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FACTORIZED DIFFERENCE SCHEME FOR 2D FRACTIONAL IN TIME DIFFUSION EQUATION

Sandra Hodžić

A factorized finite-difference scheme for numerical approximation of initial-boundary value problem for two-dimensional fractional in time diffusion equation is proposed. Its stability is investigated and a convergence rate estimate is obtained.

1. INTRODUCTION

In recent years there has been increasing interest in modeling the physical and chemical processes with equations involving fractional derivatives and integrals. Fractional partial differential equations emerge in modeling diverse processes occurring in viscoelastic media, disordered materials, media of fractal geometry, as well as in the mathematical modeling of biological, social and economic phenomena (see [**7, 11, 12**]).

One kind of these equations is the time-fractional diffusion equation, which can be obtained from the standard diffusion equation by replacing classical time derivative with fractional derivative of order $\alpha \in (0, 1)$. It corresponds to anomalous sub-diffusion which has been investigated by many authors. When the initial-boundary value problems for time fractional diffusion equation are approximated by finite difference methods, in multidimensional case a similar phenomenon is observed as for the classical parabolic equation: explicit scheme is numerically unstable, while the implicit scheme is stable but inefficient, since at each time layer an elliptic problem needs to be solved.

In this article we consider the first initial-boundary value problem for two-dimensional fractional in time diffusion equation. The problem is approximated

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by the factorized finite difference scheme, which can be regarded as a kind of alternating-direction-implicit (ADI) scheme (see [**13**]). In [**4**] the same problem is approximated by additive scheme - another type of ADI scheme. In [**2, 8**] multidimensional evolution equations with fractional in space derivatives are approximated with diverse ADI schemes.

The paper is organized as follows. In Section 2 we introduce the Riemann-Liouville fractional derivative. In Section 3 we define several function spaces containing functions with fractional derivatives. In Section 4 we define the first initial-boundary value problem for two-dimensional fractional in time diffusion equation. In Section 5 we define the factorized difference scheme by approximating the considered problem and prove its stability. In Section 6 we investigate the convergence of the proposed difference scheme.

2. FRACTIONAL DERIVATIVES

Let u be a function defined on interval $[a,b]$, and $k-1<\alpha<k$, $k\in\mathbb{N}$. Then the left Riemann-Liouville fractional derivative of order α is defined as

$$D^{\alpha}_{a+}u(t)=\frac{1}{\Gamma(k-\alpha)}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\int_a^t\frac{u(\tau)}{(t-\tau)^{\alpha+1-k}}\,\mathrm{d}\tau \tag{1}$$

and the right Riemann-Liouville fractional derivative is defined analogously

$$D^{\alpha}_{b-}u(t)=\frac{(-1)^k}{\Gamma(k-\alpha)}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\int_t^b\frac{u(\tau)}{(t-\tau)^{\alpha+1-k}}\,\mathrm{d}\tau, \tag{2}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Notice that if the function $u(t)$ has continuous derivative of order k in $[a,b]$, then for $\alpha\to k$ or $\alpha\to k-1$, the left (right) Riemann-Liouville derivative converges to a standard k– or $(k-1)$–order derivative of $u(t)$.

Since the integral is present in the definition of the fractional order derivatives, it is apparent that these derivatives are nonlocal operators.

For functions of many variables, the partial Riemann-Liouville fractional derivatives are defined in an analogous manner, for example

$$D^{\alpha}_{t,a+}u(x,t)=\frac{1}{\Gamma(k-\alpha)}\frac{\partial^k}{\partial t^k}\int_a^t\frac{u(x,\tau)}{(t-\tau)^{\alpha+1-k}}\,\mathrm{d}\tau,\quad k-1<\alpha<k,\quad k\in\mathbb{N}.$$

