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DIRICHLET–TO–ROBIN MAPS ON FINITE NETWORKS

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Our aim is to characterize those matrices that are the response matrix of a semi–positive definite Schrödinger operator on a circular planar network. Our findings generalize the known results and allow us to consider both non-singular and non diagonally dominant matrices as response matrices. To this end, we define the Dirichlet–to–Robin map associated with a Schrödinger operator on general networks, and we prove that it satisfies the *alternating property* which is essential to characterize the response matrices.

1. INTRODUCTION

Inverse boundary value problems were introduced by Alberto Calderón in 1950, although this work was not published until the 80's, see [**7**]. Its first applications are found in geophysical electrical prospection and electrical impedance tomography. The corresponding mathematical problem is whether it is possible to determine the conductivity of a body by means of boundary measurements. This problem is exponentially ill–posed, since its solution is highly sensitive to changes in the boundary data.

Inverse boundary value problems have been considered both in the continuum and discrete fields. The Dirichlet–to–Neumann map plays an important role in both frameworks, since it is the main tool that allows us to determine from boundary measurements the information in the interior of the domain.

We are here concerned with the discrete version of this problem. This point of view is important in applications since finite network models arise in finite volume discretizations of the elliptic partial differential equation that model the continuos inverse problem, see [**5, 6, 11**]. In this framework, the characterization

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of the networks that allow the recovery of the conductance is essential. When the combinatorial Laplacian is considered, the matrix associated with the Dirichlet–to–Neumann map is known as the response matrix of the network, which is a singular, symmetric and diagonally dominant M–matrix; see [**9**]. Curtis et al. characterized in [**8**], those singular, symmetric and diagonally dominant M–matrices that are the response matrix of a circular planar network.

In this paper, we first define the Dirichlet–to–Robin map associated with a Schrödinger operator on a general network, and then we derive some of its properties. These properties are analogs of those that characterize the Dirichlet–to–Neumann maps, see [**8**] for circular planar networks and [**11**] for the continuous case. In particular, we prove that the Dirichlet–to–Robin map of a general network is a self–adjoint, positive semi–definite operator, whose kernel is related with the second normal derivative of the Green operator, and it is negative off–diagonal and positive on the diagonal. Furthermore, we prove the *alternating property* for the Dirichlet–to–Robin map that is essential to characterize the response matrices. The proof of this property follows the guidelines of the continuous case and makes a deep use of the monotonicity property for Schrödinger operators. We remark that the alternating property is an important tool to built recovery algorithms for the conductances as it is shown in [**1**] for the case of spider networks.

Second, we extend the characterization given in [**8**] by removing the hypotheses of singularity and diagonal dominance. Therefore, we obtain a wider class of matrices that may be the response matrix of a circular planar network.

One of the most surprising facts about the conductance recovery is that a response matrix can be the response matrix of a family of Schrödinger operators associated with different conductances and potentials. This occurs even when the Dirichlet–to–Neumann map associated with the combinatorial Laplacian is considered. This lack of uniqueness is due to the fact that the eigenfunction corresponding to the lowest eigenvalue of the response matrix can be extended to the network as a weight in infinite ways. Therefore, by choosing a specific extension we arrive at a unique Schrödinger operator whose Dirichlet–to–Robin map corresponds to the initial matrix.

Summarizing, here we have proved the alternating property for non–singular response matrices for arbitrary networks. On one hand, the alternating property is an structural property that the response matrices associated with a wider family of difference operator must fulfill. These operators are discrete Schrödinger operators that correspond to the discrete version of the ones studied in the continuous case. On the other hand, since the continuos bi–dimensional case is completely characterized, the literature for the discrete case is mainly devoted to the study of planar finite networks. Therefore, this work can be also considered as an advance toward the recovery of the conductance for non-planar networks that can approximate multidimensional domains.

2. PRELIMINARIES

Let $\Gamma = (V, c)$ be a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V. Let E be the set of edges of the network Γ. Each edge (x, y) has been assigned a *conductance* $c(x, y)$, where $c :\ V \times V \longrightarrow [0, +\infty)$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. Then, $x, y \in V$ are *adjacent*, $x \sim y$, iff $c(x, y) > 0$. We denote by $V(x)$, the *set of neighbours of* $x \in V$; that is, the set of vertices adjacent to x. Observe that $x \notin V(x)$.

The set of functions on a subset $F \subseteq V$, denoted by $\mathcal{C}(F)$, and the set of non–negative functions on F, $\mathcal{C}^+(F)$, are naturally identified with $\mathbb{R}^{|F|}$ and the positive cone of $\mathbb{R}^{|F|}$, respectively. Note that if $f \in \mathcal{C}(F)$, we can extend f to a function on V by defining $f(x) = 0$ for all $x \in V \setminus F$.

We denote by $\int_F u(x)\mathrm{d}x$ or simply by $\int_F u$ the value $\sum\limits_{x\in F} u(x)$. Moreover, if F is a non empty subset of V, its characteristic function is denoted by $\chi_{_F}$. When $F = \{x\}$, its characteristic function will be denoted by ε_x. If $u \in \mathcal{C}(V)$, we define the *support* of u as $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$.

If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is. Any function $\omega \in \mathcal{C}^+(\bar{F})$ such that $\mathsf{supp}(\omega) = \bar{F}$ and $\int_{\bar{F}} \omega^2 = 1$ is called *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. We denote by k_F the function $k_F(x) = \sum\limits_{y\in F} c(x, y)$.

We define the *normal derivative of* $u \in \mathcal{C}(\bar{F})$ *on* F as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_{_F}}\right)(x) = \int_F c(x, y)\,\big(u(x) - u(y)\big)\mathrm{d}y, \quad \text{for any } x \in \delta(F).$$

Given $S, T \subset V$, we define

$$\mathcal{C}(S \times T) = \{f : V \times V \longrightarrow \mathbb{R} : f(x, y) = 0 \text{ if } (x, y) \notin S \times T\}.$$

In particular, any function $K \in \mathcal{C}(F \times F)$ is called a kernel on F.

If K is a kernel on F, for each $x, y \in F$ we denote by K^x and K_y the functions of $\mathcal{C}(F)$ defined by $K_x(y) = K^y(x) = K(x, y)$. The *integral operator associated with* K is the endomorphism $\mathscr{K} : \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$ that assigns to each $f \in \mathcal{C}(F)$, the function $\mathscr{K}(f)(x) = \int_F K(x, y)\, f(y)\mathrm{d}y$ for all $x \in V$. Conversely, given an endomorphism $\mathscr{K} : \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$, the associated kernel is given by $K(x, y) = \mathscr{K}(\varepsilon_y)(x)$. Clearly, kernels and operators can be identified with matrices, after giving a label on the vertex set. In addition, a function $u \in \mathcal{C}(F)$ can be identified with the kernel $K(x, x) = u(x)$ and $K(x, y) = 0$ otherwise and hence with a diagonal matrix, that will be denoted by D_u.

