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DISCRETE HERMITE-HADAMARD INEQUALITY AND ITS APPLICATIONS

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We state and prove the Hermite-Hadamard inequality for a function defined on a time scale, which has all the points as isolated. We illustrate our result with various examples.

1. INTRODUCTION

The Hermite-Hadamard inequality [**7, 8**] states that if $f : I \to \mathbb{R}$ is a convex function, then the following inequality is satisfied:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left(\int_a^b f(t)\mathrm{d}t\right) \leq \frac{f(a)+f(b)}{2},$$

where $a, b \in I$ and I is an interval in $\mathbb{R}$.

In the theory of convex functions, the Hermite-Hadamard inequality plays an important role. It has been used as a tool to obtain many nice results in integral inequalities, approximation theory, optimization theory and numerical analysis. It has been developed for different classes of convexity, such as quasi-convex functions, log-convex, r-convex functions, p-functions [**6**], and recently for discrete functions [**1**]. For the history of its developments in many directions, we refer the reader to a paper by Mitrinović and Lacković [**10**]. For the generalizations and applications in probability, we refer the reader to a paper by Merkle [**9**].

Mozyrska and Torres first introduced the convexity of a function defined on a time scale (a nonempty closed subset of $\mathbb{R}$) in their paper [**11**]. This short paper can be considered as the establishment of the foundation of convex functions on time scales.

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Here we introduce convexity by means of a midpoint condition of a function defined on a time scale which has all the points as isolated. We state and prove the Hermite-Hadamard inequality for such a class of functions. We call this new inequality as discrete Hermite-Hadamard inequality.

The paper is organized as follows: In Section 2, we define a convex real valued function on a discrete time scale $\mathbb{T}$ which time points may not be uniformly distributed on a time line. We state the midpoint condition for a function defined on $\mathbb{T}$. We then prove four equivalent statements for convex functions on $\mathbb{T}$. Section 3 starts with the proof of one kind of a substitution rule for integrals on any time scale. Then with the use of the substitution rules we prove the Hermite-Hadamard inequality for convex functions defined on $\mathbb{T}$. Three corollaries follow the main result of the paper. In the third corollary, we give an alternate proof to the Hermite-Hadamard inequality for functions defined on the set of real numbers $\mathbb{R}$. Last section of the paper is devoted to some interesting examples of the Hermite-Hadamard inequality. We close the paper with two remarks for the future direction of the research in this area.

2. PRELIMINARIES

Let $\mathbb{T} = \{0 = t_0, t_1, t_2, t_3, ...\}$ be the set of positive real numbers such that $t_i < t_j$ for $i < j$. We assume that $|\mathbb{T}| = \aleph_0$. We then define the operators $\sigma(t_i) = t_{i+1}$, $\mu(t_i) = \sigma(t_i) - t_i$, $\rho(t_i) = t_{i-1}$, and $\nu(t_i) = t_i - \rho(t_i)$ for $t_i \in \mathbb{T}$, which are known as the forward jump, the forward graininess, the backward jump, and the backward graininess operators, respectively. The time scale $\mathbb{T}$ can be considered as a discrete time scale. If $\mathbb{T}$ is the set of integers (i.e. $\mathbb{T} = \mathbb{Z}$), then $\sigma(t) = t + 1$, $\mu(t) = 1$, $\rho(t) = t - 1$ and $\nu(t) = 1$, for all $t \in \mathbb{T}$.

Definition 1. *Let f be a real valued function defined on $\mathbb{T}$. Then the Δ-derivative and the ∇-derivative of f are defined, respectively, as*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \qquad f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)},$$

where $t \in \mathbb{T}$. We define a second order derivative as $f^{\Delta\Delta} := (f^{\Delta})^{\Delta}$. The Δ-integral and the ∇-integral of f are defined as

$$\int_a^b f(\tau)\Delta\tau = \sum_{s\in[a,b)\cap\mathbb{T}} f(s)\mu(s), \qquad \int_a^b f(\tau)\nabla\tau = \sum_{s\in(a,b]\cap\mathbb{T}} f(s)\nu(s),$$

respectively, where $a, b \in \mathbb{T}$.

