

SOME RESULTS FOR CARLITZ'S q -BERNOULLI NUMBERS AND POLYNOMIALS

Yuan He

A further investigation for Carlitz's q -Bernoulli numbers and polynomials is performed, and several new formulae for these numbers and polynomials are established by applying summation transform techniques. Special cases as well as immediate consequences of the main results are also presented.

1. INTRODUCTION

The classical Bernoulli polynomials $B_n(x)$ are usually defined by the following exponential generating function:

$$(1.1) \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi).$$

In particular, the rational numbers $B_n = B_n(0)$ are called the classical Bernoulli numbers. These numbers and polynomials play important roles in many different branches of mathematics including number theory, combinatorics, special functions and analysis. Numerous interesting properties for them can be found in many books; see, for example, [9, 23, 30]).

In the present paper, we consider Carlitz's q -Bernoulli numbers $\beta_n(q)$ and q -Bernoulli polynomials $\beta_n(x, q)$, which are respectively given by means of (see, e.g., [5, 6])

$$(1.2) \quad \beta_0(q) = 1, \quad q(q\beta(q) + 1)^n - \beta_n(q) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

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and

$$(1.3) \quad \beta_n(x, q) = (q^x \beta(q) + [x]_q)^n = \sum_{k=0}^n \binom{n}{k} q^{kx} \beta_k(q) [x]_q^{n-k} \quad (n \geq 0),$$

where $q \in \mathbb{C}$ with $|q| < 1$ and \mathbb{C} being complex number field, and $[x]_q$ is the q -number defined by (see, e.g., [3, 11])

$$(1.4) \quad [x]_q = \frac{1 - q^x}{1 - q} = 1 + q + \dots + q^{x-1}.$$

Obviously, $\beta_n(q) = \beta_n(0, q)$ and $\lim_{q \rightarrow 1} [x]_q = x$.

Since the above Carlitz's q -Bernoulli numbers and q -Bernoulli polynomials first appeared, different properties for them have been well studied by many authors; see, for example, [18, 19, 20, 31, 33]. In fact, Carlitz's q -Bernoulli numbers and polynomials can be defined by the following exponential generating functions (see, e.g., [24, 27]):

$$(1.5) \quad \sum_{m=0}^{+\infty} q^m e^{[m]_q t} (1 - q - q^m t) = \sum_{n=0}^{+\infty} \beta_n(q) \frac{t^n}{n!} \quad (|t + \log q| < 2\pi),$$

and

$$(1.6) \quad \sum_{m=0}^{+\infty} q^m e^{[x+m]_q t} (1 - q - q^{x+m} t) = \sum_{n=0}^{+\infty} \beta_n(x, q) \frac{t^n}{n!} \quad (|t + \log q| < 2\pi).$$

From (1.5) and (1.6), one can easily get

$$(1.7) \quad \lim_{q \rightarrow 1} \beta_n(q) = B_n \quad \text{and} \quad \lim_{q \rightarrow 1} \beta_n(x, q) = B_n(x).$$

If the left-hand side of (1.6) is denoted by $F_q(t, x)$ then the Mellin transform gives

$$(1.8) \quad \frac{1}{\Gamma(s)} \int_0^{+\infty} F_q(-t, x) t^{s-2} dt = \sum_{n=0}^{+\infty} \frac{q^{x+2n}}{[x+n]_q^s} + \frac{1-q}{s-1} \sum_{n=0}^{+\infty} \frac{q^n}{[x+n]_q^{s-1}}$$

with $s \in \mathbb{C}$ and $x \neq 0, -1, -2, \dots$. Based on the observation on (1.8), the q -Hurwitz zeta function can be defined by (see, e.g., [24])

$$(1.9) \quad \zeta_q(s, x) = \sum_{n=0}^{+\infty} \frac{q^n}{[x+n]_q^s} + (1-q) \left(\frac{2-s}{s-1} \right) \sum_{n=0}^{+\infty} \frac{q^n}{[x+n]_q^{s-1}},$$

where $s \in \mathbb{C}$ with $\text{Re}(s) > 1$ and $x \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$. In particular, the case $x = 1$ and $q \rightarrow 1$ in (1.9) respectively, gives the q -zeta function given by SATOH [27] and the classical Hurwitz zeta function $\zeta(s, x)$:

$$(1.10) \quad \zeta(s, x) = \sum_{n=0}^{+\infty} \frac{1}{(n+x)^s} \quad (s \in \mathbb{C}, \text{Re}(s) > 1; x \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}).$$

Recently, CHOI, ANDERSON and SRIVASTAVA [8] systematically studied Carlitz's q -Bernoulli numbers and polynomials, and discovered some interesting properties between Carlitz's q -Bernoulli numbers and polynomials and some related numbers and polynomials and functions. Inspired by their work, in this paper, we perform a further investigation for Carlitz's q -Bernoulli numbers and polynomials, and give some new formulae for these numbers and polynomials by applying summation transform techniques. We determine that various known results including the recent one presented in [4] are derived as special cases.

