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BLOW-UP FOR DISCRETE REACTION-DIFFUSION EQUATIONS ON NETWORKS

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In this paper, we discuss the conditions under which blow-up occurs for the solutions of reaction-diffusion equations on networks. The analysis of this class of problems includes the existence of blow-up in finite time and the determination of the blow-up time and the corresponding blow-up rate.
In addition, when the solution blows up, we give estimates for the blow-up time and also provide the blow-up rate. Finally, we show some numerical illustrations which describe the main results.

1. INTRODUCTION

We say that a solution u to the equation blows up (or is a thermal runaway) at time T, if $|u(x_n, t_n)| \to +\infty$ for some sequence $(x_n, t_n) \to (a, T)$. Here, T is called a blow-up time and a is called a blow-up point.

There have been many papers which study the blow-up phenomenon for the solution to the reaction-diffusion equations. In fact, they show that the solution may or may not blow up in finite time, depending on the exponent q and the magnitude of the initial data. (see [**1**], [**3**], [**4**], [**7**], [**8**], [**9**], [**13**], [**14**], [**16**], [**17**], [**18**] and [**19**]).

From a similar point of view, it will be interesting to investigate the diffusion of energy or information on networks, which can be modelled by the discrete reaction-diffusion equation on networks. Here the network means a graph with edge-weight, i.e. a weighted graph (see [**5**]). On the other hand, the long time behavior (extinction and positivity) of solutions to evolution Laplace equation with absorption on networks is studied in the papers [**6**] and [**11**].

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The goal of this paper is to analyze some features of the blow-up phenomenon arising from the following discrete reaction-diffusion equation in $S \times (0, \infty)$

$$(1) \qquad \begin{cases} u_t(x,t) = \Delta_\omega u(x,t) + |u|^{q-1}(x,t)u(x,t), & (x,t) \in S \times (0,\infty) \\ u(x,t) = 0, & (x,t) \in \partial S \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \overline{S}, \end{cases}$$

where $q > 0$ and u_0 is nonnegative and nontrivial. Here S is a finite network and the operator Δ_ω is the discrete Laplacian on the network S with boundary ∂S.

Since functions on a finite network of size N can be identified with vectors in $\mathbb{R}^N$, the theoretical framework is the classical theory for systems of ODE. So, it seems that blow-up solution or global solution can be obtained by virtue of the famous Wintner's Lemma (see [**10**]). But in this paper, we follow traditional PDE techniques so called *the comparison principle*, which has been commonly accepted when dealing with blow-up theory in PDE and seems to be easier and stronger.

The reaction-diffusion equation (1) on a continuous domain $\Omega \subset \mathbb{R}^N$ has also been studied even until these days (see [**2**] and [**15**]). For example, in order to get a blow-up solution they adopted the condition such as

$$1 < q < (3N+8)/(3N-4)$$

in the paper [**15**] and

$$1 < q < (N+2)/(N-2)$$

in the paper [**2**], respectively. So it is quite natural that conditions to obtain a blow-up solution include something related to the domain $\Omega \subset \mathbb{R}^N$. But here in this paper (see Theorem 4.10), we obtain a blow-up solution under the condition

$$q > 1 \text{ and } y_0 > K^{\frac{1}{q-1}} \cdot |S|,$$

where $y_0 = \sum_{x \in \overline{S}} u_0(x)$ and $K = \max_{x \in S} \sum_{y \in \partial S} \omega(x,y)$. The constant K can be understood as a number representing the internal topology of the network S.

We organized this paper as follows: After considering some concepts on networks and the local existence of solutions to the equation (1), we discuss the comparison principles on networks in order to study the blow-up phenomenon in Section 4, in which we find out blow-up conditions of the solution to the equation (1) and the blow-up time with the blow-up rate. Finally, in Section 5, we give some numerical illustrations to exploit the main results.

2. PRELIMINARIES

In this section, we start with the theoretic graph notions frequently used throughout this paper. For more detailed information on notations, notions, and conventions, we refer the reader to [**5**].

For a graph $G = G(V, E)$ we mean a finite sets V of *vertices* (or *nodes*) with a set E of two-element subsets of V (whose elements are called *edges*). The set of vertices and edges of a graph G are sometimes denoted by $V(G)$ and $E(G)$, or simply V and E, respectively. Conventionally, we denote by $x \in V$ or $x \in G$ the facts that x is a vertex in G.

A graph G is said to be *simple* if it has neither multiple edges nor loops, and G is said to be *connected* if for every pair of vertices x and y, there exists a sequence (called a *path*) of vertices $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that x_{j-1} and x_j are connected by an edge (called *adjacent*) for $j = 1, \ldots, n$.

A graph $G' = G'(V', E')$ is said to be a *subgraph* of $G = G(V, E)$ if $V' \subset V$ and $E' \subset E$. In this case, G is a *host graph* of G'. If E' consists of all the edges from E which connect the vertices of V' in its host graph G, then G' is called an *induced* subgraph. It is noted that an induced subgraph of a connected host graph may not be connected.

A *weight* on a graph G is a symmetric function $\omega : V \times V \to [0, \infty)$ satisfying that

$$\omega(x, y) > 0 \quad \text{if and only if } \{x, y\} \in E.$$

Here, $\{x, y\}$ denotes the edge connecting the vertices x and y. Then we call a graph G with a weight ω a *network*.