We also use the notation $\mathbb{T}^{\kappa}$ which is defined as

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \sup\mathbb{T} < \infty \\ \mathbb{T}, & \sup\mathbb{T} = \infty. \end{cases}$$

For further reading on time scales, we refer the reader to an excellent book on the analysis of time scales [**2**].

Throughout this study, we focus on the discrete time scales. Let $\mathbb{T}$ be any discrete time scale and $a, b \in \mathbb{T}$ with $a < b$. $[a,b]_{\mathbb{T}}$ means $[a,b] \cap \mathbb{T}$. We define

$$\mathbb{T}_{[a,b]} = \left\{ t \mid t = \frac{b-u}{b-a} \text{ for } u \in [a,b]_{\mathbb{T}} \right\}.$$

Note that $\mathbb{T}_{[a,b]} \subset [0,1]$. We also want to point out that there exists a bijective (one-to-one and onto) map between $[a,b]_{\mathbb{T}}$ and $\mathbb{T}_{[a,b]}$.

Definition 2. *$f : \mathbb{T} \to \mathbb{R}$ is called convex on $\mathbb{T}$ if for every $x, y \in \mathbb{T}$ with $x < y$, the following inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

is satisfied for all $\lambda \in \mathbb{T}_{[x,y]}$.

Definition 3. *We define the midpoint of a and b as the n^{th} element in a finite time scale $[a,b]_{\mathbb{T}}$ with cardinality $2n-1$.*

For the notation, we denote the midpoint of a and b by $m_{[a,b]}$ in $[a,b]_{\mathbb{T}}$, and the corresponding midpoint of 0 and 1 by $m_{[0,1]}$ in $\mathbb{T}_{[a,b]}$. If $\mathbb{T} = \mathbb{Z}$, then Definition 3 reduces to the standard definition of midpoint on $\mathbb{Z}$, namely $m_{[a,b]} = \dfrac{a+b}{2}$ and $m_{[0,1]} = \dfrac{1}{2}$.

Definition 4. *$f : \mathbb{T} \to \mathbb{R}$ satisfies the midpoint condition if*

$$f(m_{[a,b]}) \leq m_{[0,1]} f(a) + (1 - m_{[0,1]}) f(b), \tag{1}$$

for every $a, b \in \mathbb{T}$ with the cardinality of $[a,b]_{\mathbb{T}}$ an odd number.

Remark 1. If we want to be more precise for the number $m_{[0,1]}$ in (1), then we can write it as $m_{[0,1]} = \dfrac{b - m_{[a,b]}}{b-a}$. Note that if $\mathbb{T} = \mathbb{Z}$, then the inequality (1) becomes $f\left(\dfrac{a+b}{2}\right) \leq \dfrac{f(a)+f(b)}{2}$, as stated in [**1**].

The following two theorems are crucial in the proof of the next theorem about some equivalent criteria of convexity for real valued functions defined on $\mathbb{T}$.

Theorem 5 (Taylor's Theorem [**2**])**.** *Let $n \in \mathbb{N}$. Suppose that f is n-times differentiable on $\mathbb{T}^{\kappa^{n-1}}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$, and define the functions h_k by*

$$h_0(r,s) \equiv 1 \text{ and } h_{k+1}(r,s) = \int_s^r h_k(\tau, s) \Delta\tau \text{ for } k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_\alpha^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau.$$

Theorem 6 (Mean Value Theorem [**3**]). *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is differentiable on $[a,b)_{\mathbb{T}}$. Then there exists $\xi,\tau \in [a,b)_{\mathbb{T}}$ such that*

$$f^{\Delta}(\tau) \leq \frac{f(b)-f(a)}{b-a} \leq f^{\Delta}(\xi).$$

Theorem 7. *Let $f:\mathbb{T}\to\mathbb{R}$. The following are equivalent:*

(i) *f is convex on $\mathbb{T}$.*

(ii) *f satisfies the midpoint condition* (1).