3. SOME FUNCTION SPACES

We define some function spaces, norms and inner products that we use thereafter. Let Ω be an open domain in $\mathbb{R}^n$. By $C^k(\Omega)$ and $C^k(\bar{\Omega})$ we denote the spaces of k times continuously differentiable functions in Ω and $\bar{\Omega}$, respectively. Further,

$\dot{C}^\infty(\Omega) = C_0^\infty(\Omega)$ stands for the space of infinitely differentiable functions with compact support in Ω. As usual, the space of measurable functions whose squares are Lebesgue integrable in Ω is denoted by $L^2(\Omega)$. The inner product and the norm in this space are defined by

$$(u,v)_\Omega = (u,v)_{L^2(\Omega)} = \int_\Omega uv \,\mathrm{d}\Omega, \qquad \|u\|_\Omega = \|u\|_{L^2(\Omega)} = (u,u)_\Omega^{1/2}.$$

We also use $H^\alpha(\Omega)$ and $\dot{H}^\alpha(\Omega) = H_0^\alpha(\Omega)$ to denote the usual Sobolev spaces [**10**] with norms denoted by $\|u\|_{H^\alpha(\Omega)}$.

For $\alpha > 0$ we set

$$|u|_{C_+^\alpha[a,b]} = \|D_{a+}^\alpha u\|_{C[a,b]}, \qquad |u|_{C_-^\alpha[a,b]} = \|D_{b-}^\alpha u\|_{C[a,b]},$$
$$\|u\|_{C_\pm^\alpha[a,b]} = \big(\|u\|^2_{C^{[\alpha]^-}[a,b]} + |u|^2_{C_\pm^\alpha[a,b]}\big)^{1/2},$$
$$|u|_{H_+^\alpha(a,b)} = \|D_{a+}^\alpha u\|_{L^2(a,b)}, \qquad |u|_{H_-^\alpha(a,b)} = \|D_{b-}^\alpha u\|_{L^2(a,b)}$$

and

$$\|u\|_{H_\pm^\alpha(a,b)} = \big(\|u\|^2_{H^{[\alpha]^-}(a,b)} + |u|^2_{H_\pm^\alpha(a,b)}\big)^{1/2},$$

where $[\alpha]^-$ denotes the largest integer $< \alpha$. Then we define $C_\pm^\alpha[a,b]$ as the space of functions $u \in C^{[\alpha]^-}[a,b]$ with the finite norms $\|u\|_{C_\pm^\alpha[a,b]}$. The space $H_\pm^\alpha(a,b)$ is defined analogously, while the space $\dot{H}_\pm^\alpha(a,b)$ is defined as the closure of $\dot{C}^\infty(a,b)$ with the respect to the norm $\|\cdot\|_{H_\pm^\alpha(a,b)}$. As for $\alpha = k \in \mathbb{N}\cup\{0\}$ fractional derivative reduces to standard k-order derivative, we have $C_\pm^k[a,b] = C^k[a,b]$ and $H_\pm^k(a,b) = H^k(a,b)$.

Lemma 1 (See [**9**]). *For $\alpha > 0$, $\alpha \neq k + 1/2$, $k \in \mathbb{N}$, the spaces $\dot{H}_+^\alpha(a,b)$, $\dot{H}_-^\alpha(a,b)$ and $\dot{H}^\alpha(a,b)$ are equal and their seminorms as well as norms are equivalent.*

For vector valued functions which map a real interval $(0,T)$ (or $[0,T]$) into Banach space X, we introduce the spaces $C^k([0,T],X)$, $k \in \mathbb{N}\cup\{0\}$ and $H^\alpha((0,T),X)$, $\alpha \geq 0$, in the usual way [**10**]. In analogous manner, we define the spaces $C_\pm^\alpha([0,T],X)$ and $H_\pm^\alpha((0,T),X)$.

Throughout the paper, by C we denote positive generic constant which may take different values in different formulas.