When K is a kernel on $\bar{F}$, for each $x \in \delta(F)$ and each $y \in \bar{F}$, we denote by $\left(\dfrac{\partial K}{\partial \mathsf{n}_x}\right)(x, y)$ the value $\left(\dfrac{\partial K^y}{\partial \mathsf{n}_{_F}}\right)(x)$, whereas for each $x \in \bar{F}$ and each $y \in \delta(F)$

we denote by $\left(\dfrac{\partial K}{\partial \mathsf{n}_y}\right)(x,y)$ the value $\left(\dfrac{\partial K_x}{\partial \mathsf{n}_{F}}\right)(y)$. Clearly, $\dfrac{\partial K}{\partial \mathsf{n}_x} \in \mathcal{C}(\delta(F)\times \bar{F})$ and $\dfrac{\partial K}{\partial \mathsf{n}_y} \in \mathcal{C}(\bar{F}\times \delta(F))$ and hence both are kernels on $\bar{F}$.

Lemma 1. *If K is a kernel on $\bar{F}$, then it satisfies $\dfrac{\partial^2 K}{\partial \mathsf{n}_x \partial \mathsf{n}_y} = \dfrac{\partial^2 K}{\partial \mathsf{n}_y \partial \mathsf{n}_x} \in \mathcal{C}(\delta(F)\times \delta(F))$. Moreover, for any $x,y\in\delta(F)$*

$$\left(\frac{\partial^2 K}{\partial \mathsf{n}_x \partial \mathsf{n}_y}\right)(x,y) = k_{F}(x)k_{F}(y)K(x,y) - k_{F}(x)\int_F c(y,z)K(x,z)\mathrm{d}z - k_{F}(y)\int_F c(x,z)K(z,y)\mathrm{d}z + \int_F\int_F c(x,u)c(y,z)K(u,z)\,\mathrm{d}u\,\mathrm{d}z.$$

In addition, $\left(\dfrac{\partial^2 K}{\partial \mathsf{n}_x \partial \mathsf{n}_y}\right)$ is a symmetric kernel when K is.

The *combinatorial Laplacian* of Γ is the linear operator $\mathscr{L}:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function defined for all $x\in V$ as

$$\mathscr{L}(u)(x) = \int_V c(x,y)\big(u(x)-u(y)\big)\mathrm{d}y.$$

Given $q\in\mathcal{C}(V)$ the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathscr{L}_q:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function $\mathscr{L}_q(u)=\mathscr{L}(u)+qu$. It is well–known that any Schrödinger operator is self–adjoint. The relation between the values of the Schrödinger operator with potential q on F and the values of the normal derivative at $\delta(F)$ is given by the *First Green Identity*, proved in [**3**]

$$\int_F v\,\mathscr{L}_q(u) = \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_{F}(x,y)(u(x)-u(y))(v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y + \int_F quv - \int_{\delta(F)} v\,\frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}},$$

where $u,v\in\mathcal{C}(\bar{F})$ and $c_{F} = c\cdot\chi_{(\bar{F}\times\bar{F})\setminus(\delta(F)\times\delta(F))}$. A direct consequence of the above identity is the so–called *Second Green Identity*

$$\int_F \big(v\,\mathscr{L}_q(u) - u\,\mathscr{L}_q(v)\big) = \int_{\delta(F)}\left(u\,\frac{\partial v}{\partial \mathsf{n}_{\mathsf{F}}} - v\,\frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}}\right), \quad \text{for all} \quad u,v\in\mathcal{C}(\bar{F}).$$

In particular, taking $v=\chi_{\bar{F}}$ in the Second Green Identity we obtain the so–called *Gauss Theorem*

$$\int_F \mathscr{L}(u) = -\int_{\delta(F)}\frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}}, \quad \text{for all} \quad u\in\mathcal{C}(\bar{F}). \tag{1}$$

A function $u\in\mathcal{C}(\bar{F})$ is called *q–harmonic on F* iff $\mathscr{L}_q(u)=0$ on F. We define the *energy associated with F and q* as the symmetric bilinear form $\mathcal{E}_q^F:\mathcal{C}(\bar{F})\times\mathcal{C}(\bar{F})\longrightarrow\mathbb{R}$ given for any $u,v\in\mathcal{C}(\bar{F})$ by

$$\mathcal{E}_q^F(u,v) = \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_{F}(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\mathrm{d}x\,\mathrm{d}y + \int_{\bar{F}} q\,u\,v. \tag{2}$$

In particular, when $q = 0$, the energy will be denoted by $\mathcal{E}^F$. From the First Green Identity, for any $u, v \in \mathcal{C}(\bar{F})$ we get that

$$\mathcal{E}_q^F(u,v) = \int_F v\mathscr{L}_q(u) + \int_{\delta(F)} v \left[\frac{\partial u}{\partial \mathsf{n}_F} + qu\right]. \tag{3}$$

Some of the present authors characterized in [**4**] when the energy is positive semi–definite by defining the potential associated with a weight. For any weight $\sigma \in \Omega(\bar{F})$, the so–called *potential associated with* σ is the function in $\mathcal{C}(\bar{F})$ defined as $q_\sigma = -\sigma^{-1}\mathscr{L}(\sigma)$ on F, $q_\sigma = -\sigma^{-1}\dfrac{\partial \sigma}{\partial \mathsf{n}_F}$ on $\delta(F)$.

Observe that $q_\sigma = 0$ iff $\sigma = a\chi_{\bar{F}}$, with $a > 0$. More generally, from Gauss Theorem, we obtain that $\int_{\bar{F}} \sigma\, q_\sigma = 0$, which implies that q_σ must take positive and negative values, except when $\sigma = a\chi_{\bar{F}}$, $a > 0$.

We end this section with some terminology following the guidelines of [**8**]. Given $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ disjoint subsets of $\delta(F)$, there exist k paths, $\gamma_1, \dots, \gamma_k$, such that γ_i starts at s_i ends at t_i and $\gamma_i \setminus \{s_i, t_i\} \subset F$, since F is connected. The pair $(S;T)$ is called *connected through* Γ, when there exist k paths connecting S and T that are mutually disjoint.

The network $\Gamma = (\bar{F}, c_F)$ is called a *circular planar network* if it can be embedded in a closed disc D in the plane so that the vertices in F lie in $\overset{\circ}{D}$ and vertices in $\delta(F)$ lie on the circumference $C = \partial D$. In this case, the vertices in $\delta(F)$ can be labelled in the clockwise circular order. The pair $(S;T)$ of boundary vertices is called a *circular pair* if the set $(s_1, \dots, s_k; t_1, \dots, t_k)$ is in circular order. When the network is not circular planar, we can label the boundary nodes, say $\delta(F) = \{p_1, \dots, p_n\}$, where $n = |\delta(F)|$. In this case, a subset $\{s_1, \dots, s_k\}$ of boundary nodes is called an *ordered set* if there exists a non decreasing function $\sigma : \{1, \dots, k\} \longrightarrow \{1, \dots, n\}$ such that $s_j = p_{\sigma(j)}$. The pair $(S;T)$ is called *ordered pair* if $(s_1, \dots, s_k; t_1, \dots, t_k)$ is an ordered set. Notice that in the definition of ordered set we are not assuming that the vertices in S nor T are different, but $S \cap T = \emptyset$.

3. DOOB TRANSFORM

In order to study the positive semi–definiteness of Schrödinger operators, some of the present authors introduced the so–called Doob transform which is a useful tool in the framework of Dirichlet forms, see [**3**]. In the next result, we adapt this technique for a network with boundary; that is, $\Gamma = (\bar{F}, c_F)$.