(iii) *$f^{\Delta^2}(t) \geq 0$ for all $t\in\mathbb{T}$.*

(iv) *$f(x) \geq f(y) + (x-y)f^{\Delta}(y)$ for all $x,y\in\mathbb{T}$ with $x>y$,*
(or $f(x) \geq f(y)+(x-y)f^{\nabla}(y)$ for all $x<y$).

Proof. We prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).
(i) $\Rightarrow$ (ii): Let $a,b\in\mathbb{T}$ with $a<b$ and $[a,b]_{\mathbb{T}}$ have odd number of time points. This implies that $[a,b]_{\mathbb{T}}$ has a midpoint $m_{[a,b]}$. Since f is convex, by choosing $\lambda = \frac{b-m_{[a,b]}}{b-a}$, we obtain

$$f(m_{[a,b]}) \leq \left(\frac{b-m_{[a,b]}}{b-a}\right) f(a) + \left(\frac{m_{[a,b]}-a}{b-a}\right) f(b).$$

Then the midpoint condition (1) follows.

Next we prove that (ii) implies (iii). Let $t\in\mathbb{T}$. Then $\sigma(t)\in\mathbb{T}$. Applying the midpoint condition at $\sigma(t)$, we have

$$f(\sigma(t)) \leq \frac{\sigma^2(t)-\sigma(t)}{\sigma^2(t)-t} f(t) + \frac{\sigma(t)-t}{\sigma^2(t)-t} f(\sigma^2(t)).$$

Simple algebra implies that $f^{\Delta^2}(t)\geq 0$.

Next we prove that (iii) $\Rightarrow$ (i). Let $x,y\in\mathbb{T}$ with $x<y$. Fix $\lambda\in\mathbb{T}_{[x,y]}$. Define $x_0=\lambda x+(1-\lambda)y$. Using Taylor's theorem (Theorem 5) at x_0 we have

$$f(y) = \sum_{i=0}^{1} h_i(y,x_0) f^{\Delta^i}(x_0) + \sum_{\tau=x_0}^{\rho^2(y)} h_1(y,\sigma(\tau)) f^{\Delta^2}(\tau)(\sigma(\tau)-\tau).$$

Since $f^{\Delta^2}(\tau)\geq 0$ on $\mathbb{T}$ and $h_1(y,\sigma(\tau)) = y-\sigma(\tau)\geq 0$ on $\mathbb{T}$, we have

$$f(y) \geq f(x_0) + (y-x_0) f^{\Delta}(x_0). \tag{2}$$

Using the Mean Value Theorem (Theorem 6) for f on $[x,x_0]_{\mathbb{T}}$, there exists $\tau\in[x,x_0)_{\mathbb{T}}$ such that

$$\frac{f(x_0)-f(x)}{x_0-x} \leq f^{\Delta}(\tau).$$

Since $f^{\Delta^2}(t) \geq 0$ on $\mathbb{T}$, we have $f^{\Delta}(\tau) \leq f^{\Delta}(x_0)$. Therefore we obtain

$$f(x) \geq f(x_0) + (x - x_0)f^{\Delta}(x_0). \tag{3}$$

If we multiply the inequality (2) by $1-\lambda$ and the inequality (3) by λ and add them side by side, we obtain

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $\lambda \in \mathbb{T}_{[x,y]}$.
Now we prove (i) $\Rightarrow$ (iv). Given any $x, y \in \mathbb{T}$ such that $y < x$, by convexity of f on $[y, x]$, we have

$$f(\sigma(y)) \leq \frac{\sigma(y) - y}{x - y} f(x) + \frac{x - \sigma(y)}{x - y} f(y).$$

After rearranging the terms we have

$$f^{\Delta}(y) \leq \frac{f(x) - f(y)}{x - y}.$$

This simplifies to $f(x) \geq f(y) + (x-y)f^{\Delta}(y)$ for all $x > y$. The same argument works to show that

$$f(x) \geq f(y) + (x-y)f^{\nabla}(y) \text{ for all } x < y.$$

Finally we prove (iv) $\Rightarrow$ (iii). Given $f(x) \geq f(y) + (x-y)f^{\Delta}(y)$ for all $x > y$. By choosing $x = \sigma^2(y)$, we obtain $f^{\Delta^2}(y) \geq 0$. Since y is arbitrary, (iii) follows.
This completes the proof.

