal if and only if every component of G is complete and H is chordal, or vice versa,*
- *interval if and only if every component of G is complete and H is interval, or vice versa,*

- *circular-arc if and only if one of the following conditions holds*:
  (i) *G is complete and H is a circular-arc graph, or vice versa,*
  (ii) *G is 2-complete and H is a connected co-chain graph, or vice versa,*
  (iii) *each component of G is complete and H is interval, or vice versa.*

Proof. Again, we prove the three equivalences in the order as stated in the theorem.

Suppose first that $G \boxtimes H$ is chordal. Each graph G and H must also be chordal since they are induced subgraphs of $G \boxtimes H$. Suppose now that not all components of G are complete and no all components of H are complete. Therefore there is a component of G and a component of H each having an induced P_3. But $P_3 \boxtimes P_3$ contains an induced 4-cycle, and is therefore not chordal, a contradiction. Thus, all components of one of the factors must be complete.

To show sufficiency, let $G_1, \ldots, G_k$ be the components of G, let $H_1, \ldots, H_\ell$, be the components of H, and suppose that G_i is complete for $i = 1, \ldots, k$, and H is chordal. Note that the components of $G \boxtimes H$ are of the form $G_i \boxtimes H_j$ for $1 \leq i \leq k$, $1 \leq j \leq \ell$. Every component $G_i \boxtimes H_j$ of $G \boxtimes H$ is chordal since it is the result of applying a sequence of true twin additions to a chordal graph, namely H_j. (The operation of adding a true twin is easily seen to preserve chordality.) Since each component of $G \boxtimes H$ is chordal and chordal graphs are closed under disjoint union, we conclude that $G \boxtimes H$ is chordal.

Suppose now that $G \boxtimes H$ is interval. Again, G and H must be interval since they are induced subgraphs of the product. Necessity follows immediately from the chordal case. To conclude the proof for the interval case, assume that every component of G is complete and H is interval. In that case, the strong product $G \boxtimes H$ can be obtained as disjoint union of graphs each of which is the result of applying a sequence of true twin additions to the interval graph H. Since the operations of disjoint union and true twin addition preserve the class of interval graphs, we conclude that $G \boxtimes H$ is interval.

It remains to analyze the circular-arc case.

Necessity. Suppose that $G \boxtimes H$ is circular-arc. Then, G and H are induced subgraphs of $G \boxtimes H$ and therefore circular-arc as well. Suppose that G and H are both connected. Since $G \boxtimes H$ is 1-perfectly orientable, by Theorem 4, either G is complete, or G is 2-complete and H is co-chain. So we are in cases (i) or (ii), respectively.

Now, if not both factors are connected, the product $G \boxtimes H$ is disconnected. Since $G \boxtimes H$ is circular-arc, by Lemma 6 we know that all its components are interval. Moreover, since for every component G_i of G and every component H_j of H their product $G_i \boxtimes H_j$ is a component of $G \boxtimes H$ we infer that all components of G are interval, and similarly for H. Therefore, G and H are interval. Since $G \boxtimes H$ is a disjoint union of interval graphs, it is interval, and in particular chordal. Thus we can apply the already stablished characterization for the chordal case, and so one of G and H must be a disjoint union of complete graphs. This concludes the proof of the forward implication.

Sufficiency. We will show that if one of (i), (ii), or (iii) holds, then $G \boxtimes H$ is circular-arc.

If condition (i) holds, say G is complete and H is circular-arc, then the product $G \boxtimes H$ is the result of applying a sequence of true twin additions to a circular-arc graph, namely H, and so it is circular-arc.

Suppose now that (ii) holds, say G is 2-complete and H is a connected co-chain graph. By Lemma 17, we may assume that both factors are true-twin-free. Therefore, $G \cong P_3$ and, by Lemma 5 $H \in \{K_1\} \cup \{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$. Notice first that $P_3 \boxtimes K_1 \cong P_3$ is circular-arc. Since $R_n * K_1$ is an induced subgraph of R_{n+2}, it is enough to show that $P_3 \boxtimes R_n$ is circular-arc for all $n \geq 1$.