For an induced subgraph S of a $G = G(V, E)$, the (vertex) *boundary* ∂S of S is the set of all vertices $z \in V \setminus S$ but are adjacent to some vertex in S, i.e.

$$\partial S := \{z \in V \setminus S \mid z \sim y \text{ for some } y \in S\}.$$

By $\overline{S}$, we denote a subgraph of G whose vertices are consisting of those in S or ∂S and whose edges set are formed by the edges between vertices in S and edges between a vertex in S and a vertex in ∂S. By the way, these last edges are usually known as *boundary edges*.

Throughout this paper, a subgraph S in our concern is assumed to be an induced subgraph which is simple and connected. From now on, for simplicity, by a network S with boundary ∂S we mean a subgraph $\overline{S}$ of G, associated with the weight ω.

The *degree* $d_\omega x$ of a vertex x in a network S (with boundary ∂S) is defined as

$$d_\omega x := \sum_{y \in \overline{S}} \omega(x, y).$$

The *discrete Laplacian* Δ_ω on a network S of a function $u : \overline{S} \to \mathbb{R}$ is defined by

$$\Delta_\omega u(x) := \sum_{y \in \overline{S}} [u(y) - u(x)]\omega(x, y)$$

for each $x \in S$.

We now briefly discuss the existence and uniqueness of a solution for

$$(2)\qquad \begin{cases} u_t = \Delta_\omega u + |u|^{q-1}u & \text{in } S \times (0,\infty), \\ u = 0 & \text{on } \partial S \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{on } \overline{S}, \end{cases}$$

where $q \geq 1$ and $u_0 : \overline{S} \to \mathbb{R}$ is a function satisfying that $u(z) = 0$, for all $z \in \partial S$.

By $C(S)$ we mean the set of all functions $u : \overline{S} \to \mathbb{R}$ satisfying that $u(z) = 0$, for all $z \in \partial S$.

For $u_0 \in C(S)$, $q \geq 1$ and $t_0 > 0$, consider a Banach space

$$X_{t_0} = \{u : \overline{S} \times [0,t_0] \to \mathbb{R} \mid u(x,\cdot) \in C\left([0,t_0]\right) \text{ for each } x \in \overline{S}\}$$

with the norm $\|u\|_{X_{t_0}} := \max_{x\in\overline{S}} \max_{0\leq t\leq t_0} |u(x,t)|$. Then it is clear that $C(S) \subset X_{t_0}$.

Then it is easy to see that the operator $D{:}X_{t_0} \to X_{t_0}$, defined by

$$D[u](x,t) := \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,s)\mathrm{d}s + \int_0^t |u|^{q-1}(x,s)u(x,s)\mathrm{d}s, & S \times [0,t_0] \\ 0, & \partial S \times [0,t_0], \end{cases}$$

is well-defined. In the next lemma we show that this operator is contractive on the closed ball so that we obtain the existence and uniqueness of solutions to the equation (2) in the time interval $[0,t_0]$, for t_0 small enough, as a consequence of Banach's fixed point theorem. In fact, For $u_0 \in C(S)$, it is noted that u is a solution of the initial value problem

$$\begin{cases} u_t = \Delta_\omega u + |u|^{q-1}u & \text{in } S \times (0,\infty), \\ u = 0 & \text{on } \partial S \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{on } \overline{S}, \end{cases}$$

if and only if it is a *fixed point* of D, that is, $D[u] = u$.

Lemma 2.1. *The operator D is a contraction on the closed ball*

$$B(u_0, 2\|u_0\|_{X_{t_0}}) := \left\{u \in X_{t_0} \mid \| \ u - u_0 \ \|_{X_{t_0}} \leq 2\|u_0\|_{X_{t_0}}\right\}$$

if t_0 is small enough.

Proof. Consider u and $v \in B(u_0, 2\|u_0\|_{X_{t_0}})$. Then for $(x,t) \in S \times [0,t_0]$,

$$\begin{aligned} &|D[u](x,t) - D[v](x,t)| \\ &= \left|\int_0^t \Delta_\omega(u-v)(x,s)\mathrm{d}s + \int_0^t |u|^{q-1}(x,s)u(x,s) - |v|^{q-1}(x,s)v(x,s)\mathrm{d}s\right| \\ &\leq \left[2|\overline{S}| \max_{(x,y)\in E} \omega(x,y)\|u-v\|_{X_{t_0}} + q \max_{x\in\overline{S}} \max_{0\leq s\leq t} |\eta(x,s)|^{q-1}\|u-v\|_{X_{t_0}}\right] t \\ &\leq \left[2|\overline{S}| \max_{(x,y)\in E} \omega(x,y)\|u-v\|_{X_{t_0}} + q(3\|u_0\|_{X_{t_0}})^{q-1}\|u-v\|_{X_{t_0}}\right] t_0 \\ &= C_1 t_0 \|u-v\|_{X_{t_0}}, \end{aligned}$$

where $|\eta(x,s)| < \max\{\|u\|_{X_{t_0}}, \|v\|_{X_{t_0}}\}$ for $(x,s) \in \overline{S} \times [0,t]$, derived from the mean value theorem and $C_1 = 2|\overline{S}| \max_{(x,y)\in E} \omega(x,y) + q(3\|u_0\|_{X_{t_0}})^{q-1}$. Moreover, it is easy to see that the above inequality still holds for $(x,t) \in \partial S \times [0,t_0]$. Hence choosing t_0 sufficiently small, we obtain a contraction on the closed ball $B(u_0, 2\|u_0\|_{X_{t_0}})$ into itself. The proof is thus complete. $\square$

The local existence and uniqueness is then assured and therefore the existence and uniqueness of the maximal solution for the above initial value problem, according to ODE theory, for example, Wintner's Lemma (see [**10**]).