3. HERMITE-HADAMARD INEQUALITY ON DISCRETE TIME SCALES

In this section we prove discrete Hermite-Hadamard inequality for convex functions defined on a discrete time scale $\mathbb{T}$.

Theorem 8 (Substitution rule on time scales [**5**])**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a an rd-continuous function and ν is differentiable with rd-continuous derivative, then if $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s$$

or

$$\int_a^b f(t)\nu^{\nabla}(t)\nabla t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\nabla} s.$$

Next we state and prove the substitution rule for a strictly decreasing function $\nu : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ can now be any time scale, and not strictly isolated.

Theorem 9. *Assume $\nu : \mathbb{T} \to \mathbb{R}$ be strictly decreasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)(-\nu^{\Delta})(t)\,\Delta t = \int_{\nu(b)}^{\nu(a)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\nabla} s$$

and

$$\int_a^b f(t)(-\nu^{\nabla})(t)\,\nabla t = \int_{\nu(b)}^{\nu(a)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\Delta} s.$$

Proof. We prove the second equality. In the proof we use the basics of the dual time scales introduced in [**4**] and the substitution method for a strictly increasing function (Theorem 8). We start on the right side of the equality:

$$\begin{aligned}\int_{\nu(b)}^{\nu(a)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\Delta} s &= \int_{-\nu(a)}^{-\nu(b)} (f \circ \nu^{-1})^*(s)\overline{\nabla} s \\ &= \int_{-\nu(a)=(u^{-1}\circ\nu)(a)}^{-\nu(b)=(u^{-1}\circ\nu)(a)} f(u^{-1} \circ \nu)^{-1}(s)\overline{\nabla} s \\ &= \int_a^b f(t)(u^{-1} \circ \nu)^{\nabla}(t)\nabla t = \int_a^b f(t)(-\nu)^{\nabla}\nabla t,\end{aligned}$$

where $\overline{\mathbb{T}}$ represents the dual time scale, $u(s) := -s$ and

$$f(\nu^{-1})^*(s) = f(\nu^{-1}(-s)) = f((\nu^{-1} \circ u)(s)) = f((u^{-1} \circ \nu)^{-1}(s)).$$

Remark 2. We note that the statement of Theorem 2.3 (ii) in the paper [**5**] is not correct since $\widetilde{\mathbb{T}}$ was defined as $-\nu(\mathbb{T})$ for a strictly decreasing function ν.

We use the following notation in the proof of the next theorem: Let $a, b \in \mathbb{T}, a < b$ and the cardinality of $[a, b]_{\mathbb{T}}$ be an odd number, say $k + 1$. Let $t \in \mathbb{T}_{[a,b]}$. Then there exists an $n \in \mathbb{N} \cup \{0\}$ such that $t = \sigma^n(0)$. We denote $\hat{t}$ by $\sigma^{k-n}(0)$. Similarly, let $u \in [a, b]_{\mathbb{T}}$. Then there exists a $\ell \in \mathbb{N} \cup \{0\}$ such that $u = \sigma^{\ell}(a)$. We denote $\hat{u}$ by $\sigma^{k-\ell}(a)$. We also note that $\hat{\hat{u}} = u$ and $\hat{\hat{t}} = t$.

Next we illustrate this new notation with an example.

Example 1. Let $[a, b]_{\mathbb{T}} = \{x_0 = a, x_1, x_2, x_3, x_4, x_5, x_6 = b\}$, where $x_i < x_{i+1}$ for $0 \leq i \leq 5$. Then we have $\hat{x_i} = \sigma^{6-i}(a) = x_{6-i}$ for $0 \leq i \leq 6$. It follows that

$$\mathbb{T}_{[a,b]} = \{t_0 = 0, t_1 = \frac{b - x_5}{b - a}, t_2 = \frac{b - x_4}{b - a}, t_3 = \frac{b - x_3}{b - a}, t_4 = \frac{b - x_2}{b - a}, t_5 = \frac{b - x_1}{b - a}, t_6 = 1\}.$$

Hence we have $\hat{t_i} = \sigma^{6-i}(0) = t_{6-i}$ for $0 \leq i \leq 6$.