Proposition 2 (Doob Transform)**.** *Given $\sigma \in \Omega(\bar{F})$, then for any $u \in \mathcal{C}(\bar{F})$ the following identities hold:*

$$\mathscr{L}(u)(x) = \frac{1}{\sigma(x)}\int_{\bar{F}} c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)} - \frac{u(y)}{\sigma(y)}\right)\mathrm{d}y - q_\sigma(x)\,u(x),\;\; x \in F,$$

$$\left(\frac{\partial u}{\partial \mathsf{n}_F}\right)(x) = \frac{1}{\sigma(x)}\int_{F} c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)} - \frac{u(y)}{\sigma(y)}\right)\mathrm{d}y - q_\sigma(x)\,u(x),\;\; x \in \delta(F).$$

In addition, for any $u,v\in\mathcal{C}(\bar F)$ we get that

$$\mathcal{E}^F(u,v)=\frac{1}{2}\int_{\bar F}\int_{\bar F}c_{_F}(x,y)\sigma(x)\sigma(y)\Big(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\Big)\Big(\frac{v(x)}{\sigma(x)}-\frac{v(y)}{\sigma(y)}\Big)\mathrm{d}x\,\mathrm{d}y-\int_{\bar F}q_\sigma u\,v.$$

Proof. First observe that

$$\sigma(x)\big(u(x)-u(y)\big)=\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)+\big(\sigma(x)-\sigma(y)\big)\,u(x),$$

for any $x,y\in\bar F$. So, if $x\in F$, then

$$\begin{aligned}\mathscr{L}(u)(x)&=\int_{\bar F}c(x,y)\big(u(x)-u(y)\big)\,\mathrm{d}y\\&=\frac{1}{\sigma(x)}\int_{\bar F}c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)\mathrm{d}y+\frac{u(x)}{\sigma(x)}\int_{\bar F}c(x,y)\big(\sigma(x)-\sigma(y)\big)\mathrm{d}y\\&=\frac{1}{\sigma(x)}\int_{\bar F}c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)\mathrm{d}y-q_\sigma(x)\,u(x),\end{aligned}$$

whereas if $x\in\delta(F)$, then

$$\begin{aligned}\frac{\partial u}{\partial \mathsf{n}_{_F}}(x)&=\int_F c(x,y)\big(u(x)-u(y)\big)\,\mathrm{d}y\\&=\frac{1}{\sigma(x)}\int_F c(x,y)\sigma(x)\sigma(y)\Big(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\Big)\mathrm{d}y-q_\sigma(x)\,u(x).\end{aligned}$$

Finally, from Identity (3), we get that

$$\begin{aligned}\mathcal{E}^F(u,v)=&\int_F\frac{v(x)}{\sigma(x)}\int_{\bar F}c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)\mathrm{d}y\,\mathrm{d}x\\&+\int_{\delta(F)}\frac{v(x)}{\sigma(x)}\int_F c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)\mathrm{d}y\,\mathrm{d}x-\int_{\bar F}q_\sigma u\,v\\=&\int_{\bar F}\int_{\bar F}c_{_F}(x,y)\sigma(x)\sigma(y)\frac{v(x)}{\sigma(x)}\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)\mathrm{d}x\,\mathrm{d}y-\int_{\bar F}q_\sigma u\,v.\end{aligned}$$

Therefore, the last claim is consequence of the following identities,

$$\begin{aligned}&\int_{\bar F}\int_{\bar F}c_{_F}(x,y)\sigma(x)\sigma(y)\frac{v(x)}{\sigma(x)}\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)\mathrm{d}x\,\mathrm{d}y\\&=\int_{\bar F}\int_{\bar F}c_{_F}(y,x)\sigma(y)\sigma(x)\frac{v(y)}{\sigma(y)}\left(\frac{u(y)}{\sigma(y)}-\frac{u(x)}{\sigma(x)}\right)\mathrm{d}y\,\mathrm{d}x\\&=-\int_{\bar F}\int_{\bar F}c_{_F}(x,y)\sigma(x)\sigma(y)\frac{v(y)}{\sigma(y)}\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where we have taken into account the symmetry of $c_{_F}$.

Corollary 3. *If there exist $\sigma \in \Omega(\bar{F})$ and $\lambda \geq 0$ such that $q = q_\sigma + \lambda\chi_{\delta(F)}$, then the Energy $\mathcal{E}_q^F$ is positive semi–definite. Moreover, it is not strictly definite iff $\lambda = 0$, in which case $\mathcal{E}_q^F(v,v) = 0$ iff $v = a\sigma$, $a \in \mathbb{R}$.*

We remark that given $q \in \mathcal{C}(V)$, there exist unique $\sigma \in \Omega(V)$ and $\lambda \in \mathbb{R}$ such that $q = q_\sigma + \lambda$. This property is a consequence of the Perron–Frobenius Theorem applied to the matrix $t\mathsf{I} - \mathsf{K}$, for $t \in \mathbb{R}$ large enough, where K is the matrix associated with the operator $K = \mathscr{L}_q$ on F and $K = \dfrac{\partial}{\mathsf{n}_F} + q$ on $\delta(F)$.

So, $q = q_\sigma + \lambda$ does not represent any restriction on the potentials. Moreover, the Doob transform allows us to characterize when K is positive semidefinite: this occurs iff $\lambda \geq 0$ and, in this case, K is singular iff $\lambda = 0$. Therefore, $q = q_\sigma$ corresponds to positive semidefinite and singular operators.

On the other hand, it is true that for $\lambda > 0$, $q = q_\sigma + \lambda\chi_{\delta(F)}$ determine a positive definite operator and hence there exist also a weight $\omega \in \Omega(V)$ and a value $\mu > 0$ such that

$$q = q_\sigma + \lambda\chi_{\delta(F)} = q_\omega + \mu.$$

However, the relation between σ and ω, and λ and μ is not easy to find. In fact, we only know that $\mu < \lambda$.

From now on, we will work with potentials given by a weight $\sigma \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$ such that $q = q_\sigma + \lambda\chi_{\delta(F)}$; so that the corresponding Schrödinger operator is positive semi–definite. Notice, that in this case the weight σ is q–harmonic on F; that is, $\mathscr{L}_q(\sigma) = 0$ on F.

In [**3**, Proposition 4.10], some of the present authors proved the following version of the minimum principle that will be useful in what follows.

Proposition 4 (Monotonicity Property)**.** *If $u \in \mathcal{C}(\bar{F})$ is such that $\mathcal{L}_q(u) \geq 0$ on F and $u \geq 0$ on $\delta(F)$ then either $u > 0$ on F or $u = 0$ on $\bar{F}$.*

Let us consider the following *Dirichlet problem*: Given $f \in \mathcal{C}(F)$ and $g \in \mathcal{C}(\delta(F))$ find $u \in \mathcal{C}(\bar{F})$ satisfying

$$\mathscr{L}_q(u) = f \quad \text{on} \quad F \quad \text{and} \quad u = g \quad \text{on} \quad \delta(F). \tag{4}$$

The existence and uniqueness of solution for Problem (4) were proved in the above–mentioned paper. In fact, the *Dirichlet Principle* tell us that for any data $f \in C(F)$ and $g \in C(\delta(F))$, Problem (4) has a unique solution; see [**4**, Proposition 3.3].