3. DISCRETE VERSION OF COMPARISON PRINCIPLES

We devote this section to proving the comparison principle for the discrete reaction-diffusion equation (2) in order to study the blow-up occurrence and global existence, which we begin in the next section.

Define

$$S_T := S \times (0,T) \text{ and } \overline{S}_T := \overline{S} \times [0,T).$$

By $C^1(\overline{S}_T)$, we denote set of all functions $u : \overline{S} \times [0,T) \to \mathbb{R}$, satisfying that for each $x \in \overline{S}$, $u(x,\cdot)$ is continuous on $[0,T)$ and continuously differentiable $(0,T)$.

Now, we state the comparison principles and some related corollaries.

Theorem 3.2 (Comparison Principle: Discrete Version)**.** *Let $T > 0$ be arbitrarily given* (*may be infinite*) *and $q \geq 1$. Suppose that u and $v \in C^1(\overline{S}_T)$ satisfy the inequality*

$$(3) \qquad \begin{cases} u_t - \Delta_\omega u - |u|^{q-1}u \geq v_t - \Delta_\omega v - |v|^{q-1}v & \textit{in} \quad S_T, \\ u \geq v & \textit{on } \partial S \times [0,T), \\ u(\cdot,0) \geq v(\cdot,0) & \textit{on} \quad \overline{S}. \end{cases}$$

Then $u \geq v$ on $\overline{S} \times [0,T)$.

Proof. Let $T' > 0$ be arbitrary given with $T' < T$. For each $k \in \mathbb{R}$, let $\tilde{u}, \tilde{v} : \overline{S} \times [0,T'] \to \mathbb{R}$ be the functions defined by

$$\tilde{u}(x,t) := e^{-kt}u(x,t);\ \tilde{v}(x,t) := e^{-kt}v(x,t).$$

From (3) we have, applying the mean value theorem,

$$e^{kt}(\tilde{u}_t - \tilde{v}_t)(x,t) - e^{kt}\Delta_\omega(\tilde{u} - \tilde{v})(x,t) + e^{kt}[k - q\xi^{q-1}(x,t)e^{(q-1)kt}](\tilde{u} - \tilde{v})(x,t) \geq 0$$

for all $(x,t) \in S \times (0,T']$, where $|\xi(x,t)| < \max\{|\tilde{u}(x,t)|, |\tilde{v}(x,t)|\}$. Multiplying e^{-kt} on both sides, we have

$$(4) \quad (\tilde{u}_t - \tilde{v}_t)(x,t) - \Delta_\omega(\tilde{u} - \tilde{v})(x,t) + [k - q\xi^{q-1}(x,t)e^{(q-1)kt}](\tilde{u} - \tilde{v})(x,t) \geq 0$$

for all $(x,t) \in S \times (0,T']$. Let $\lambda : \overline{S} \times [0,T'] \to \mathbb{R}$ be a function by

$$\lambda(x,t) := \tilde{u}(x,t) - \tilde{v}(x,t).$$

Since $\lambda(x,\cdot)$ is continuous on $[0,T']$ for each $x \in \overline{S}$, λ has a minimum value on $\overline{S} \times [0,T']$, that is,

$$\lambda(x_0,t_0) = \min_{x\in\overline{S}} \min_{0\le t\le T'} \lambda(x,t).$$

Then we have only to show that $\lambda(x_0,t_0) \ge 0$. Suppose that $\lambda(x_0,t_0) < 0$, on the contrary. Since $\lambda \ge 0$ on both $\partial S \times [0,T']$ and $\overline{S} \times \{0\}$, we have $(x_0,t_0) \in S \times (0,T']$. It follows from the differentiability of $\lambda(x,\cdot)$ in $(0,T']$ for each $x \in \overline{S}$ that

$$\lambda_t(x_0,t_0) \le 0. \tag{5}$$

It is clear that

$$\lambda(y,t_0) - \lambda(x_0,t_0) \ge 0 \text{ for all } y \in S$$

which implies

$$\Delta_\omega \lambda(x_0,t_0) \ge 0. \tag{6}$$

Now we note that

$$\begin{aligned} |q\xi^{q-1}(x_0,y_0)e^{(q-1)kt_0}| &< qe^{(q-1)kt_0} \max\{|\tilde{u}(x_0,t_0)|^{q-1}, |\tilde{v}(x_0,t_0)|^{q-1}\} \\ &= q\max\{|u(x_0,t_0)|^{q-1}, |v(x_0,t_0)|^{q-1}\}. \end{aligned}$$

Then we choose $k > 0$ sufficiently large so that $k - q\xi^{q-1}(x_0,t_0)e^{(q-1)kt_0} > 0$, which gives

$$[k - q\xi^{q-1}(x_0,t_0)e^{(q-1)kt_0}]\lambda(x_0,t_0) < 0. \tag{7}$$

Combining (5), (6) and (7), we obtain

$$\lambda_t(x_0,t_0) - \Delta_\omega\lambda(x_0,t_0) + [k - q\xi^{q-1}e^{(q-1)kt_0}]\lambda(x_0,t_0) < 0,$$

which contradicts (4). Therefore, $u \ge v$ on $S \times (0,T']$. Since $T' < T$ is arbitrarily given, we finally get $u \ge v$ on $S \times (0,T)$. □

According to the above comparison principle, it is easy to obtain the uniqueness of solutions to the equation (2), which is stated as follows.