4. PROBLEM FORMULATION

Let $0 < \alpha < 1$, $\Omega = (0,1)\times(0,1)$ and $Q = \Omega \times (0,T)$. We consider the time fractional diffusion equation

$$D_{t,0+}^\alpha u - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y,t), \quad (x,y,t) \in Q, \tag{3}$$

subject to homogeneous initial and boundary conditions

$$u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \in (0,T), \tag{4}$$

$$u(x, y, 0) = 0, \quad (x, y) \in \bar{\Omega}. \tag{5}$$

Initial-boundary value problem (3)–(5) is often called sub-diffusion problem.

For $f \in L^2(Q)$, the problem (3)–(5) is well posed in the space $\dot{H}^{1,\alpha/2}(Q) = L^2((0,T), \dot{H}^1(\Omega)) \cap \dot{H}^{\alpha/2}((0,T), L^2(\Omega))$ and its weak solution satisfies the a priori estimate (see [**6, 9**])

$$\|u\|_{H^{1,\alpha/2}(Q)} \leq C\|f\|_{L^2(Q)}.$$

5. FINITE DIFFERENCE APPROXIMATION

In the domain $\bar{Q} = [0,1] \times [0,1] \times [0,T]$, we define the uniform mesh $\bar{Q}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$, where $\bar{\omega}_h = \{(x_i, y_j) = (ih, jh) \,|\, i,j = 0,1,\ldots,n;\ h = 1/n\}$ and $\bar{\omega}_\tau = \{t_k = k\tau \,|\, k = 0,1,\ldots,m;\ \tau = T/m\}$. We also define $\omega_h = \bar{\omega}_h \cap \Omega$, $\gamma_h = \bar{\omega}_h \setminus \omega_h$, $\omega_{1h} = \bar{\omega}_h \cap (0,1] \times (0,1)$, $\omega_{2h} = \bar{\omega}_h \cap (0,1) \times (0,1]$, $\omega_\tau = \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^- = \bar{\omega}_\tau \cap [0,T)$ and $\omega_\tau^+ = \bar{\omega}_\tau \cap (0,T]$. We use standard notation from the theory of the finite difference schemes (see [**13**]):

$$v = v(x,y,t), \quad \hat{v} = v(x,y,t+\tau), \quad v^k = v(x,y,t_k), \quad (x,y) \in \bar{\omega}_h,$$

$$v_x = \frac{v(x+h,y,t) - v(x,y,t)}{h} = v_{\bar{x}}(x-h,y,t),$$

$$v_y = \frac{v(x,y+h,t) - v(x,y,t)}{h} = v_{\bar{y}}(x,y-h,t).$$

For a function u defined on $\bar{Q}$ which satisfies the homogeneous initial condition, we approximate the left Riemann-Liouville fractional derivative $D^\alpha_{t,0+}u(x,y,t_k)$ by (see [**3**]):

$$(D^\alpha_{t,0+,\tau}u)^k = \frac{1}{\Gamma(2-\alpha)} \sum_{\ell=0}^{k-1} (t^{1-\alpha}_{k-\ell} - t^{1-\alpha}_{k-\ell-1}) u^\ell_t.$$

The following result holds:

Lemma 2 (See [**14**]). *Suppose that* $u \in C^2[0,t]$, $t \in \omega_\tau^+$. *Then*

$$|D^\alpha_{t,0+}u - D^\alpha_{t,0+,\tau}u| \leq \tau^{2-\alpha} \frac{1}{1-\alpha} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha}) \right] \max_{0 \leq s \leq t} |u''(s)|.$$

We approximate the initial-boundary value problem (3)–(5) with the following factorized finite difference scheme:

$$\big((I + \theta\tau^\alpha A_1)\,(I + \theta\tau^\alpha A_2)\, D^\alpha_{t,0+,\tau} v\big)^k - \Delta_h v^{k-1} = \bar{f}^k, \quad (x,y) \in \omega_h, \tag{6}$$

$k = 1, 2, \ldots, m,$ subject to homogeneous boundary and initial conditions:

$$v(x,y,t) = 0, \quad (x,y,t) \in \gamma_h \times \omega_\tau^+, \tag{7}$$

(8) $$v(x,y,0)=0, \quad (x,y)\in\bar{\omega}_h,$$

where I is the identity operator, θ is positive parameter, $A_1v=-v_{x\bar{x}}$, $A_2v=-v_{y\bar{y}}$ and $\Delta_h v=v_{x\bar{x}}+v_{y\bar{y}}=-Av$.