2. THE STATEMENT OF THE RESULTS

We begin by describing the falling factorial $(x)_k$ of order k and rising factorial $x^{(k)}$ of order k ($x \in \mathbb{C}$ and k non-negative integer):

$$(2.1) \quad (x)_k = x(x-1)(x-2)\dots(x-k+1) \quad (k \geq 1), \quad (x)_0 = 1,$$

and

$$(2.2) \quad x^{(k)} = x(x+1)(x+2)\dots(x+k-1) \quad (k \geq 1), \quad x^{(0)} = 1.$$

We now recall the following addition theorem of Carlitz's q -Bernoulli polynomials (see, e.g., [8]),

$$(2.3) \quad \beta_n(x+y, q) = \sum_{k=0}^n \binom{n}{k} q^{kx} \beta_k(y, q) [x]_q^{n-k} \quad (n \geq 0).$$

Clearly, $\beta_n(y, q) = \beta_n(-x + (x+y), q)$ for non-negative integer n , so from (2.3) we obtain that for non-negative integers m, n ,

$$(2.4) \quad \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} \frac{\beta_{n+k+r}(y, q)}{(n+k)_r} [x]_q^{m-k} = \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} \frac{[x]_q^{m-k}}{(n+k)_r} \\ \times \sum_{i=0}^{n+k+r} \binom{n+k+r}{i} q^{-ix} \beta_i(x+y, q) [-x]_q^{n+k+r-i}.$$

Since $[x]_q = (-q^x)[-x]_q$ then from (2.4) we get

$$(2.5) \quad \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} \frac{\beta_{n+k+r}(y, q)}{(n+k)_r} [x]_q^{m-k} = \sum_{k=0}^m \binom{m}{k} q^{(m+n)x} \frac{(-1)^{m-k}}{(n+k)_r} \\ \times \sum_{i=0}^{n+k+r} \binom{n+k+r}{i} q^{-ix} \beta_i(x+y, q) [-x]_q^{m+n+r-i}.$$

If we change the order of the summations in the right hand side of (2.5) then

$$(2.6) \quad \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} \frac{\beta_{n+k+r}(y, q)}{(n+k)_r} [x]_q^{m-k} \\ = \sum_{i=0}^{m+n+r} q^{(m+n-i)x} \beta_i(x+y, q) [-x]_q^{m+n+r-i} \sum_{k=0}^m \binom{m}{k} \binom{n+k+r}{i} \frac{(-1)^{m-k}}{(n+k)_r}.$$

Observe that for non-negative integers n, k, r ,

$$(2.7) \quad (n+k)_r = (n+k)(n+k-1)\cdots(n+k-r+1) = r! \cdot \binom{n+k}{r}.$$

So from (2.6) and (2.7), we discover

$$(2.8) \quad r! \sum_{k=0}^m \binom{m}{k} \binom{n+k}{r} q^{(n+k)x} \frac{\beta_{n+k+r}(y, q)}{(n+k)_r} [x]_q^{m-k} \\ = \sum_{i=0}^{m+n+r} q^{(m+n-i)x} \beta_i(x+y, q) [-x]_q^{m+n+r-i} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{n+k+r}{i}.$$

Notice that for a complex number s and non-negative integers p, h (cf. the identity of Wu described in [10, 14, 29]),

$$(2.9) \quad \sum_{k=0}^p (-1)^{p+k} \binom{p}{k} \binom{k+h+s}{h} = \operatorname{res}_x (1+x)^{s+h} x^{-h+p-1} = \binom{s+h}{h-p}.$$

Hence, by applying (2.9) to (2.8), we obtain

$$(2.10) \quad \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} \beta_{n+k}(y, q) [x]_q^{m-k} \\ = \sum_{k=0}^n \binom{n}{k} q^{(n-k)x} \beta_{m+k}(x+y, q) [-x]_q^{n-k}.$$

Since Carlitz's q -Bernoulli polynomials satisfy the symmetric distribution (see, e.g., [8])

$$(2.11) \quad \beta_n(1-x, q^{-1}) = (-q)^n \beta_n(x, q) \quad (n \geq 0),$$

so by setting $x+y+z=1$ in (2.10), in view of (2.11) and $[x]_q = (-q^x)[-x]_q$, we immediately get the following result.