Corollary 3.3 (Uniqueness). *The equation* (2) *has a unique solution.*

The following theorem describes solutions which have non-trivial initial conditions in S.

Theorem 3.4 (Strict Comparison Principle: Discrete Version). *Under the same hypotheses as in Theorem* 3.2 *above, we have* $u > v$ *on* $S \times (0,T)$ *if* $u_0(x_0) > v_0(x_0)$ *for some* $x_0 \in S$.

Proof. Note that $u \geq v$ on $\overline{S} \times [0, T)$ by Theorem 3.2. Let $T' > 0$ be arbitrary given $T' < T$. Let $\lambda : \overline{S} \times [0, T'] \to \mathbb{R}$ be a function defined by

$$\lambda(x, t) := u(x, t) - v(x, t).$$

Then $\lambda(x, t) \geq 0$ on $\overline{S} \times [0, T']$ and summing the equations on S, we have, for each $t \in (0, T']$,

$$\sum_{x \in S} \lambda_t(x, t) \geq \sum_{x \in S} \Delta_\omega \lambda(x, t) + \sum_{x \in S} [u^q(x, t) - v^q(x, t)].$$

We note that

$$\begin{aligned}
\sum_{x \in S} \Delta_\omega \lambda(x, t) &= \sum_{x \in S} \sum_{y \in \overline{S}} [\lambda(y, t) - \lambda(x, t)] \omega(x, y) \\
&= \sum_{x \in S} \sum_{y \in S} [\lambda(y, t) - \lambda(x, t)] \omega(x, y) + \sum_{x \in S} \sum_{y \in \partial S} [\lambda(y, t) - \lambda(x, t)] \omega(x, y) \\
&\geq - \sum_{x \in S} \sum_{y \in \partial S} \lambda(x, t) \omega(x, y) \geq -K \sum_{x \in S} \lambda(x, t),
\end{aligned}$$

where $K = \max_{x \in S} \sum_{y \in \partial S} \omega(x, y)$ and

$$u^q(x, t) - v^q(x, t) = q \xi^{q-1}(x, t) [u(x, t) - v(x, t)]$$

on $\overline{S} \times [0, T']$, where $|\xi(x, t)| < m := \max_{(x,t) \in \overline{S} \times [0, T']} \{|u(x, t)|, |v(x, t)|\}$. Then it follows that

$$\sum_{x \in S} \lambda_t(x, t) \geq -K' \sum_{x \in S} \lambda(x, t), \; t \in (0, T'], \tag{8}$$

where $K' := K + q m^{q-1}$. This implies

$$\sum_{x \in S} \lambda(x, t) \geq C_0 \, e^{-K' t}, \; t \in (0, T'], \tag{9}$$

where $C_0 := \sum_{x \in S} \lambda(x, 0) > 0$.

Now, suppose that there exists a $(x^*, t^*) \in S \times (0, T']$ such that $\lambda(x^*, t^*) = 0$, that is,

$$\lambda(x^*, t^*) = \min_{x \in S} \min_{0 < t \leq T'} \lambda(x, t).$$

Then it follows from the differentiability of $\lambda(x, \cdot)$ in $(0, T']$ for each $x \in S$ that

$$\lambda_t(x^*, t^*) \leq 0.$$

From the first inequality in (3), we have

$$\begin{aligned} 0 \geq \lambda_t(x^*,t^*) &\geq \Delta_\omega \lambda(x^*,t^*) + [u^q(x^*,t^*) - v^q(x^*,t^*)] \\ &= \sum_{y\in\overline{S}} \lambda(y,t^*)\omega(x^*,y) \geq 0, \end{aligned}$$

which implies that $\lambda(y,t^*) = 0$ for all $y \in \overline{S}$ with $y \sim x^*$. Now, for any $x \in \overline{S}$, there exists a path

$$x^* \sim x_1 \sim \cdots \sim x_{n-1} \sim x_n = x,$$

since $\overline{S}$ is connected. By applying the same argument as above inductively we see that $\lambda(x,t^*) = 0$ for every $x \in \overline{S}$. This gives a contradiction to (9). Since $T' < T$ is arbitrarily given, we finally get $u > v$ on $S \times (0,T)$. □

We note that the comparison principle as in Theorem 3.2 is not true in general for $0 < q < 1$, so that the uniqueness of solutions (2) is not guaranteed. But, we present here a similar version of the comparison principle for $0 < q < 1$, under a little bit stronger boundary conditions.

Theorem 3.5. *Let $T > 0$ (T may be $+\infty$) and $q > 0$. Suppose that u and $v \in C^1(\overline{S}_T)$ satisfy the inequality*

$$(10) \qquad \begin{cases} u_t(x,t) - \Delta_\omega u(x,t) - u(x,t)|u(x,t)|^{q-1} \\ \geq v_t(x,t) - \Delta_\omega v(x,t) - v(x,t)|v(x,t)|^{q-1}, & (x,t) \in S_T, \\ u(x,t) > v(x,t), & (x,t) \in \partial S \times [0,T), \\ u(\cdot,0) > v(\cdot,0), & x \in \overline{S}. \end{cases}$$

Then $u(x,t) \geq v(x,t)$ for all $(x,t) \in S \times (0,T)$.

Proof. Let $T' > 0$ and $\delta > 0$ be arbitrary given with $T' < T$ and $0 < \delta < \min_{(v,t)\in\Gamma}(u-v)(x,t)$, respectively where $\Gamma := \{(x,t) \in \overline{S} \times [0,T'] \,|\, t = 0 \text{ or } x \in \partial S\}$, called a parabolic boundary.