When the right-hand side f is continuous function, we set $\bar{f}=f$, otherwise we have to use some averaged value, for example $\bar{f}=T_1T_2f$, where T_1 and T_2 are the Steklov averaging operators:

$$T_1f(x,y,t)=\int_{-1/2}^{1/2} f(x+hs,y,t)\,\mathrm{d}s, \quad T_2f(x,y,t)=\int_{-1/2}^{1/2} f(x,y+hs,t)\,\mathrm{d}s.$$

Let us note that the finite difference scheme (6)-(8) is numerically efficient, unlike the standard implicit scheme [**15**]. Indeed, to compute the values of solution v at the time layer $t=t_k$ it is necessary to invert the operators $(I+\theta\tau^\alpha A_1)$ and $(I+\theta\tau^\alpha A_2)$. By suitable ordering of mesh nodes each of these operators can be represented by a tridiagonal matrix. This way, the required solutions are obtained by two applications of the Thomas algorithm.

We define the following discrete inner products and norms:

$$(v,w)_h=(v,w)_{L^2(\omega_h)}=h^2\sum_{(x,y)\in\omega_h} vw, \quad \|v\|_h=\|v\|_{L^2(\omega_h)}=(v,v)_h^{1/2},$$

$$(v,w)_{ih}=(v,w)_{L^2(\omega_{ih})}=h^2\sum_{(x,y)\in\omega_{ih}} vw, \|v\|_{ih}=\|v\|_{L^2(\omega_{ih})}=(v,v)_{ih}^{1/2}, i=1,2,$$

$$|v|^2_{H^1(\omega_h)}=\|v_{\bar{x}}\|^2_{1h}+\|v_{\bar{y}}\|^2_{2h}, \quad \|v\|^2_{H^1(\omega_h)}=|v|^2_{H^1(\omega_h)}+\|v\|^2_h,$$

$$\|v\|_{L^2(Q_{h\tau})}=\left(\tau\sum_{k=1}^m \|v^k\|^2_h\right)^{1/2},$$

$$\|v\|_{B^{1,\alpha/2}(Q_{h\tau})}=\left[\tau\sum_{k=1}^m \|v^k\|^2_{H^1(\omega_h)}+\tau\sum_{k=1}^m\left(D^\alpha_{t,0+,\tau}\big(\|v\|^2_h\big)\right)^k\right]^{1/2}.$$

For every function $v(t)$ defined on the mesh $\bar{\omega}_\tau$, which satisfies the initial condition $v(0)=0$, the following equality holds (see [**3**])

$$\tau\sum_{k=1}^m\left(D^\alpha_{t,0+,\tau}(v^2)\right)^k=\frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^m\left(t^{1-\alpha}_{m-k+1}-t^{1-\alpha}_{m-k}\right)\left(v^k\right)^2.$$

In particular, from here it follows that the norm $\|v\|_{B^{1,\alpha/2}(Q_{h\tau})}$ is well defined.