Associated with the Dirichlet problem we can consider the following semi homogenous problems, that allow us to introduce the concept of Green and Poisson operators:

Given $f \in \mathcal{C}(F)$ find $u \in \mathcal{C}(F)$ satisfying

$$\mathscr{L}_q(u) = f \quad \text{on} \quad F \quad \text{and} \quad u = 0 \quad \text{on} \quad \delta(F), \tag{5}$$

and given $g \in \mathcal{C}(\delta(F))$ find $v \in \mathcal{C}(\bar{F})$ satisfying

$$\mathscr{L}_q(v) = 0 \quad \text{on} \quad F \quad \text{and} \quad v = g \quad \text{on} \quad \delta(F). \tag{6}$$

Clearly, if u and v are the unique solutions of (5) and (6) respectively, then $u+v$ is the unique solution of Problem (4).

We define the *Green operator* for F as the operator $\mathscr{G}_q\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$ assigning to any $f \in \mathcal{C}(F)$ the unique solution, $u_f \in \mathcal{C}(F)$, of Problem (5). The kernel associated with the Green operator is called *Green kernel* and denoted by G_q. Then, $G_q : F \times F \longrightarrow \mathbb{R}$ is a symmetric kernel and $G_q(x,y) = \mathscr{G}_q(\varepsilon_y)(x)$.

On the other hand, we define the *Poisson operator* for F as the operator $\mathscr{P}_q\colon \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\bar{F})$ assigning to any $g \in \mathcal{C}(\delta(F))$ the unique solution, u_g, of Problem (6). The kernel associated with the Poisson operator is called *Poisson kernel* and denoted by P_q. Then, $P_q : \bar{F} \times \delta(F) \longrightarrow \mathbb{R}$ and $P_q(x,y) = \mathscr{P}_q(\varepsilon_y)(x)$. From Proposition 4, we obtain that $G_q(x,y) > 0$ for any $x, y \in F$ and $P_q(x,y) > 0$ for any $x \in F, y \in \delta(F)$. Moreover, $\mathscr{P}_q(\sigma\chi_{\delta(F)}) = \sigma$ on $\bar{F}$, since $\mathscr{L}_q(\sigma) = 0$ on F, which means that weight $\sigma \in \Omega(\bar{F})$ is the unique solution of the Dirichlet problem

$$\mathscr{L}_q(u) = 0 \quad \text{on} \quad F \quad \text{and} \quad u = \sigma \quad \text{on} \quad \delta(F).$$

Therefore, given $f \in \mathcal{C}(F)$ and $g \in \mathcal{C}(\delta(F))$ the unique solution $u \in \mathcal{C}(\bar{F})$ of Problem (4) is given by

$$u = \mathscr{G}_q(f) + \mathscr{P}_q(g).$$

Moreover, we have the following relation between the Green and Poisson kernels that was proved in [**3**]:

$$P_q(x,y) = \varepsilon_y(x) - \frac{\partial G_q}{\partial \mathsf{n}_y}(x,y).$$

4. THE DIRICHLET–TO–ROBIN MAP

In this section we define the *Dirichlet–to–Robin map* on general networks and we study its main properties. This map is naturally associated to a Schrödinger operator, and generalizes the concept of *Dirichlet–to–Neumann map* for the case of the combinatorial Laplacian.

The map $\Lambda_q\colon \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any function $g \in \mathcal{C}(\delta(F))$ the function $\Lambda_q(g) = \dfrac{\partial u_g}{\partial \mathsf{n}_F} + qg$ is called *Dirichlet–to–Robin map*, where $u_g = \mathscr{P}_q(g)$.

The Poisson kernel is directly related to the Dirichlet–to–Robin map Λ_q, as it is shown on the proposition below.

Proposition 5. *The Dirichlet–to–Robin map, Λ_q, is a self–adjoint, positive semi–definite operator whose associated quadratic form is given by*

$$\int_{\delta(F)} g\Lambda_q(g) = \mathcal{E}_q^F(u_g, u_g).$$

Moreover, λ is the lowest eigenvalue of Λ_q and its associated eigenfunctions are multiple of σ. In addition, the kernel $N \in \mathcal{C}(\delta(F) \times \delta(F))$ of Λ_q is

$$N = \frac{\partial P_q}{\partial \mathsf{n}_x} + q = k_F + q - \frac{\partial^2 G_q}{\partial \mathsf{n}_x \partial \mathsf{n}_y},$$

which is symmetric, negative off–diagonal and positive on the diagonal.

Proof. From (3) we get that for any $f, g \in \mathcal{C}(\delta(F))$,

$$\int_{\delta(F)} f\Lambda_q(g) = \mathcal{E}_q^F(u_f, u_g) = \mathcal{E}_q^F(u_g, u_f) = \int_{\delta(F)} g\Lambda_q(f)$$

and hence Λ_q is self–adjoint and positive semi–definite. Moreover, for every $x \in \delta(F)$, using that $\mathscr{P}_q(\sigma\chi_{\delta(F)}) = \sigma$ on $\bar{F}$ it is easily seen that

$$\Lambda_q(\sigma\chi_{\delta(F)})(x) = \frac{\partial \sigma}{\partial \mathsf{n}_F}(x) + q_\sigma(x)\sigma(x) + \lambda\sigma(x) = \lambda\sigma(x),$$

since $q_\sigma = -\sigma^{-1}\dfrac{\partial \sigma}{\partial \mathsf{n}_F}$ on $\delta(F)$.

On the other hand, from Proposition 2 and taking into account that $u_g = g$ on $\delta(F)$, we get

$$\begin{aligned}\mathcal{E}_q^F(u_g, u_g) &= \mathcal{E}^F(u_g, u_g) + \int_{\bar{F}} q u_g^2 = \mathcal{E}^F(u_g, u_g) + \int_{\bar{F}} q_\sigma u_g^2 + \lambda\int_{\delta(F)} g^2 \\ &= \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_F(x,y)\sigma(x)\sigma(y)\left(\frac{u_g(x)}{\sigma(x)} - \frac{u_g(y)}{\sigma(y)}\right)^2 \mathrm{d}x\,\mathrm{d}y + \lambda\int_{\delta(F)} g^2 \geq \lambda\int_{\delta(F)} g^2.\end{aligned}$$

The equality holds iff $u_g = a\sigma$; that is, iff $g = a\sigma$.

Suppose that g is a non–zero eigenfunction corresponding to the eigenvalue α. Then, by the definition of eigenvalue and the first part of the proposition, we have

$$\alpha\int_{\delta(F)} g^2 = \int_{\delta(F)} g\Lambda_q(g) = \mathcal{E}_q^F(u_g, u_g) \geq \lambda\int_{\delta(F)} g^2$$

which implies that $\alpha \geq \lambda$.