Now, define a function $\lambda : \overline{S} \times (0,T'] \to \mathbb{R}$ be a function defined by

$$\lambda(x,t) := [u(x,t) - v(x,t)] - \delta, \ (x,t) \in \overline{S} \times (0,T']$$

Then $\lambda(x,t) > 0$ on Γ. Now, we suppose that $\min_{x\in S, 0<t\leq T'} \lambda(x,t) < 0$. Then there exists $(x_0,t_0) \in S \times (0,T']$ such that

(i) $\lambda(x_0,t_0) = 0$,

(ii) $\lambda(y,t_0) \geq \tau(x_0,t_0) = 0$, $y \in S$,

(iii) $\lambda(x,t) > 0$, $(x,t) \in S \times (0,t_0)$.

Then $\tau_t(x_0,t_0) \leq 0$ and since $u(y,t_0) - u(x_0,t_0) \geq v(y,t_0) - v(x_0,t_0)$, we obtain

$$\Delta_\omega u(x_0,t_0) \geq \Delta_\omega v(x_0,t_0).$$

Hence, the equation (10) gives

$$\begin{aligned} 0 &\geq \lambda_t\left(x_0,t_0\right) \geq \left|u\left(x_0,t_0\right)\right|^{q-1}u\left(x_0,t_0\right)-\left|v\left(x_0,t_0\right)\right|^{q-1}v\left(x_0,t_0\right) \\ &= \left|v\left(x_0,t_0\right)+\delta\right|^{q-1}\left(v\left(x_0,t_0\right)+\delta\right)-\left|v\left(x_0,t_0\right)\right|^{q-1}v\left(x_0,t_0\right)>0, \end{aligned}$$

which leads a contradiction. Hence, $\lambda(x,t)\geq 0$ for all $(x,t)\in S\times(0,T']$ so that we have $u(x,t)\geq v(x,t)$ for all $(x,t)\in S\times(0,T)$, since δ and T' are arbitrary.

Definition 3.6. *A real-valued function u is a super-solution of the equation* (2) *if it satisfies*

$$\begin{cases} u_t \geq \Delta_\omega u + |u|^{q-1}u & \textit{in} \quad S\times(0,\infty),\\ u\geq 0 & \textit{on}\ \partial S\times(0,\infty),\\ u(\cdot,0)\geq u_0 & \textit{on} \quad \overline{S}. \end{cases}$$

A sub-solution is defined similarly by reversing the inequalities.

The followings are easy consequences of the above theorems.

Corollary 3.7. *Let u be a super-solution to the equation* (2)*. Then we have $u\geq 0$ on $\overline{S}\times[0,\infty)$. Moreover, $u>0$ on $S\times(0,\infty)$ if u_0 is non-trivial on S.*

Corollary 3.8. *Let u be a super-solution and v be a sub-solution to the equation* (2)*, respectively. Then $u\geq v$ on $\overline{S}\times[0,\infty)$. Moreover, $u>v$ on $S\times(0,\infty)$ if $u_0(x_0)>v_0(x_0)$ for some $x_0\in S$.*

4. BLOW-UP AND GLOBAL EXISTENCE

Throughout this section we assume that the initial data $u_0\in C(S)$ are non-negative on $\overline{S}$ and non-trivial on S. As seen in the previous sections, we see that a solution to the equation (2) with $T=\infty$ is nonnegative on $\overline{S}\times[0,\infty)$. Hence, the reaction term $|u|^{q-1}u$ of the equation (2) can be written as u^q.

Now, we study whether a solution to

$$\begin{cases} u_t = \Delta_\omega u + u^q & \text{in} \quad S\times(0,\infty),\\ u = 0 & \text{on}\ \partial S\times(0,\infty),\\ u(\cdot,0) = u_0 & \text{on} \quad \overline{S}, \end{cases} \tag{11}$$

exists globally in time, or blows up in finite time. If the blow-up occurs, we want to find out the blow-up time (see [**1**]) and rate (see [**4**]). For general references on blow-up problems we refer to surveys [**1**] and [**12**].

First we introduce the concept of blow-up and begin to discuss the sufficient conditions to guarantee that the solution blows up.

Definition 4.9 (Blow-up in a finite time T)**.** *We say that a solution u to the equation* (11) *blows up* (*or is a thermal runaway*) *at $t=T$, if there exists $x\in S$ such that $|u(x,t)|\to+\infty$ as $t\nearrow T$. In this case, we say that the solution blows up in finite time if T is finite.*

Theorem 4.10. *Let u be a solution of the equation* (11) *and consider* $y_0 = \sum_{x\in\overline{S}} u_0(x)$ *and* $K = \max_{x\in S} \sum_{y\in\partial S} \omega(x,y)$. *Then*:

(i) *If* $0 < q \le 1$, *then the solution is global.*

(ii) *If* $q > 1$ *and* $y_0 > K^{\frac{1}{q-1}} \cdot |S|$, *then the solution blows up.*

Moreover, the blow-up time T in (ii) *satisfies*

$$T \le \frac{1}{K(q-1)} \ln \left[1 - K \cdot \left(\frac{|S|}{y_0}\right)^{q-1}\right]^{-1}.$$

Proof. First, we prove (i). Consider the following ODE problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = z^q(t), t > 0, \\ z(0) = \max_{x\in\overline{S}} u_0(x) + 1. \end{cases}$$