Lemma 3 (See [**4**]). *For $0<\alpha<1$ and arbitrary function $v(t)$ defined on the mesh $\bar{\omega}_\tau$ the following inequality is valid*

(9) $$v^k\left(D^\alpha_{t,0+,\tau}v\right)^k\ge\frac{1}{2}\left(D^\alpha_{t,0+,\tau}(v^2)\right)^k+\frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}\left(v_t^{k-1}\right)^2.$$

Theorem 1. *Let $\alpha\in(0,1)$ and $\theta\ge\dfrac{\Gamma(2-\alpha)}{2(1-2^{-\alpha})}$. Then the finite difference scheme* (6)–(8) *is absolutely stable and its solution satisfies the following a priori estimate*:

(10) $$\|v\|_{B^{1,\alpha/2}(Q_{h\tau})}\le C\|\bar{f}\|_{L^2(Q_{h\tau})}.$$

Proof. Taking the inner product of (6) with v^k, we obtain

$$\left(v^k, BD^{\alpha}_{t,0+,\tau}v^k\right)_h + \left(v^k, Av^k\right)_h - \left(v^k, A(v^k - v^{k-1})\right)_h = \left(v^k, \bar{f}^k\right)_h,$$

where $A = A_1 + A_2 = -\Delta_h$ and $B = (I + \theta\tau^{\alpha}A_1)(I + \theta\tau^{\alpha}A_2)$. Operators A and B are positive and self-adjoint, so the corresponding energy norms (see [**13**]) $\|v\|_A = (Av, v)_h^{1/2}$ and $\|v\|_B = (Bv, v)_h^{1/2}$ are well defined.

Using the Lemma 3, Cauchy-Schwarz and ε inequalities, and taking into account that $v^k = \dfrac{v^k + v^{k-1}}{2} + \dfrac{v^k - v^{k-1}}{2}$, we further obtain

$$\frac{1}{2}\,D^{\alpha}_{t,0+,\tau}\|v^k\|^2_B + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}\|v^{k-1}_t\|^2_B + \frac{1}{2}\|v^k\|^2_A + \frac{1}{2}\|v^{k-1}\|^2_A - \frac{\tau^2}{2}\|v^{k-1}_t\|^2_A$$
$$\leq \frac{1}{4\varepsilon}\|\bar{f}^k\|^2_h + \varepsilon\|v^k\|^2_h.$$

From operator inequalities

$$B = I + \theta\tau^{\alpha}A + \theta^2\tau^{2\alpha}A_1A_2 \geq I, \qquad B \geq \theta\tau^{\alpha}A,$$

the equality

$$\|v\|_A = |v|_{H^1(\omega_h)}$$

and discrete Poincaré inequality (see [**13**])

$$\|v\|_h \leq \frac{1}{2\sqrt{2}}|v|_{H^1(\omega_h)}$$

it follows that

$$\frac{1}{2}\,D^{\alpha}_{t,0+,\tau}\|v^k\|^2_h + \left(\theta\frac{(1-2^{-\alpha})}{\Gamma(2-\alpha)} - \frac{1}{2}\right)\tau^2\|v^{k-1}_t\|^2_A + \frac{1}{2}|v^k|^2_{H^1(\omega_h)} + \frac{1}{2}|v^{k-1}|^2_{H^1(\omega_h)}$$
$$\leq \frac{1}{4\varepsilon}\|\bar{f}^k\|_h + \frac{\varepsilon}{8}|v^k|^2_{H^1(\omega_h)}.$$

For $\theta \geq \dfrac{\Gamma(2-\alpha)}{2(1-2^{-\alpha})}$, by choosing $\varepsilon = 2$ and summing for $k = 1, 2, \ldots, m$, we obtain the a priori estimate (10) with $C = \dfrac{3}{4}$. □

In the Theorem 1 it was assumed that the parameter θ must be greater than some value which depends on α. Fig. 1 shows the graph of the function $\alpha \mapsto \dfrac{\Gamma(2-\alpha)}{2(1-2^{-\alpha})}$.