Let $V(P_3) = \{u_1, u_2, u_3\}$ where u_1 and u_3 are the two leaves. Assuming the notation as in the definition of rafts, let $V(R_n) = X \cup Y$, where $X = \{x_0, x_1, \ldots, x_n\}$ and $Y = \{y_0, y_1, \ldots, y_n\}$ are the two parts of the raft. Vertices in $P_3 \boxtimes R_n$ will be said to be *left*, resp. *right*, depending on whether their second coordinate is in X or in Y, respectively.

Fig. 1 shows a schematic representation of $P_3 \boxtimes R_n$. We partition the vertex set of the graph in the following way: 6 singletons, namely $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{b_1\}$, $\{b_2\}$, $\{b_3\}$, where $a_i = (u_i, x_0)$ and $b_i = (u_{4-i}, y_0)$, and 6 cliques of size k each, namely A_1, A_2, A_3, B_1, B_2, and B_3, defined as follows: for $i \in \{1, 2, 3\}$, we have $A_i = \{u_i\} \times (X \setminus \{x_0\})$ and $B_i = \{u_{4-i}\} \times (Y \setminus \{y_0\})$. Bold lines between certain pairs of sets mean that every possible edge between the two sets is present. If the corresponding line is not bold, then only some of the edges between the two sets are present.

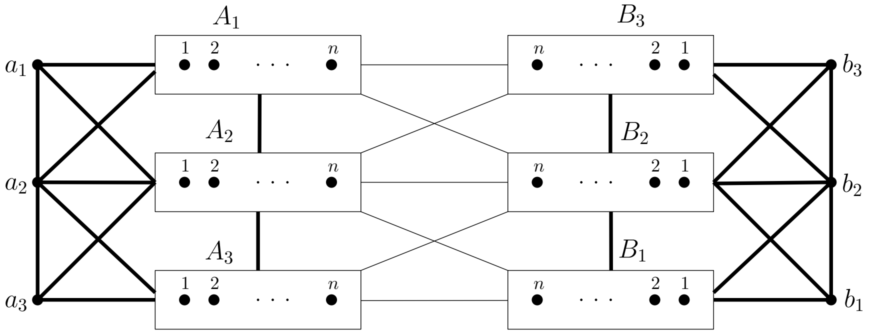

Figure 1. A schematic representation of $P_3 \boxtimes R_n$

To describe such edges, we introduce the following ordering of the vertices within each of the 6 cliques $A_1, A_2, A_3, B_1, B_2, B_3$ of size n. Note that for every $1 \leq i < j \leq n$, we have that $N_{R_n}[x_i] \subset N_{R_n}[x_j]$ and $N_{R_n}[y_i] \subset N_{R_n}[y_j]$. We order the vertices in the 6 cliques accordingly, that is, for each clique of the form A_i, the linear ordering of its vertices is $(u_i, x_1), \ldots, (u_i, x_n)$; for each clique of the form B_i, the linear ordering of its vertices is $(u_{4-i}, y_1), \ldots, (u_{4-i}, y_n)$. To keep the notation light, we will slightly abuse the notation, speaking of "vertex i in clique C" (for $i \in \{1, \ldots, n\}$ and $C \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$) when referring to the i-th vertex in the linear ordering of C.

The edges of graph G can be now concisely described as follows. We will say that two cliques K_i and K_j (where K_ℓ is either $A_\ell, B_\ell, \{a_\ell\}$, or $\{b_\ell\}$ for some ℓ) are *adjacent* if $|i - j| \leq 1$. The closed neighborhood of a_i is the union of the cliques A_j adjacent to a_i and $\{a_1\} \cup \{a_2\} \cup \{a_3\}$. The neighborhood of b_i is the union of the cliques B_j adjacent to b_i and $\{b_1\} \cup \{b_2\} \cup \{b_3\}$. For each vertex i in a left clique, say A_j, its closed neighborhood consists of the vertices a_i in its adjacent cliques $\{a_i\}$, all the vertices belonging to some left clique adjacent to A_j, and of vertices $\{n - i + 1, \ldots, n\}$ in each right clique adjacent to B_{4-j}. For each vertex i in a right clique, say B_j, its closed neighborhood consists of the vertices b_i in its adjacent cliques $\{b_i\}$, all vertices belonging to some right clique adjacent to B_j, and of vertices $\{n - i + 1, \ldots, n\}$ in each left clique adjacent to A_{4-j}.