This implies that $\hat{t_i} = \dfrac{b - x_i}{b - a}$. One simple algebra step implies that

$$x_i = a\hat{t_i} + (1 - \hat{t_i})b.$$

On the other hand, $\hat{x_i} = x_{6-i} = at_i + (1 - t_i)b$.

Theorem 10. *Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a convex function on $[a,b]_{\mathbb{T}}$. Then*

$$f(m_{[a,b]}) \leq \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla(t) - \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t$$
$$\leq m_{[0,1]}f(a) + (1-m_{[0,1]})f(b),$$

where $g : [a,b]_{\mathbb{T}} \to [a,b]_{\mathbb{T}}$ is defined by $g(u) = \hat{u}$ and $k : [a,b]_{\mathbb{T}} \to \mathbb{R}^+$ is defined by

$$k(x) := \begin{cases} \dfrac{g(x) - m_{[a,b]}}{g(x) - x}, & x \neq m_{[a,b]} \\ 1/2, & x = m_{[a,b]}. \end{cases} \tag{4}$$

Proof. Fix $t \in \mathbb{T}_{[a,b]}$. Then there exists $x \in [a,b]_{\mathbb{T}}$ such that $x = ta + (1-t)b$. As we pointed out in Example 1, $\hat{x} = \hat{t}a + (1-\hat{t})b$. Denote $\hat{x}$ by y, i.e. $y = \hat{t}a + (1-\hat{t})b$. Note that $m_{[a,b]} = m_{[x,y]}$ using the definition of the hat operator.

Let $\xi : [a,b]_{\mathbb{T}} \to \mathbb{T}_{[a,b]}$ be a linear function defined as $\xi(u) = \dfrac{b-u}{b-a}$. Hence we have $\xi(x) = t$ and $\xi(y) = \hat{t}$. If $x \neq m_{[a,b]}$, then by convexity of f we have

$$f(m_{[a,b]}) \leq \frac{y - m_{[a,b]}}{y - x} f(x) + \frac{m_{[a,b]} - x}{y - x} f(y). \tag{5}$$

If $x = m_{[a,b]}$, then it reduces to $x = y = m_{[a,b]}$. Clearly we have

$$f(m_{[a,b]}) = \frac{1}{2}f(x) + \frac{1}{2}f(y). \tag{6}$$

We combine (5) and (6) using the function k

$$f(m_{[a,b]}) \leq k(x)f(x) + k(y)f(y).$$

Next we integrate each side of the above inequality from 0 to 1 on $\mathbb{T}_{[a,b]}$ and we obtain

$$f(m_{[a,b]})\int_{\mathbb{T}_{[a,b]}} \tilde{\Delta} t \leq \int_{\mathbb{T}_{[a,b]}} k(x)f(x)\tilde{\Delta} t + \int_{\mathbb{T}_{[a,b]}} k(y)f(y)\tilde{\Delta} t$$
$$= \int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(t))f(\xi^{-1}(t))\tilde{\Delta} t + \int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(\hat{t}))f(\xi^{-1}(\hat{t}))\tilde{\Delta} t.$$

Here we first claim that

$$\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(t))f(\xi^{-1}(t))\tilde{\Delta} t = \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla t. \tag{7}$$

Let's define $F := k \cdot f$. Then we have $F(\xi^{-1}(t)) = k(\xi^{-1}(t))f(\xi^{-1}(t))$.

Next, we apply the substitution rule (Theorem 9) to the integral on the left side of the equality in (7).

$$\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(t))f(\xi^{-1}(t))\tilde{\Delta} t = \int_{\mathbb{T}_{[a,b]}} F(\xi^{-1}(t))\tilde{\Delta} t = \int_a^b F(t)\frac{1}{b-a}\nabla t.$$

This finishes the proof of our first claim.