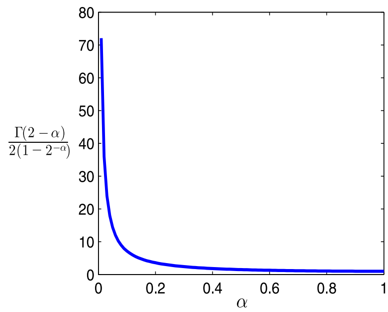

Fig. 1

6. CONVERGENCE OF THE DIFFERENCE SCHEME

Let u be the solution of the initial-boundary-value problem (3)–(5) and v the solution of the difference scheme (6)–(8) with $\bar{f} = T_1 T_2 f$. The error $z = u - v$ is defined on the mesh $\bar{\omega}_h \times \bar{\omega}_\tau$. Putting $v = -z + u$ into (6)–(8) it follows that error satisfies

$$\big((I + \theta\tau^\alpha A_1)\,(I + \theta\tau^\alpha A_2)\, D^\alpha_{t,0+,\tau} z \big)^k - \Delta_h z^{k-1} = \psi^k, \qquad (x,y) \in \omega_h, \quad k = 1, 2, \ldots, m, \tag{11}$$

$$z = 0, \quad (x,y) \in \gamma_h, \quad t \in \bar{\omega}_\tau, \tag{12}$$

$$z^0 = z(x,y,0) = 0, \quad (x,y) \in \omega_h, \tag{13}$$

where

$$\begin{aligned}
\psi^k &= (I + \theta\tau^\alpha A_1)\,(I + \theta\tau^\alpha A_2)\, D^\alpha_{t,0+,\tau} u^k - \Delta_h u^{k-1} - T_1 T_2 f^k \\
&= \big((I + \theta\tau^\alpha A_1)\,(I + \theta\tau^\alpha A_2)\, D^\alpha_{t,0+,\tau} u - T_1 T_2 D^\alpha_{t,0+} u \big)^k \\
&\quad + \left[\left(T_1 T_2 \frac{\partial^2 u}{\partial x^2} \right)^k - u^{k-1}_{x\bar{x}} \right] + \left[\left(T_1 T_2 \frac{\partial^2 u}{\partial y2} \right)^k - u^{k-1}_{y\bar{y}} \right] \\
&= \xi_1^k + \xi_2^k + \eta^k_{1,x} + \eta^k_{2,y} + \zeta^k_{1,x} + \zeta^k_{2,y} + \chi^k_{1,x} + \chi^k_{2,y} + \mu^k_{1,x} + \mu^k_{2,y},
\end{aligned}$$

where

$$\begin{aligned}
&\xi_1 = D^\alpha_{t,0+,\tau} u - D^\alpha_{t,0+} u, \quad \xi_2 = D^\alpha_{t,0+} u - T_1 T_2 D^\alpha_{t,0+} u, \\
&\eta_1 = T_2 \frac{\partial u}{\partial x}(x - h/2, y, t) - u_{\bar{x}}(x,y,t), \quad \eta_2 = T_1 \frac{\partial u}{\partial y}(x, y - h/2, t) - u_{\bar{y}}(x,y,t), \\
&\zeta_1 = \tau u_{\bar{t}\bar{x}}, \quad \zeta_2 = \tau u_{\bar{t}\bar{y}}, \\
&\chi_1 = -\theta\tau^\alpha D^\alpha_{t,0+,\tau} u_{\bar{x}}, \quad \chi_2 = -\theta\tau^\alpha D^\alpha_{t,0+,\tau} u_{\bar{y}}, \\
&\mu_1 = \frac{1}{2}\,\theta^2\tau^{2\alpha} D^\alpha_{t,0+,\tau} u_{\bar{x}y\bar{y}}, \quad \mu_2 = \frac{1}{2}\,\theta^2\tau^{2\alpha} D^\alpha_{t,0+,\tau} u_{x\bar{x}\bar{y}}.
\end{aligned}$$