For any two adjacent clique A_i, B_j, the vertices of $A_i \cup B_j$ induce a special co-chain graph, called a semiraft. Given a non-negative integer $n \geq 0$, the *semiraft of order* n is the graph S_n consisting of two disjoint cliques on n vertices each, say $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ together with additional edges between X and Y such that for every $1 \leq i, j \leq n$, vertex x_i is adjacent to vertex y_j if and only if $i + j \geq n$.

As shown by the interval representation given in Fig. 2, every semiraft is an interval graph.

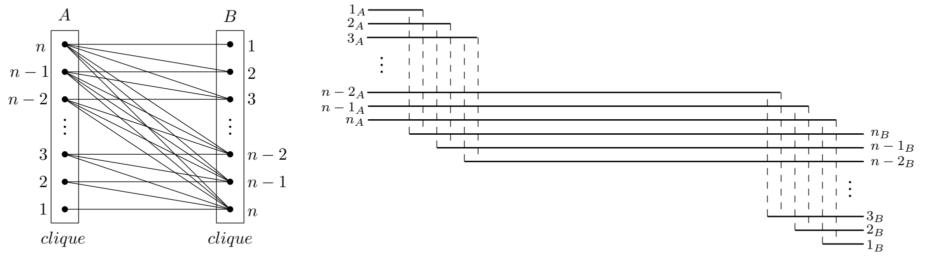

Figure 2. The semiraft S_n and its interval representation.

Suppose first that $n = 1$. A circular-arc representation of $P_3 \boxtimes R_1$ is depicted in Fig. 3. (The rectangles P and Q also depicted in Fig. 3 are not part of the representation, they will be used later on in the proof.)

Suppose now that $n > 1$. We will give a circular-arc representation of $P_3 \boxtimes R_n$ similar to that of $P_3 \boxtimes R_1$ shown in Fig. 3, combined with the interval representations of semirafts represented by Fig. 2. The circular-arc representation of $P_3 \boxtimes R_n$ is the same as in Fig. 3, but this time instead of each clique $C \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$ being represented by a single arc, it will consist of n arcs. If we were to "zoom in" at the rectangles marked as P and Q in Fig. 3 to see how the arcs representing the four cliques interact, then we would see the representations shown in Fig. 4 and 5 below.

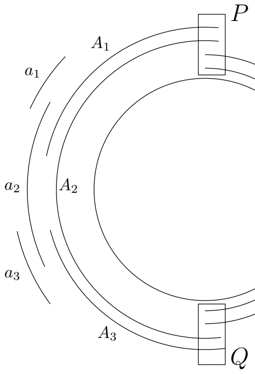

Figure 3. A circular-arc representation of $P_3 \boxtimes R_1$.

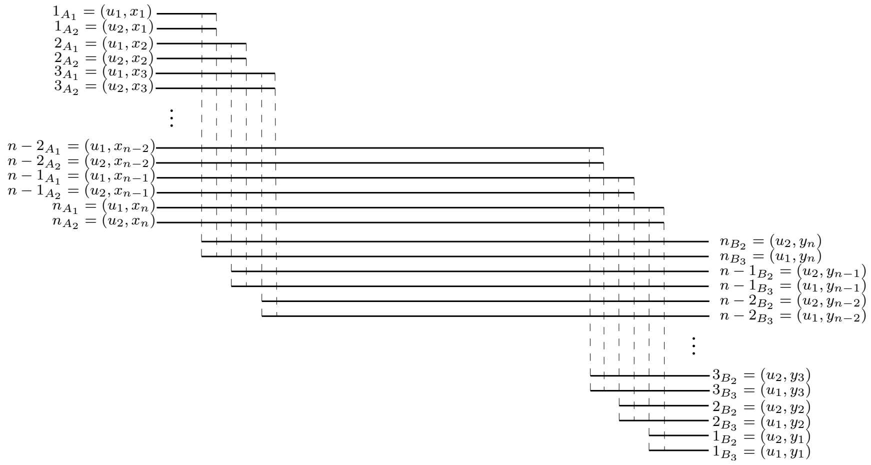

Figure 4. Intersection of cliques A_1, A_2, B_2, and B_3.