Taking now $g = \varepsilon_y$, $y \in \delta(F)$, we obtain that $u_g = P_q^y$ and therefore

$$\Lambda_q(\varepsilon_y) = \frac{\partial P_q^y}{\partial \mathsf{n}_F} + q\varepsilon_y.$$

Using this and equality $P_q(x,y) = \varepsilon_y(x) - \dfrac{\partial G_q}{\partial \mathsf{n}_y}(x,y)$ we easily see that

$$\Lambda_q(\varepsilon_y) = \frac{\partial}{\partial \mathsf{n}_F}\left(\varepsilon_y - \frac{\partial G_q^y}{\partial \mathsf{n}_y}\right) + q\varepsilon_y = k_F\varepsilon_y + q\varepsilon_y - \frac{\partial^2 G_q}{\partial \mathsf{n}_x \partial \mathsf{n}_y}.$$

Finally, for any $x, y \in \delta(F)$ with $x \neq y$, notice that $P_q(x,y) = \varepsilon_y(x) = 0$. In this case, we get that

$$\begin{aligned}N(x,y) = \Lambda_q(\varepsilon_y)(x) &= \frac{\partial P_q}{\partial \mathsf{n}_x}(x,y) \\ &= \sum_{z\in F} c(x,z)\big(P_q(x,y) - P_q(z,y)\big) = -\sum_{z\in F} c(x,z)P_q(z,y) < 0,\end{aligned}$$

since $P_q(z,y)>0$. Moreover, as $\Lambda_q(\sigma)=\lambda\sigma$ on $\delta(F)$, then for any $y\in\delta(F)$

$$\sum_{x\in\delta(F)}\Lambda_q(\varepsilon_y)(x)\sigma(x)=\lambda\sigma(y),$$

and hence

$$\Lambda_q(\varepsilon_y)(y)=\lambda-\sigma(y)^{-1}\sum_{\substack{x\in\delta(F)\\ x\neq y}}\Lambda_q(\varepsilon_y)(x)\sigma(x)>0,$$

where we used the fact that $\Lambda_q(\varepsilon_y)(x)<0$ for any $x,y\in\delta(F)$ and $x\neq y$ as shown above. □

The kernel of Dirichlet–to–Robin map is closely related to the Schur complement of $\big(\mathscr{L}_q\big)_{|F}$ in $\mathscr{L}_q$; see [**10**] and [**8**, Theorem 3.2] for the combinatorial Laplacian and the Dirichlet–to–Neumann map. Notice that the Robin problem

$$\mathscr{L}_q(u)=f \quad\text{on}\quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_F}+qu=g \quad\text{on } \delta(F),$$

has the following matrix expression

$$\mathsf{L}=\begin{bmatrix}\mathsf{D} & -\mathsf{C}\\ -\mathsf{C}^\top & \mathsf{M}\end{bmatrix}\begin{bmatrix}\mathsf{v}_\delta\\ \mathsf{v}\end{bmatrix}=\begin{bmatrix}\mathsf{g}\\ \mathsf{f}\end{bmatrix},$$

where D is the diagonal matrix whose diagonal entries are given by k_F+q, $\mathsf{C}=\big(c(x,y)\big)_{\substack{x\in\delta(F)\\ y\in F}}$, M is the matrix associated with $\big(\mathscr{L}_q\big)_{|F}$, v_δ, v, f, g are the vectors determined by $u_{|_{\delta(F)}}$, $u_{|_F}$, f and g, respectively. Then, M is invertible and $\mathsf{M}^{-1}=\big(G_q(x,y)\big)_{x,y\in F}$. Moreover, the Schur complement of M in L is

$$\mathsf{L}/\mathsf{M}=\mathsf{D}-\mathsf{C}\mathsf{M}^{-1}\mathsf{C}^\top=\big(N(x,y)\big)_{x,y\in\delta(F)},$$

since we have the equality given for N in Proposition 5 and the following equality from Lemma 1

$$\mathsf{C}\mathsf{M}^{-1}\mathsf{C}^\top=\left(\frac{\partial^2 G_q}{\partial\mathsf{n}_x\partial\mathsf{n}_y}(x,y)\right)_{x,y\in\delta(F)},$$

where we have taken into account that G_q is symmetric and zero on $\delta(F)\times F$.

Now we show that the Dirichlet–to–Robin map has the alternating property, which may be considered as a generalization of the monotonicity property; see [**11**, Theorem 2.1] for the continuous version of this property.

Theorem 6 (Alternating paths)**.** *Suppose that $\delta(F)=A\cup B$, where A and B are disjoint subsets. Let $g\in\mathcal{C}(B)$ and $s_1,\dots,s_k\in A$ such that for every index $i=1,\dots,k$*

$$(-1)^{i+1}\Lambda_q(g)(s_i)>0. \tag{7}$$

Then there exist $t_1, \dots, t_k \in B$ *such that for every index* $i = 1, \dots, k$

$$\Lambda_q(g)(s_i) g(t_{k-i+1}) < 0. \tag{8}$$

Moreover, for any $i = 1, \dots, k$*, there exists a path* γ_i *from* s_i *to* t_{k-i+1} *such that* $\gamma_i \setminus \{s_i, t_{k-i+1}\} \subset F$ *and* $g(t_{k-i+1}) u_{g_{|(\gamma_i \setminus \{s_i\})}} > 0$*, where* $u_g = \mathscr{P}_q(g)$.

Proof. Observe that as $g \in \mathcal{C}(B)$, for any $x \in A$, we have that $\Lambda_q(g)(x) = \dfrac{\partial u_g}{\partial \mathsf{n}}(x)$. By (7), $0 < \dfrac{\partial u_g}{\partial \mathsf{n}}(s_1) = -\int_F c(s_1, y) u_g(y)\, \mathrm{d}y$. Then, there exists $y \in F \cap V(s_1)$ such that $u_g(y) < 0$.

Let W be the connected component of $\{z \in F : u_g(z) < 0\}$ containing y. Suppose that $\overline{W} \cap B = \emptyset$; that is, $\overline{W} \subset F \cup A$. We consider $v = u_g \chi_{\overline{W}}$; then $\mathscr{L}_q(v) = 0$ on W, $v \geq 0$ on $\delta(W)$, then from the monotonicity principle $v \geq 0$ on W which is a contradiction. Therefore, $\overline{W} \cap B \neq \emptyset$ and hence $u_g \geq 0$ on $\delta(W) \cap F$. If $u_g \geq 0$ on $\delta(W) \cap B$, we get that $\mathscr{L}_q(u_g) = 0$ on W, $u_g \geq 0$ on $\delta(W)$, so $u_g \geq 0$ on W applying again the monotonicity principle which is a contradiction. So, there exists $t_k \in \delta(W) \cap B$ such that $u_g(t_k) < 0$ which means $g(t_k) < 0$. As $t_k \in \delta(W)$, there exists $z_1 \in W$, so $u_g(z_1) < 0$, such that $t_k \sim z_1$. As W is a connected subset we can join t_k and s_1 by a path $\gamma_1 = \{s_1 \sim y \sim \ldots \sim z_1 \sim t_k\}$ such that $\{y, \dots, z_1\} \subset W$ and hence $u_{g_{|(\gamma_1 \setminus \{s_1\})}} < 0$.

We can repeat this argument to produce paths γ_j such that γ_j joins s_j to a point $t_{k+1-j} \in B$ such that $\gamma_j \setminus \{s_j, t_{k+1-j}\} \subset F$ and $(-1)^j u_g(z) < 0$ for all $z \in \gamma_j \setminus \{s_j\}$.