Lemma 4. *Let* $\theta \geq \dfrac{\Gamma(2-\alpha)}{2(1-2^{-\alpha})}$. *The finite difference scheme* (11)–(13) *is absolutely stable and the following a priori estimate holds*:

$$\|z\|_{B^{1,\alpha/2}(Q_{h\tau})} \leq C \Big[\sum_{i=1}^{2} \tau \sum_{k=1}^{m} \Big(\|\xi_i^k\|_h^2 + \|\eta_i^k\|_{ih}^2 + \|\zeta_i^k\|_{ih}^2 + \|\chi_i^k\|_{ih}^2 + \|\mu_i^k\|_{ih}^2 \Big) \Big]^{1/2}. \tag{14}$$

Proof. Taking the inner product of (11) with z^k, we obtain

$$\begin{aligned}
&\big(z^k, (I + \theta\tau^\alpha A_1)\,(I + \theta\tau^\alpha A_2)\, D^\alpha_{t,0+,\tau} z^k \big)_h - \big(z^k, \Delta_h z^{k-1} \big)_h - \big(z^k, \xi_1^k \big)_h \\
&= \big(z^k, \xi_2^k \big)_h + \big(z^k, \eta^k_{1,x} \big)_h + \big(z^k, \eta^k_{2,y} \big)_h + \big(z^k, \zeta^k_{1,x} \big)_h + \big(z^k, \zeta^k_{2,y} \big)_h \\
&\quad + \big(z^k, \chi^k_{1,x} \big)_h + \big(z^k, \chi^k_{2,y} \big)_h + \big(z^k, \mu^k_{1,x} \big)_h + \big(z^k, \mu^k_{2,y} \big)_h
\end{aligned}$$

Next, we estimate the terms at the right-hand side:

$$\begin{aligned}
\left|\left(z^k, \xi_2^k\right)_h\right| &\le \frac{\varepsilon_1}{2}\left\|z^k\right\|_h^2 + \frac{1}{2\varepsilon_1}\left\|\xi_2^k\right\|_h^2 \le \frac{\varepsilon_1}{16}\left|z^k\right|_{H^1(\omega_h)}^2 + \frac{1}{2\varepsilon_1}\left\|\xi_2^k\right\|_h^2, \quad \varepsilon_1 > 0,\\
\left|\left(z^k, \eta_{1,x}^k\right)\right| &= \left|\left(\eta_1^k, z_{\bar{x}}^k\right)_{1h}\right| \le \frac{\varepsilon_1}{2}\left\|z_{\bar{x}}^k\right\|_{1h}^2 + \frac{1}{2\varepsilon_1}\left\|\eta_1^k\right\|_{1h}^2,\\
\left|\left(z^k, \eta_{2,y}^k\right)\right| &= \left|\left(\eta_2^k, z_{\bar{y}}^k\right)_{2h}\right| \le \frac{\varepsilon_1}{2}\left\|z_{\bar{y}}^k\right\|_{2h}^2 + \frac{1}{2\varepsilon_1}\left\|\eta_2^k\right\|_{2h}^2,
\end{aligned}$$

and analogously for the other terms. If we transform the left-hand side terms as in the proof of Theorem 1, we obtain

$$\begin{aligned}
\frac{1}{2}D_{t,0+,\tau}^{\alpha}\|z^k\|^2 + \frac{1}{2}|z^k|_{H^1(\omega_h)}^2 \le \frac{1}{4\varepsilon}\|\xi_1^k\|_h^2 &+ \left(\frac{\varepsilon}{8} + \frac{\varepsilon_1}{16} + 4\frac{\varepsilon_1}{2}\right)|z^k|_{H^1(\omega_h)}^2\\
&+ \frac{1}{2\varepsilon_1}\big(\|\xi_2^k\|_h^2 + \|\eta_1^k\|_{1h}^2 + \|\eta_2^k\|_{2h}^2 + \|\zeta_1^k\|_{1h}^2 + \|\zeta_2^k\|_{2h}^2\\
&+ \|\chi_1^k\|_{1h}^2 + \|\chi_2^k\|_{2h}^2 + \|\mu_1^k\|_{1h}^2 + \|\mu_2^k\|_{2h}^2\big).
\end{aligned}$$

Result follows for sufficiently small ε and ε_1 after summation for $k = 1, 2, \ldots, m$. □

In order to obtain the error bound of finite difference scheme (6)–(8), it is sufficient to estimate the right-hand side terms in (14).