Then we have

$$\begin{cases} z(t) = \left[(1-q)t + z^{1-q}(0)\right]^{\frac{1}{1-q}}, & q \ne 1, \\ z(t) = z(0) e^t, & q = 1, \end{cases}$$

for every $t \ge 0$.
Take $v(x,t) := z(t)$ for all $x \in \overline{S}$ and $t \ge 0$. Then it is easy to see that $v(x,t) > u(x,t)$, $(x,t) \in \partial S \times (0,+\infty)$, $v(x,0) = z(0) > \max_{x\in\overline{S}} u_0(x)$, $x \in \overline{S}$, and

$$v_t(x,t) - \Delta_\omega v(x,t) - v^q(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} z(t) - z^q(t) = 0.$$

Thus, $0 \le u(x,t) \le v(x,t) = z(t)$, $(x,t) \in \overline{S} \times (0,+\infty)$ by Theorem 3.5. This implies that u must be global.

Secondly, we prove (ii). Assume that $q > 1$. Then summing on S the equation (11), we get

$$\sum_{x\in S} u_t(x,t) \ge -K \sum_{x\in S} u(x,t) + |S|^{1-q} \left[\sum_{x\in S} u(x,t)\right]^q.$$

Here, we used Jensen's inequality and the same technique as in the proof of Theorem 3.4. Multiplying $\left[\sum_{x\in S} u(x,t)\right]^{-q}$ on both sides of the inequality and letting $\eta(t) := \left[\sum_{x\in S} u(x,t)\right]^{1-q}$, we have

$$\eta'(t) \le K(q-1)\eta(t) + (1-q)|S|^{1-q}.$$

Using the differential inequality, we have

$$\eta(t) \le e^{K(q-1)t} \left[\eta(0) + \frac{|S|^{1-q}}{K} (e^{K(1-q)t} - 1)\right].$$

Let $y(t) := \sum_{x\in S} u(x,t)$. Then

$$y^{1-q}(t) \le e^{K(q-1)t}\left[y_0^{1-q} - \frac{|S|^{1-q}}{K} + \frac{|S|^{1-q}}{K}e^{K(1-q)t}\right].$$

This is equivalent to

$$y^{q-1}(t) \ge \frac{1}{\left(y_0^{1-q} - \frac{|S|^{1-q}}{K}\right)e^{K(q-1)t} + \frac{|S|^{1-q}}{K}}. \tag{12}$$

Thus, if $y_0^{q-1} > K\cdot|S|^{q-1}$, the solution blows up and we have the following estimate for the blow-up time:

$$T \le \frac{1}{K(q-1)}\ln\left[1 - K\cdot\left(\frac{|S|}{y_0}\right)^{q-1}\right]^{-1}.$$

Remark. In (ii) above, if the condition $y_0 > K^{\frac{1}{q-1}}\cdot|S|$ is dropped, then the solution can be global. For example, consider a function

$$v(x,t) := \alpha\phi_1(x),\ x\in\overline{S},\ t>0,$$

where $0<\alpha\ll 1$ and

$$\begin{cases} -\Delta_\omega\phi_1(x) = \lambda_1\phi_1(x), & x\in S,\\ \phi_1(x) = 0, & x\in\partial S.\end{cases}$$

Note that it is well-known that $\lambda_1 > 0$ and $\phi_1(x) > 0$ for all $x \in S$. (see [**5**]). Then

$$v_t(x,t) - \Delta_\omega v(x,t) - v^q(x,t) = \alpha\phi_1(x)[\lambda_1 - \alpha^{q-1}\phi_1^{q-1}(x)] > 0$$

on $S\times(0,\infty)$. Now, let $u(x,t)$ be a solution to the equation (11). Then $v(x,t) = u(x,t) = 0$ on $\partial S\times(0,\infty)$. If $u_0(x) \le v(x,0) = \alpha\phi_1(x)$ for a sufficiently small $\alpha > 0$, then it follows from Theorem 3.2 that

$$0 \le u(x,t) \le v(x,t)$$

on $\overline{S}\times[0,\infty)$, which implies that $u(x,t)$ must be global.

Now, we derive the blow-up rate for the solution of the equation (11).

Theorem 4.11. *Let $q>1$ and u be a solution to the equation* (11) *blowing up at finite time T. Then:*

(i) (*The lower bound*)

$$\max_{x\in\overline{S}} u(x,t) \ge \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}(T-t)^{-\frac{1}{q-1}},\quad t>0.$$

(ii) (*The upper bound*)

$$\max_{x\in\overline{S}} u(x,t) \le \left[(q-1)(T-t) - \frac{1}{2}d(q-1)^2(T-t)^2\right]^{-\frac{1}{q-1}},\quad t>0,$$

where $d = \max_{x\in\overline{S}} d_\omega x$.