Corollary 7. *Suppose that the network* $\Gamma = (\bar{F}, c_{_F})$ *is circular planar and* $\delta(F) = A \cup B$*, where* A *and* B *are disjoint subsets. Let* $g \in \mathcal{C}(B)$ *and* $u_g = \mathscr{P}_q(g)$*, if there is a set of different points* $\{p_1, \dots, p_k\} \in A$ *in circular order such that for every index* $i = 1, \dots, k$

$$(-1)^{i+1} \Lambda_q(g)(p_i) > 0 \tag{9}$$

then there is a set of different points $\{q_1, \dots, q_k\} \in B$ *in circular order such that for every index* $i = 1, \dots, k$

$$\Lambda_q(g)(p_i) g(q_{k-i+1}) < 0. \tag{10}$$

Moreover, for any $i = 1, \dots, k$*, there exists a path* γ_i *from* p_i *to* q_{k-i+1} *such that* $\gamma_i \setminus \{p_i, q_{k-i+1}\} \subset F$*,* $g(q_{k-i+1}) u_{g_{|(\gamma_i \setminus \{p_i\})}} > 0$ *in such a way that* $(p_1, \dots, p_k; q_1, \dots, q_k)$ *is connected through* Γ.

Proof. The paths built on the proof of Theorem 6 do not intersect if the network is planar and the vertices $p_j \in A$ are in circular order, since the values for the function u_g in γ_j have different sign than the ones on γ_{j-1} and γ_{j+1}. Then, $q_j \in B$ are the points given by the last theorem and are also in circular order.

Theorem 8 (Strong alternating paths)**.** *Suppose that $\delta(F) = A \cup B$, where A and B are disjoint subsets. Let $g \in \mathcal{C}(B)$ and $u_g = \mathscr{P}_q(g)$. If there is a set of points $\{s_1, \dots, s_k\} \in A$ such that for every index $i = 1, \dots, k$*

$$\Lambda_q(g)(s_i) = 0 \tag{11}$$

then there is a set of points $\{t_1, \dots, t_k\} \in B$ such that for every index $i = 1, \dots, k$

$$(-1)^i g(t_i) \geq 0. \tag{12}$$

Moreover, for any $i = 1, \dots, k$, there exists a path P_i from $s_i \sim x_1^i \sim \dots \sim x_{n_i}^i \sim t_i$ such that $P_i \setminus \{s_i, t_i\} \subset F$ and there exists $j_i \in \{1, \dots, n_i + 1\}$ such that $u_g(x_\ell^i) = 0$ for all $\ell = 0, \dots, j_i - 1$ and $g(t_i) u_g(x_\ell^i) > 0$ for all $\ell = j_i, \dots, n_i + 1$, where $x_0^i = s_i$ and $x_{n_i+1}^i = t_i$.

Proof. By (11), $0 = \dfrac{\partial u_g}{\partial \mathsf{n}}(s_1) = -\int_F c(s_1, y) u_g(y)\, \mathrm{d}y$. Then, either $u_g = 0$ on $V(s_1) \cap F$ or there exists $y_1 \in F$ such that $u_g(y_1) < 0$ (in this case there also exists $z_1 \in F$ such that $u_g(z_1) > 0$). If $u_g = 0$ on $V(s_1) \cap F$, consider $x_1 \in V(s_1) \cap F$, then $0 = \mathscr{L}_q(u_g)(x_1) = -\sum_{y \in \bar{F}} c(x_1, y) u_g(y)$. Again, either $u_g = 0$ on $V(x_1)$ or there exists $y_2 \in F \cup B$ such that $u_g(y_2) < 0$ (in this case there exists also $z_2 \in F \cup B$ such that $u_g(z_2) > 0$). If $y_2 \in B$, this will be the vertex t_1.

Otherwise, let W be the connected component of $\{z \in F : u_g(z) < 0\}$ and proceeding as in the proof of Theorem 6 we get the result. □

Having fixed a label in the boundary, we say that the network has the *alternating property* if for any ordered set $\{s_1, \dots, s_k\}$ satisfying the hypothesis of Theorem 6, then the vertices $\{t_1, \dots, t_k\}$, given by the Theorem, are also in order. Next result also tells us a property of the Dirichlet–to–Robin map of networks having the alternating property which is related with totally nonnegative matrices.

Theorem 9. *Let Γ be a network having the alternating property and let the pair $(x_1, \dots, x_k; y_1, \dots, y_k)$ be an ordered pair on $\delta(F)$. Let $M = (m_{ij})$ be the $k \times k$ matrix with entries defined by $m_{ij} = \dfrac{\partial P_q}{\partial \mathsf{n}_x}(x_i, y_j)$. Then,*

$$(-1)^{\frac{k(k+1)}{2}} \mathsf{det}(M) \geq 0. \tag{13}$$

Proof. Clearly we can suppose that the vertices $(x_1, \dots, x_k)$ are different and that the vertices $(y_1, \dots, y_k)$ are different too. The proof is by induction on k. For $k = 1$, from Proposition 5, the result is true. Next we assume that the result is true for all $(k-1) \times (k-1)$ submatrices and prove that it is true for $k \times k$ matrices. If the result is not true, then we have a sequence of distinct vertices $(x_1, \dots, x_k; y_1, \dots, y_k)$ such that

$$(-1)^{\frac{k(k+1)}{2}} \mathsf{det}(M) < 0. \tag{14}$$

Consider the matrix M^{-1} with entries (h_{ij}). Then,

$$h_{ij} = (-1)^{i+j} \frac{\mathsf{det}(M_{ij})}{\mathsf{det}(M)}, \tag{15}$$

where M_{ij} is the (i,j) minor of M. By induction hypothesis, (14) and (15),

$$(-1)^{i+j+k+1} h_{ij} = (-1)^{i+j+\frac{k(k-1)}{2}+\frac{k(k+1)}{2}+1} h_{ij} \geq 0. \tag{16}$$

Since M is nonsingular, for fixed i there must be some j for which

$$(-1)^{i+j+k+1} h_{ij} > 0. \tag{17}$$

Now let $w = (w_i)_{i=1}^k$ be the vector $w_i = (-1)^{i+1}$ and $z = M^{-1}w$. Then using (16) and (17) it is easy to verify that

$$(-1)^{i+k} z_i > 0. \tag{18}$$

So we have a vector z such that

$$w_i = \sum_{j=1}^{k} \frac{\partial P_q}{\partial \mathsf{n}_x}(x_i, y_j) z_j \tag{19}$$

and $(-1)^{k+1} z_i w_i > 0$.

Let the function $g \in \delta(F)$, defined as $g(y_j) = z_j$, $j = 1, \ldots, k$ and $g = 0$ otherwise then,

$$\Lambda_q g(x_i) = \sum_{j=1}^{k} \frac{\partial P_q}{\partial \mathsf{n}_x}(x_i, y_j) g(y_j) = w_i,$$

by the choice of g, the definition of Λ_q and equation (19). Then by Theorem 6, there exists a set of vertices $\{t_1, \ldots, t_k\} \in \delta(F)$ such that the pair $(x_1, \ldots, x_k; t_1, \ldots, t_k)$ is an ordered pair and

$$\omega_i g(t_{k-i+1}) < 0,$$

or equivalenty, such that $(-1)^k w_i g(t_i) > 0$. Therefore, for any $i = 1, \ldots, k$, $t_i \in \{y_1, \ldots, y_k\}$, since otherwise we get $g(t_i) = 0$. Moreover, $\{y_1, \ldots, y_k\}$ and $\{t_1, \ldots, t_k\}$ are ordered subsets and hence there exists i such that $t_i = y_i$. Then, we will have $0 < (-1)^k w_i g(t_i) = (-1)^k w_i g(y_i) = (-1)^k w_i z_i$, which is a contradiction with the fact that $(-1)^{k+1} z_i w_i > 0$.