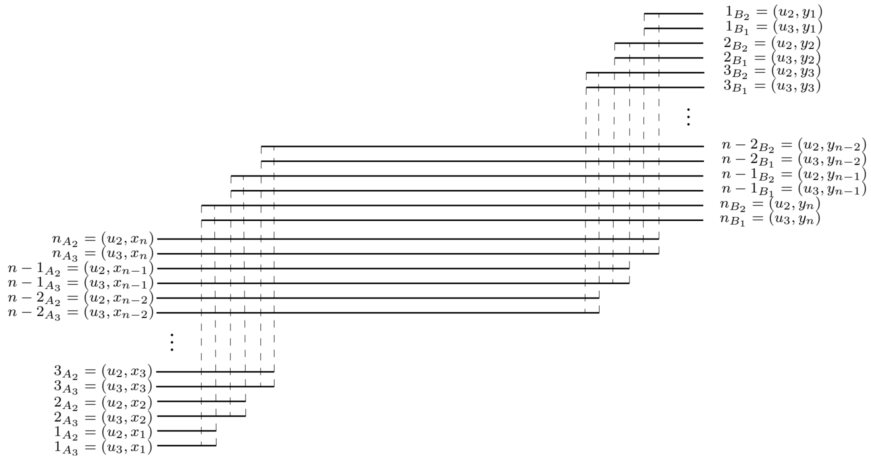

Figure 5. Intersection of cliques A_2, A_3, B_1, and B_2.

This gives a circular-arc representation of the graph $P_3 \boxtimes R_n$. This implies that if G is 2-complete and H is co-chain, the strong product $G \boxtimes H$ is circular-arc, concluding this part of the proof.

Finally, if condition (iii) holds, say each component of G is complete and H is interval, then each component of the strong product, $G_i \boxtimes H_j$, is interval, since it can be obtained by applying a sequence of true twin additions to an interval graph, H_j. It follows from Lemma 6 that $G \boxtimes H$ is circular-arc. ☐

Hartinger and Milanič showed in [**11**] that for each $n \geq 1$, the graph $P_3 \boxtimes R_n$ is 1-perfectly orientable. Theorem 18 (and its proof) imply that for each $n \geq 1$, the graph $P_3 \boxtimes R_n$ is circular-arc. Since the class of circular-arc graphs is a subclass of the class of 1-perfectly orientable graphs, this gives an alternative proof of the fact that graphs of the form $P_3 \boxtimes R_n$ are 1-perfectly orientable.

Since the strong product is distributive (up to isomorphism) with respect to the disjoint union, Theorem 18 implies the following.

Corollary 19. *Let* $\mathcal{C}_\boxtimes$, $\mathcal{I}_\boxtimes$, *resp.* $\mathcal{CA}_\boxtimes$, *denote the sets of* (*isomorphism classes of*) *nontrivial direct product graphs that are chordal, interval, resp. circular-arc. Then*:

$$\begin{aligned}
\mathcal{C}_\boxtimes =&\Big\{ \mathop{+}_{i=1}^{k} (G \boxtimes K_{n_i}) : G \text{ chordal}, \\
&\qquad |V(G)| \geq 2,\ k \geq 1,\ n_i \geq 1\ \forall i = 1, \ldots, k,\ \sum_{i=1}^{k} n_i \geq 2 \Big\}, \\
\mathcal{I}_\boxtimes =&\Big\{ \mathop{+}_{i=1}^{k} (G \boxtimes K_{n_i}) : G \text{ interval}, \\
&\qquad |V(G)| \geq 2,\ k \geq 1,\ n_i \geq 1\ \forall i = 1, \ldots, k,\ \sum_{i=1}^{k} n_i \geq 2 \Big\}, \\
\mathcal{CA}_\boxtimes =&\{ G \boxtimes K_n : G \text{ circular-arc},\ n \geq 2,\ |V(G)| \geq 2\} \cup \\
&\{ G \boxtimes H : G \text{ 2-complete},\ H \text{ connected and co-chain},\ |V(H)| \geq 2\} \cup \\
&\Big\{ \mathop{+}_{i=1}^{k} (G \boxtimes K_{n_i}) : G \text{ circular-arc}, \\
&\qquad |V(G)| \geq 2,\ k \geq 1,\ n_i \geq 1\ \forall i = 1, \ldots, k,\ \sum_{i=1}^{k} n_i \geq 2 \Big\}.
\end{aligned}$$

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