Theorem 2. *Let the solution u of the initial-boundary value problem* (3)–(5) *belongs to the space $C^2([0,T], C(\bar{\Omega})) \cap C^1([0,T], H^3(\Omega))$ and $\theta \ge \dfrac{\Gamma(2-\alpha)}{2(1-2^{-\alpha})}$. Then the solution v of the finite difference scheme* (6)–(8) *with $\bar{f} = T_1T_2f$ converges to u and the following convergence rate estimate holds*:

$$\|u - v\|_{B^{1,\alpha/2}(Q_{h\tau})} = O(h^2 + \tau^\alpha).$$

Proof. From Lemma 2 it immediately follows that

$$\left(\tau\sum_{k=1}^{m}\|\xi_1^k\|_h^2\right)^{1/2} \le C\tau^{2-\alpha}\,\|u\|_{C^2([0,T],C(\bar{\Omega}))}. \tag{15}$$

From integral representation

$$\begin{aligned}
u(x,y) - T_1T_2u(x,y) = \frac{1}{h^2}\int_{x-h/2}^{x+h/2}\int_{y-h/2}^{y+h/2}&\left(\int_{x'}^{x}\int_{y'}^{y}\frac{\partial^2 u}{\partial x\partial y}(x'',y'')\,\mathrm{d}y''\mathrm{d}x''\right.\\
-\int_{x'}^{x}\int_{x''}^{x}\frac{\partial^2 u}{\partial x^2}(x''',y')\,\mathrm{d}x'''\mathrm{d}x'' &- \left.\int_{y'}^{y}\int_{y''}^{y}\frac{\partial^2 u}{\partial y^2}(x',y''')\,\mathrm{d}y'''\mathrm{d}y''\right)\mathrm{d}y'\mathrm{d}x'
\end{aligned}$$

one obtains

$$\left(\tau\sum_{k=1}^{m}\|\xi_2^k\|_h^2\right)^{1/2} \le Ch^2\,\|u\|_{C_+^{\alpha}([0,T],H^2(\Omega))}. \tag{16}$$

Using the Bramble-Hilbert lemma [**1**] and methodology presented in [**5**], one obtains

$$\left(\tau\sum_{k=1}^{m}\|\eta_i^k\|_{ih}^2\right)^{1/2} \le Ch^2\,\|u\|_{C([0,T],H^3(\Omega))},\quad i=1,2. \tag{17}$$

Terms ζ_i, χ_i and μ_i can be estimated directly:

$$\left(\tau\sum_{k=1}^{m}\|\zeta_i^k\|_{ih}^2\right)^{1/2} \le C\tau\,\|u\|_{H^1((0,T),C^1(\bar\Omega))},\quad i=1,2, \tag{18}$$

$$\left(\tau\sum_{k=1}^{m}\|\chi_i^k\|_{ih}^2\right)^{1/2} \le C\tau^{\alpha}\,\|u\|_{C^1([0,T],C^1(\bar\Omega))},\quad i=1,2, \tag{19}$$

$$\left(\tau\sum_{k=1}^{m}\|\mu_i^k\|_{ih}^2\right)^{1/2} \le C\tau^{2\alpha}\,\|u\|_{C^1([0,T],H^3(\Omega))},\quad i=1,2. \tag{20}$$

Result follows from (15)–(20).

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University of Belgrade,
Faculty of Mathematics,
Studentski trg 16, 11000 Belgrade
Serbia
e-mail: sandra@matf.bg.ac.rs

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