5. CHARACTERIZATION OF THE DIRICHLET–TO–ROBIN MAP

Our next objective is to characterize the kernels on $\delta(F)$ that are the kernels associated to a Dirichlet–to–Robin map. The results in the previous section show that the matrices associated with these kernels are necessarily symmetric M–matrices, in fact a particular class of M–matrices. Recall that a symmetric

M–matrix is a positive semidefinite symmetric matrix whose off–diagonal entries are non–positive, see [**2**]. For the case of the combinatorial Laplacian of a network, Curtis et al. characterized in [**8**] those singular, symmetric and diagonally dominant M–matrices that are the response matrices of a circular planar network. This case corresponds to $\lambda = 0$ and σ a constant weight. Now, our proposal is to generalized the above–mentioned result to the general case, in order to include a wider class of M–matrices.

We first observe that the response matrices do not identify the associated difference operator; that is, a given singular, symmetric and diagonally dominant M–matrix can be the response matrix associated with multiple Schrödinger operators of a circular planar network as the following simple example shows.

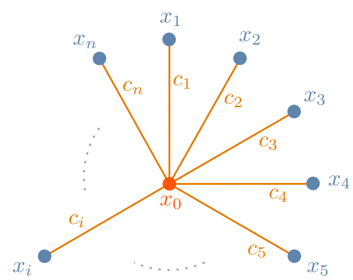

Figure 1. The star network on n vertices.

Let us consider the star network (Γ, c) on $n \geq 3$ vertices with central vertex x_0 and peripherical vertices $x_1, \dots, x_n$, see Figure 1. Let $F = \{x_0\}$, $\delta(F) = \{x_1, \dots, x_n\}$, $\sigma \in \Omega(\bar{F})$ and the conductances are given by $c(x_0, x_i) = c_i > 0$. Then, by using Doob transform given in Proposition 2 we see that

$$\mathscr{L}_{q_\sigma}(u)(x_0) = \frac{u(x_0)}{\sigma(x_0)} \sum_{i=1}^{n} c_i \sigma(x_i) - \sum_{i=1}^{n} c_i u(x_i),$$
$$\frac{\partial u}{\partial \mathsf{n}_F}(x_i) + q_\sigma(x_i) u(x_i) = \frac{c_i \sigma(x_0)}{\sigma(x_i)} u(x_i) - c_i u(x_0), \qquad i = 1, \dots, n$$

for all $u \in \mathcal{C}(\bar{F})$. Therefore, given $g \in \mathcal{C}(\delta(F))$, the unique solution of the problem $\mathscr{L}_{q_\sigma}(u) = 0$ on F and $u = g$ on $\delta(F)$ is given by $u(x_i) = g(x_i)$ for all $i = 1, \dots, n$ and

$$u(x_0) = \frac{\sigma(x_0)}{\sum_{i=1}^{n} c_i \sigma(x_i)} \sum_{i=1}^{n} c_i g(x_i).$$

As a consequence, the Dirichlet–to–Robin operator for the star network with weight σ and conductance c, denoted here by $\Lambda_{c,\sigma}$, is given by

$$\Lambda_{c,\sigma}(g)(x_i) = \frac{c_i \sigma(x_0)}{\sigma(x_i)} g(x_i) - \frac{c_i \sigma(x_0)}{\sum_{k=1}^{n} c_k \sigma(x_k)} \sum_{k=1}^{n} c_k g(x_k) \tag{20}$$

for all $g \in \mathcal{C}(\delta(F))$. Clearly, $\Lambda_{c,\sigma}(g) = 0$ iff g is a multiple of $\sigma_{|_{\delta(F)}}$.

Let $\hat{\sigma} \in \Omega(\delta(F))$ defined as $\hat{\sigma}(x_i) = \dfrac{\sigma(x_i)}{\sqrt{1 - \sigma(x_0)^2}}$, $i = 1, \dots, n$ and for any

$0 < \omega_0 < 1$, consider $\omega \in \Omega(\bar{F})$ defined as

$$\omega(x_0) = \omega_0 \quad \text{and} \quad \omega(x_i) = \sqrt{1-\omega_0^2}\,\hat{\sigma}(x_i), \quad i = 1, \ldots, n.$$

Observe that $\hat{\sigma}$ is the unique eigenfunction of $\Lambda_{c,\sigma}$ corresponding to the zero eigenvalue that is a weight on $\delta(F)$. Moreover, ω has been built by normalizing an arbitrary extension of $\hat{\sigma}$.

We now look for a conductance d on Γ such that $\Lambda_{d,\omega} = \Lambda_{c,\sigma}$. From Equation (20) this happens iff

$$\frac{c_i\sigma(x_0)g(x_i)}{\sigma(x_i)} - \frac{c_i\sigma(x_0)\sum_{k=1}^{n} c_k g(x_k)}{\sum_{k=1}^{n} c_k\sigma(x_k)} = \frac{d_i\omega_0 g(x_i)}{\alpha\sigma(x_i)} - \frac{d_i\omega_0\sum_{k=1}^{n} d_k g(x_k)}{\alpha\sum_{k=1}^{n} d_k\sigma(x_k)} \tag{21}$$

where $d_i = d(x_0, x_i)$, $i = 1, \ldots, n$ and $\alpha = \sqrt{\dfrac{1-\omega_0^2}{1-\sigma^2(x_0)}}$. If $g = \varepsilon_{x_j}$, $j = 1, \ldots, n$, then

$$\frac{c_i c_j\sigma(x_0)}{\sum_{k=1}^{n} c_k\sigma(x_k)} = \frac{d_i d_j\omega_0}{\alpha\sum_{k=1}^{n} d_k\sigma(x_k)}, \qquad i \neq j.$$

If $\beta = \dfrac{\alpha\sigma(x_0)\sum_{k=1}^{n} d_k\sigma(x_k)}{\omega_0\sum_{k=1}^{n} c_k\sigma(x_k)}$, then $\beta = \dfrac{d_i d_j}{c_i c_j}$ for all $i, j = 1, \ldots, n$ with $i \neq j$.

Therefore,

$$\frac{d_i d_j}{c_i c_j} = \frac{d_k d_j}{c_k c_j} \quad \text{for all } i, k = 1, \ldots, n \text{ with } i, k \neq j$$

and hence $d_j = \dfrac{\alpha\sigma(x_0)}{\omega_0}c_j = \sqrt{\dfrac{\sigma^2(x_0)(1-\omega_0^2)}{\omega_0^2(1-\sigma^2(x_0))}}c_j$ for all $j = 1, \ldots, n$ since $n \geq 3$. Notice that $d = c$ iff $\omega = \sigma$.

Therefore, for an arbitrary extension of the weight σ on $F = \{x_0\}$ there exists a conductance function such that the Dirichlet–to–Robin operator associated with the Schrödinger operator with the new conductances and weight coincides with the initial one.