(iii) (*The blow up rate*)

$$\lim_{t\to T^-}(T-t)^{\frac{1}{q-1}}\max_{x\in\overline{S}}u(x,t)=\left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}.$$

Proof. First, we prove (i). Note that $u>0$ on $S\times(0,\infty)$ by Theorem 3.4. Let T be the finite maximal time of existence of a blow-up solution and $x_t\in S$ be a node such that $u(x_t,t)=\max_{x\in\overline{S}}u(x,t)$ for each $t>0$. In fact, we note that $\max_{x\in\overline{S}}u(x,t)$ is differentiable almost everywhere for each $x\in S$. Then we now have $\lim\limits_{t\to T^-}u(x_t,t)=\infty$ and the equation (11) at the node x_s for almost all $s>0$ can be written as follows:

$$\begin{aligned}(13)\qquad u_t(x_s,s)&=\Delta_\omega u(x_s,s)+u^q(x_s,s)\\&=\sum_{y\in\overline{S}}[u(y,s)-u(x_s,s)]\omega(x_s,y)+u^q(x_s,s)\le u^q(x_s,s).\end{aligned}$$

Integrating (13) in (t,T), and taking into account that $q>1$, we obtain

$$T-t\ge\int_t^T\frac{u_s(x_s,s)}{u^q(x_s,s)}\mathrm{d}s=\int_{u(x_t,t)}^{\infty}\frac{1}{u^q}\mathrm{d}u=\frac{1}{q-1}u^{1-q}(x_t,t).$$

Hence we obtain

$$u(x_t,t)\ge\left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}(T-t)^{-\frac{1}{q-1}}.$$

Next, we prove (ii). As in the previous theorem, we get the following estimate:

$$\begin{aligned}u_t(x_s,s)=\Delta_\omega u(x_s,s)+u^q(x_s,s)&=\sum_{y\in\overline{S}}[u(y,s)-u(x_s,s)]\omega(x_s,y)+u^q(x_s,s)\\&\ge-\sum_{y\in\overline{S}}u(x_s,s)\omega(x_s,y)+u^q(x_s,s)\\&=-du(x_s,s)+u^q(x_s,s)=u^q(x_s,s)[1-du^{1-q}(x_s,s)],\end{aligned}$$

for each $s>0$, where $d=\max_{x\in\overline{S}}d_\omega x$. By the lower-bound of $u(x_s,s)$ in (i), we have

$$(14)\qquad u_t(x_s,s)\ge u^q(x_s,s)[1-d(q-1)(T-s)].$$

Integrating (14) over (t,T), the upper-bound can be derived by a method similar to that used in (i).

Finally, (iii) can be easily obtained by (i) and (ii).

5. EXAMPLES AND NUMERICAL ILLUSTRATIONS

In this section, we show some examples and numerical illustrations to our results in the previous section.

First, we consider a digon, that is, a graph $S = \{x_1\}$ with $\partial S = \{x_2\}$ and $\omega(x_1, x_2) = 1$. Thus, the equation (11) can be rewritten as

$$(15) \qquad \begin{cases} u_t(x_1, t) = -u(x_1, t) + u^q(x_1, t), & t \in (0, \infty) \\ u(x_2, t) = 0, & t \in (0, \infty) \\ u(x_1, 0) = u_0 > 0. \end{cases}$$

When $q = 1$, the solution $u(x_1, t) = u_0$, $t \in [0, \infty)$, which is global. For $q > 1$, then the equation (15) is well-known Bernoulli equation. The explicit solution to the equation (15) is as follows:

$$(16) \qquad u(x_1, t) = \left[(u_0^{1-q} - 1)e^{(q-1)t} + 1\right]^{\frac{1}{1-q}}.$$

If $u_0^{1-q} - 1 < 0$, then it is easy to see that $u(x_1, t)$ blows up in finite time and the blow-up time is

$$T = \frac{-\ln(1 - u_0^{1-q})}{(q-1)}.$$

Moreover, a tedious calculation makes us to get the following limit:

$$\lim_{t \to T^-} (T - t)^{\frac{1}{q-1}} u(x_1, t) = \lim_{t \to T^-} \left[(T - t)\left[(u_0^{1-q} - 1)e^{(q-1)t} + 1\right]^{-1}\right]^{\frac{1}{q-1}} = \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}.$$

Now, we consider a path on 30 vertices, that is, $S = \{x_2, \cdots x_{29}\}$

$$\circ_{x_1} \cdots\cdots \bullet_{x_2} \text{———} \bullet_{x_3} \cdots\cdots \bullet_{x_{28}} \text{———} \bullet_{x_{29}} \cdots\cdots \circ_{x_{30}}$$

Figure 1. Graph $\overline{S}$

with the boundary $\partial S = \{x_1, x_{30}\}$ and a weight ω as

$$\omega(x_j, x_{j+1}) = \begin{cases} 1, & j = 1, 2, \ldots, 29, \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.12. For the above graph $\overline{S}$, we put $q = 2.5$ in Theorem 4.10. The initial condition u_0 is given by Table 1.

Node i	$u_0(x_i)$	Node i	$u_0(x_i)$	Node i	$u_0(x_i)$
1	0	11	0	21	1
2	1	12	2	22	2
3	2	13	3	23	3
4	3	14	4	24	4
5	4	15	5	25	5
6	5	16	4	26	4
7	4	17	3	27	3
8	3	18	2	28	2
9	2	19	1	29	1
10	1	20	0	30	0

Table 1. Initial values u_0 of u

By simple calculation, we see that the initial value u_0 of Table 1 satisfies the hypothesis of Theorem 4.10 (ii) where the solution to the equation (11) blows up. Figure 2 shows the blow-up in time of a solution u beginning with u_0. The computed blow-up time T to the solution u is

$$T < 2120.$$

In particular, Figure 2 shows the blow-up occurrence to the solution at some nodes x_1, x_1