Next we claim that

$$\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(\hat{t}))f(\xi^{-1}(\hat{t}))\tilde{\Delta}t = -\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t. \tag{8}$$

Before we prove the equality (8), we point out that the function g is a bijection and $g \equiv g^{-1}$ since g^2 is an identity function. As a result of this, we have $g(\hat{t}a+(1-\hat{t})b) = ta + (1-t)b$. This observation will help us to complete the proof of the claim. By applying $w(u) = \widehat{\xi(u)}$ to the integral on the left side of the equality (8), we have

$$\begin{aligned}\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(\hat{t}))f(\xi^{-1}(\hat{t}))\tilde{\Delta}t &= \int_{\mathbb{T}_{[a,b]}} F(\xi^{-1}(\hat{t}))\tilde{\Delta}t \\ &= \int_0^1 (F\circ w^{-1})(t)\tilde{\Delta}t = -\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)F(t)\Delta t,\end{aligned}$$

where

$$\begin{aligned} w^{\Delta}(u) &= \frac{w(\sigma(u)) - w(u)}{\sigma(u)-u} = \frac{\widehat{\xi(\sigma(u))} - \widehat{\xi(u)}}{\sigma(u)-u} \\ &= \frac{\frac{b-\widehat{\sigma(u)}}{b-a} - \frac{b-\hat{u}}{b-a}}{\sigma(u)-u} = \frac{1}{(a-b)}\frac{(\widehat{\sigma(u)}-\hat{u})}{\sigma(u)-u} = \frac{g^{\Delta}(u)}{a-b} \geq 0,\end{aligned}$$

since $\widehat{\sigma(u)} < \hat{u}$. This completes the proof of our second claim.

To prove the right side of the inequality, we construct the following inequalities using convexity of f.

$$\begin{aligned} f(x) &\leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b), \\ f(y) &\leq \frac{b-y}{b-a}f(a) + \frac{y-a}{b-a}f(b).\end{aligned}$$

Next, we multiply both inequality by $k(x)$ and $k(y)$ respectively. We obtain

$$\begin{aligned} k(x)f(x) &\leq \frac{b-x}{b-a}k(x)f(a) + \frac{x-a}{b-a}k(x)f(b), \\ k(y)f(y) &\leq \frac{b-y}{b-a}k(y)f(a) + \frac{y-a}{b-a}k(y)f(b).\end{aligned}$$

Simple algebra implies the following identities

$$\begin{aligned} \frac{b-x}{b-a}k(x) + \frac{b-y}{b-a}k(y) &= \frac{b-m_{[a,b]}}{b-a}, \\ \frac{x-a}{b-a}k(x) + \frac{y-a}{b-a}k(y) &= \frac{m_{[a,b]}-a}{b-a}.\end{aligned}$$

Recall that x and y both depend on t. We let t vary over $\mathbb{T}_{[a,b]}$ and integrate each side of the last two inequalities on $\mathbb{T}_{[a,b]}$ and we add them side by side, we obtain

$$\begin{aligned}&\int_{\mathbb{T}_{[a,b]}} k(\nu^{-1}(t))f(\nu^{-1}(t))\tilde{\Delta}t + \int_{\mathbb{T}_{[a,b]}} k(\nu^{-1}(\hat{t}))f(\nu^{-1}(\hat{t}))\tilde{\Delta}t\\ &\leq \frac{b-m_{[a,b]}}{b-a}f(a) + \frac{m_{[a,b]}-a}{b-a}f(b) = m_{[0,1]}f(a) + (1-m_{[0,1]})f(b),\end{aligned}$$

where the last equality holds by means of Remark 1.

Corollary 1. *Suppose that $f : h\mathbb{Z} \to \mathbb{R}$ is a convex function with $h > 0$, $a, b \in h\mathbb{Z}, a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_{[a,b]_{h\mathbb{Z}}} f(t)\Delta t + \int_{[a,b]_{h\mathbb{Z}}} f(t)\nabla t\right] \leq \frac{f(a)+f(b)}{2}. \tag{9}$$

Proof. Here g and k simplify into $g(x) = a+b-x$ and $k(x) = 1/2$ and $g^{\Delta}(x) = -1$. Hence we have the desired inequality. ☐

When $h = 1$, we obtain the Hermite-Hadamard inequality on $\mathbb{Z}$.