In particular, if σ is constant then, $q_\sigma = 0$ on $\bar{F}$ and hence the corresponding Schrödinger operator is the combinatorial Laplacian and the Dirichlet–to–Robin map becomes the classical Dirichlet–to–Neumann map. Therefore, the above results tell us that the Dirichlet–to–Neumann map does not identify uniquely the Schrödinger operator and, in fact, it appears as the Dirichlet–to–Robin map of an infinite family of Schrödinger operators.

The following definitions follow the terminology of [**8**]. Suppose that $A = \{a_{ij}\}$ is a matrix, $S = (s_1, \ldots, s_k)$ is an ordered subset of the rows and $T =$

$(t_1,\dots,t_m)$ is an ordered subset of the columns. Then, $A(S;T)$ denotes the $k\times m$ matrix obtained by taking the entries of A which are in rows $s_1,\dots,s_k$ and columns $t_1,\dots,t_m$. If $(S;T)$ is a circular pair of indices, $A(S;T)$ is called a *circular minor* of A.

For any $n\ge 2$, $\sigma\in\Omega(\delta(F))$ and $\lambda\ge 0$, let $\Phi_{\lambda,\sigma}$ be the set of irreducible and symmetric n–matrices, M, for which λ is the lowest eigenvalue and σ is the eigenvector associated with λ, satisfying the following condition

If $M(S;T)$ is a $k\times k$ circular minor of M, then $(-1)^k\det M(S;T)\ge 0$.

This condition says that if $M\in\Phi_{\lambda,\sigma}$ and $(S;T)$ is a circular pair of indices, then the matrix $-M(S;T)$ is totally non–negative. In particular, if $M\in\Phi_{\lambda,\sigma}$, then M is an M–matrix.

If we denote by $\mu=|\delta(F)|^{-\frac{1}{2}}\chi_{\delta(F)}$, the unique constant weight on $\delta(F)$, then $\Phi_{0,\mu}=D_\sigma\Phi_{0,\sigma}D_\sigma$, for arbitrary σ. The next results were obtained in [**8**].

Lemma 10 ([**8**, Theorem 3])**.** *Suppose that M is in $\Phi_{0,\mu}$. Then, there is a circular planar graph with a conductivity c such that $M=\Lambda$, where Λ is the Dirichlet–to–Neumann map associated with the combinatorial Laplacian for the conductance c.*

Suppose that Γ is a circular planar network with n boundary vertices, and $\pi=\pi(\Gamma)$ is the set of circular pairs $(P;Q)$ which are connected through Γ. Then, Γ is called a *critical circular planar network*, if removing any edge breaks at least one of the connections $(P;Q)$ in π. A subset $\Phi_{\lambda,\sigma}(\pi)$ of $\Phi_{\lambda,\sigma}$ is defined by the following condition: For each circular pair of indices $(P;Q)$, $(P;Q)\in\pi$ iff $(-1)^k\det M(P;Q)>0$.

Lemma 11 ([**8**, Theorem 4])**.** *Suppose Γ is a critical planar graph with m edges and $\pi=\pi(\Gamma)$. Then, the map which sends conductance c to Λ is a diffeomorphism of $(\mathbb{R}^+)^m$ onto $\Phi_{0,\mu}(\pi)$.*

Given $M\in\Phi_{\lambda,\sigma}$, we say that $u\in\mathcal{C}(\bar F)$ is *M–harmonic* iff $\mathscr{L}(u)=0$ on F, where $\mathscr{L}$ is the combinatorial Laplacian whose conductance is uniquely associated with the matrix $D_\sigma(M-\lambda I)D_\sigma$ given in Lemma 11.

Theorem 12. *For any $n\ge 2$, $\sigma\in\Omega(\delta(F))$ and $\lambda\ge 0$, suppose M is in $\Phi_{\lambda,\sigma}$. Then, there is a circular planar graph with conductance c such that for any $\omega\in\Omega(\bar F)$ satisfying $\omega=k\sigma$ on $\delta(F)$, $M=\Lambda_q$, where Λ_q is the Dirichlet–to–Robin map associated with the operator $\mathscr{L}_q$, with $q=q_\omega+\lambda\chi_{\delta(F)}$ and conductances $c_\omega=\dfrac{c}{\omega\otimes\omega}$. Moreover, if $M\in\Phi_{\lambda,\sigma}(\pi)$, then there is a unique critical circular planar network with conductance c and a unique $\omega\in\Omega(\bar F)$, M–harmonic function such that $M=\Lambda_q$.*

Proof. Let $M\in\Phi_{\lambda,\sigma}$, then $\hat M=D_\sigma(M-\lambda I)D_\sigma\in\Phi_{0,\mu}$. Applying Lemma 10, there is a circular planar graph with conductances c such that $\hat M=\Lambda$, where Λ is the Dirichlet–to–Neumann map associated with the combinatorial Laplacian, $\mathscr{L}^c$, of the network. Consider $\widetilde{\omega} \in \mathcal{C}(\bar{F})$ such that $\widetilde{\omega} = \sigma$ on $\delta(F)$ and $\widetilde{\omega} > 0$ on F and define $\omega = ||\widetilde{\omega}||^{-1}\widetilde{\omega} \in \Omega(\bar{F})$. Then, $\omega = k\sigma$ on $\delta(F)$ where $k = \sqrt{\sum\limits_{x \in \delta(F)} \omega^2(x)}$. Consider the conductance $c_\omega = \dfrac{c}{\omega \otimes \omega}$, the corresponding combinatorial Laplacian, $\mathscr{L}^{c_\omega}$, q_ω the associated potential and Λ_{q_ω} the corresponding Dirichlet–to–Robin map. Then, applying the Doob transform, we obtain that $\mathcal{D}_\omega \circ \mathscr{L}^{c_\omega}_{q_\omega} \circ \mathcal{D}_\omega = \mathscr{L}^c$ and moreover $\mathcal{D}_\omega \circ \Lambda_{q_\omega} \circ \mathcal{D}_\omega = \Lambda$, and hence $M - \lambda I$ is the matrix associated with Λ_{q_ω}.

On the other hand, if $M \in \Phi_{\lambda,\sigma}(\pi)$, then Lemma 11 assures that there is a unique critical circular planar network with conductivity c such that $\hat{M} = \Lambda$. In addition, if we choose $\widetilde{\omega}$ the unique solution of the Dirichlet problem $\mathscr{L}^c(u) = 0$ on F and $u = \sigma$ on $\delta(F)$, the minimum principle assures that $\widetilde{\omega} > 0$ on F and hence $\omega = ||\widetilde{\omega}||^{-1}\widetilde{\omega} \in \Omega(\bar{F})$ is a M–harmonic function satisfying that $M = \Lambda_{q_\omega}$.

Consider now $\tau \in \Omega(\bar{F})$ such that it is M–harmonic and $\Lambda_{q_\tau} = M = \Lambda_{q_\omega}$. Then $0 = \Lambda_{q_\tau}(\tau) = \Lambda_{q_\omega}(\tau)$, which implies that $\tau = \alpha\omega$ on $\delta(F)$. Therfore, $\tau = \alpha\omega$ on $\bar{F}$ since τ is M–harmonic. Finally $\alpha = 1$ because τ and ω are weights.

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