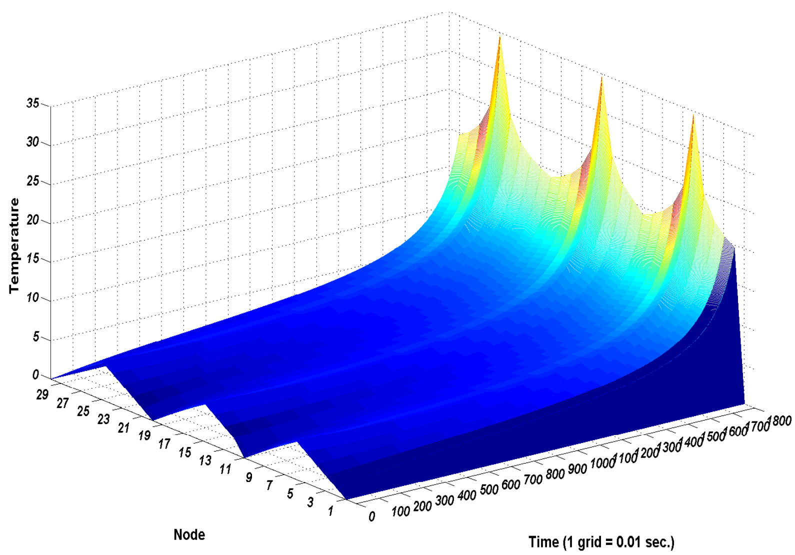

Figure 2. Blow-up solution u

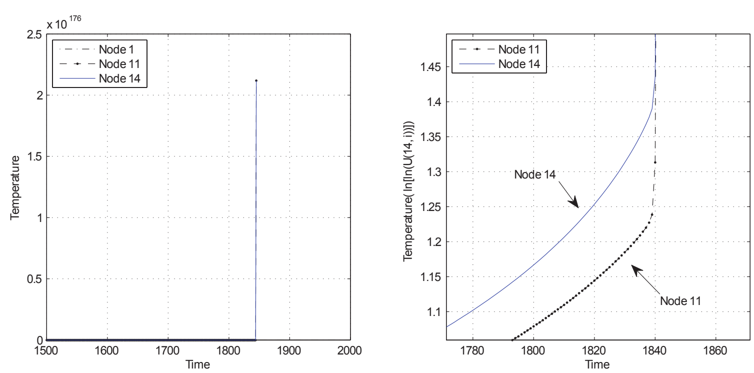

Figure 3. Behaviors of $\ln u$ at nodes x_1, x_{11} and x_{14}

Example 5.13. For the above graph $\overline{S}$ with the same initial condition u_0 in Example 5.12, we put $q = 1$. Figure 4 shows that the solution is global and the global behavior of node x_{17} and x_{21} in time.

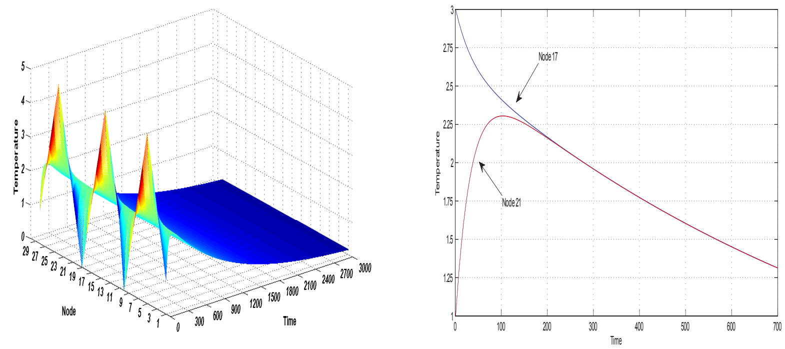

Figure 4. Global solution u and Behaviors of node x_{17} and x_{21}

Example 5.14. For the above graph $\overline{S}$, we put $q = 1.5$ in Theorem 4.10. The initial condition u_0 is given by Table 2. Then $y_0 \leq K^{\frac{1}{q-1}} \cdot |S|$. Figure 5 shows the solution u of equation (11) is global even if $q > 1$ and the behaviors the node x_8 and x_{13}.

Node i	$u_0(x_i)$	Node i	$u_0(x_i)$	Node i	$u_0(x_i)$
1	0	11	0.1	21	0.2
2	0.05	12	0.15	22	0.15
3	0.1	13	0.2	23	0.1
4	0.15	14	0.15	24	0.05
5	0.2	15	0.1	25	0
6	0.15	16	0.05	26	0.05
7	0.1	17	0	27	0.1
8	0.05	18	0.05	28	0.15
9	0	19	0.1	29	0.2
10	0.05	20	0.15	30	0

Table 2. Initial Values of u

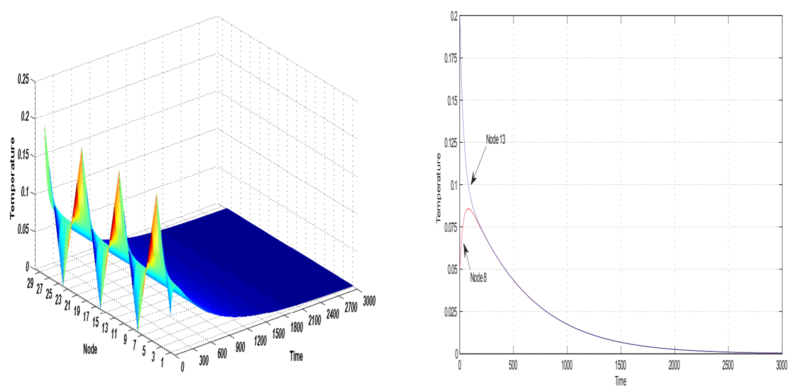

Figure 5. Global solution u and Behaviors of node x_8 and x_{13}

Conflict of Interests. The authors declares that there is no conflict of interests regarding the publication of this paper.

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