Corollary 2. [**1**] *Suppose that $f : \mathbb{Z} \to \mathbb{R}$ is a convex function on $[a,b]_{\mathbb{Z}}$ with $a, b \in \mathbb{Z}$, $a < b$, and $a+b$ an even number. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_a^b f(t)\Delta t + \int_a^b f(t)\nabla t\right] \leq \frac{f(a)+f(b)}{2}. \tag{10}$$

Next we give an alternate proof of the Hermite-Hadamard inequality on $\mathbb{R}$ (continuous Hermite-Hadamard inequality) by using the main result of this paper. For this purpose, we first state the following lemma without giving its proof.

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function on $\mathbb{R}$. Then for any $h > 0$, its restriction to $h\mathbb{Z}$ is also a convex function.*

Corollary 3. *Let f be a real convex function on the finite interval $[a,b] \subset \mathbb{R}$. Then f satisfies the continuous Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\left(\int_a^b f(t)\mathrm{d}t\right) \leq \frac{f(a)+f(b)}{2}.$$

Proof. By restricting f to $h\mathbb{Z}$ we obtain the inequality (9). Since f is convex on $[a,b]$, it is continuous on (a,b), hence integrable on $[a,b]$. When h tends to zero, the $\Delta-$integral and $\nabla-$integral converge to the Riemann integral of f on $[a,b]$. In other words,

$$\lim_{h\to 0} \sum_{t\in[a,b)_{h\mathbb{Z}}} f(t)h = \int_a^b f(t)\mathrm{d}t \text{ and } \lim_{h\to 0} \sum_{t\in(a,b]_{h\mathbb{Z}}} f(t)h = \int_a^b f(t)\mathrm{d}t.$$

Hence the result follows.

4. APPLICATIONS

(**i**). Let $f(t) = (1+h)^{t/h}$ be a function on $h\mathbb{Z}$ for some positive real number h. Since $f^{\Delta^2}(t) = f(t) \geq 0$, f satisfies Hermite-Hadamard inequality on the interval $[a,b]_{h\mathbb{Z}}$, where $a, b \in h\mathbb{Z}$.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_{[a,b]_{h\mathbb{Z}}} f(t)\Delta t + \int_{[a,b]_{h\mathbb{Z}}} f(t)\nabla t\right] \leq \frac{f(a)+f(b)}{2}, \tag{11}$$

where

$$\int_{[a,b]_{h\mathbb{Z}}} f(t)\Delta t = \sum_{t\in[a,b)_{h\mathbb{Z}}} (1+h)^{t/h}h = (1+h)^{b/h} - (1+h)^{a/h},$$
$$\int_{[a,b]_{h\mathbb{Z}}} f(t)\nabla t = \sum_{t\in(a,b]_{h\mathbb{Z}}} (1+h)^{t/h}h = (1+h)^{1+b/h} - (1+h)^{1+a/h}.$$

Hence it follows that

$$\begin{aligned}(1+h)^{\frac{a+b}{2h}} &\leq \frac{1}{2(b-a)}\big([(1+h)^{b/h} - (1+h)^{a/h}] + [(1+h)^{1+b/h} - (a+h)^{1+a/h}]\big)\\ &\leq \frac{(1+h)^{a/h} + (1+h)^{b/h}}{2}.\end{aligned}$$

Now, let $x = f(a)$ and $y = f(b)$, then the above inequality (11) simplifies into

$$\sqrt{xy} \leq \frac{2+h}{h}\left[\frac{y-x}{f^{-1}(y) - f^{-1}(x)}\right] \leq \frac{x+y}{2}, \tag{12}$$

for all $x, y \in f(h\mathbb{Z})$.

Note that inequality (12) holds for all $x, y \in f(h\mathbb{Z})$ for a given $h > 0$. Now we let h vary and take the limit as h goes to zero. Then $f(t) = (1+h)^{t/h}$ converges to e^t and the inequality turns into the well-known geometric-logarithmic-arithmetic mean inequality:

$$\sqrt{xy} \leq \frac{y-x}{\ln y - \ln x} \leq \frac{x+y}{2},$$

for all x, y positive real numbers.

(**ii**). Let $f : h\mathbb{N} \to \mathbb{R}$ be defined as $f(t) = \frac{1}{t}$ for some positive real number h. Since $f^{\Delta^2}(t) = \dfrac{2}{t(t+h)(t+2h)} \geq 0$, f is convex on $h\mathbb{N}$. Hence the Hermite-Hadamard inequality holds and we obtain

$$\frac{2}{a+b} \leq \frac{H(b,a)}{b-a} - \frac{h}{2ab} \leq \frac{\frac{1}{a} + \frac{1}{b}}{2},$$

where $a, b \in h\mathbb{N}$ and $H(b,a) = \displaystyle\int_{[a,b]_{h\mathbb{Z}}} \frac{1}{t}\Delta t$.

We take the limit as h goes to 0. Then we have

$$\frac{2}{a+b} \leq \frac{\ln(b)-\ln(a)}{b-a} \leq \frac{\frac{1}{a}+\frac{1}{b}}{2},$$

for all a, b positive real numbers.

Theorem 11. *Let $\mathbb{T}$ be a discrete time scale and f be function on $\mathbb{T}$, not necessarily convex, satisfying $\alpha \leq f^{\Delta^2}(t) \leq \beta$. Then we get*

$$\alpha U \leq \left(\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla t - \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t\right) - f(m_{[a,b]}) \leq \beta U$$
$$\alpha V \leq m_{[0,1]}f(a) + (1-m_{[0,1]})f(b) - \left(\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla t - \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t\right) \leq \beta V,$$

where

$$U = \frac{1}{(b-a)}\left[\int_{[a,b]_{\mathbb{T}}} k(t)h_2(t)\nabla t - \int_{[a,b]_{\mathbb{T}}} k(t)g^{\Delta}(t)h_2(t)\Delta t\right] - h_2(m_{[a,b]}) \text{ and}$$
$$V = m_{[0,1]}h_2(a) + (1-m_{[0,1]})h_2(b) - \frac{1}{(b-a)}\left[\int_{[a,b]_{\mathbb{T}}} k(t)h_2(t)\nabla t - \int_{[a,b]_{\mathbb{T}}} k(t)g^{\Delta}(t)h_2(t)\Delta t\right].$$

Proof. Let $h_2(t)$ be the Taylor monomial with $s = 0$. In other words it is a function on $\mathbb{T}$ whose second $\Delta-$derivative is 1. Taylor monomials are defined in Theorem 5. Let $F(t) := f(t) - \alpha h_2(t)$ and $G(t) := \beta h_2(t) - f(t)$. Since $\alpha \leq f^{\Delta^2}(t) \leq \beta$ we have $F^{\Delta^2}(t) \geq 0$ and $G^{\Delta^2}(t) \geq 0$. Therefore F and G are convex. If we apply the Hermite Hadamard inequality for both F and G, then we derive the desired inequalities.

Corollary 4. *If $\mathbb{T} = \mathbb{Z}$, then $U = \frac{(b-a)^2+2}{24}$ and $V = \frac{(b-a)^2-1}{12}$.*

Corollary 5. *If $\mathbb{T} = q^{\mathbb{N}}$, then*

$$U = \frac{1}{q(1+q)^2(b-a)}\left[q^2\sqrt{ab}(b^2-a^2) + 2n(q^2-1)(ab)^{3/2} - 2(q^2+q)ab(b-a)\right] \text{ and}$$
$$V = \frac{1}{q(1+q)^2(b-a)}\left[q\sqrt{ab}(b^2-a^2) - 2n(q^2-1)(ab)^{3/2}\right].$$

The following two remarks will state some open problems for the researchers who are interested in working in this area.

Remark 3. Midpoint condition plays an important role to prove the main result of this paper. Even though the convexity of the function on any time scale has been defined in [**1, 11**], it is still an open problem to define the midpoint condition for such a function.

Remark 4. As it is pointed out in the paper [**12**], the Hermite-Hadamard inequality for the function defined on a real interval can characterize the convexity of the function. Again this can be done as another direction of the research for the class of functions we consider in this paper.

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