Theorem 2.1. *Let m, n be non-negative integers. Then for $x+y+z=1$,*

$$(2.12) \quad (-1)^m \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} \beta_{n+k}(y, q) [x]_q^{m-k} \\ = (-1)^n \sum_{k=0}^n \binom{n}{k} q^{-(m+k)} \beta_{m+k}(z, q^{-1}) [x]_q^{n-k}.$$

It is worth noticing that the case $n = 0$ in the formula (2.10) gives the formula (2.3) and the formula (2.10) can be also derived by applying the generating function methods, see [16]. Theorem 2.1 above can be regarded as the corresponding q -analogue of a result of SUN [32], namely

$$(2.13) \quad (-1)^m \sum_{k=0}^m \binom{m}{k} x^{m-k} B_{n+k}(y) = (-1)^n \sum_{k=0}^n \binom{n}{k} x^{n-k} B_{m+k}(z).$$

If we set $x = 1$ and $y = z = 0$ in Theorem 2.1, we get that for non-negative integers m, n ,

$$(2.14) \quad (-1)^m \sum_{k=0}^m \binom{m}{k} q^{n+k} \beta_{n+k}(q) = (-1)^n \sum_{k=0}^n \binom{n}{k} q^{-(m+k)} \beta_{m+k}(q^{-1}),$$

which is a q -analogue of the familiar formula described in [13]

$$(2.15) \quad (-1)^m \sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^n \binom{n}{k} B_{m+k} \quad (m, n \geq 0).$$

For some similar results on the q -Bernoulli numbers attached to formal group to (2.14), one is referred to [27].

We next give a more general form of Theorem 2.1. In a similar consideration to (2.6), we have

$$(2.16) \quad \begin{aligned} & \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} (n+k)_r \beta_{n+k-r}(y, q) [x]_q^{m-k} \\ &= \sum_{i=0}^{m+n-r} q^{(m+n-i)x} \beta_i(x+y, q) [-x]_q^{m+n-r-i} \\ & \quad \times \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{n+k-r}{i} (n+k)_r, \end{aligned}$$

which together with (2.7) yields

$$(2.17) \quad \begin{aligned} & \sum_{k=0}^m \binom{m}{k} \binom{n+k}{r} q^{(n+k)x} \beta_{n+k-r}(y, q) [x]_q^{m-k} \\ &= \sum_{i=0}^{m+n-r} q^{(m+n-i)x} \beta_i(x+y, q) [-x]_q^{m+n-r-i} \\ & \quad \times \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{n+k-r}{i} \binom{n+k}{r}. \end{aligned}$$

Clearly, $(-m)^{(k)} = (-1)^k m(m-1) \cdots (m-k+1)$ and $(n+k)! = n! \cdot (n+1)^{(k)}$ for non-negative integers k, m , which yields

$$(2.18) \quad \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{n+k-r}{i} \binom{n+k}{r} \\ = \frac{(-1)^m n!}{i! \cdot r! \cdot (n-i-r)!} \sum_{k=0}^m \frac{(-m)^{(k)} (n+1)^{(k)}}{k! \cdot (n+1-i-r)^{(k)}}.$$

Note that for non-negative integer n and complex numbers a, b (cf. the Chu-Vandermonde summation formula stated in [3, 11]),

$$(2.19) \quad \sum_{k=0}^n \frac{(-n)^{(k)} \cdot a^{(k)}}{k! \cdot b^{(k)}} = \frac{(b-a)^{(n)}}{b^{(n)}}.$$

Hence, by applying (2.19) to (2.18), we get

$$(2.20) \quad \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{n+k-r}{i} \binom{n+k}{r} \\ = \frac{n! \cdot (i+r)(i+r-1) \cdots (i+r-m+1)}{i! \cdot r! \cdot (m+n-i-r)!} = \binom{n}{i+r-m} \binom{i+r}{i}.$$

Combining (2.17) and (2.20) gives

$$(2.21) \quad \sum_{k=0}^m \binom{m}{k} \binom{n+k}{r} q^{(n+k)x} \beta_{n+k-r}(y, q) [x]_q^{m-k} \\ = \sum_{k=0}^n \binom{n}{k} \binom{m+k}{r} q^{(n+r-k)x} \beta_{m+k-r}(x+y, q) [-x]_q^{n-k}.$$

If we set $x+y+z=1$ in (2.21), in light of (2.11) and $[x]_q = (-q^x)[-x]_q$, we get

$$(2.22) \quad \sum_{k=0}^m \binom{m}{k} \binom{n+k}{r} q^{(n+k-r)x} \beta_{n+k-r}(y, q) [x]_q^{m-k} \\ = (-1)^{m+n-r} \sum_{k=0}^n \binom{n}{k} \binom{m+k}{r} q^{-(m+k-r)} \beta_{m+k-r}(z, q^{-1}) [x]_q^{n-k}.$$

Thus, by substituting m for $m+r$ and n for $n+r$ in (2.22), we immediately obtain the following result.

Theorem 2.2. *Let m, n, r be non-negative integers. Then for $x+y+z=1$,*

$$(2.23) \quad \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} q^{(n+k)x} \beta_{n+k}(y, q) [x]_q^{m+r-k} \\ = (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} q^{-(m+k)} \beta_{m+k}(z, q^{-1}) [x]_q^{n+r-k}.$$

Clearly, the case $r = 0$ in Theorem 2.2 gives Theorem 2.1. On the other hand, if we set $q \rightarrow 1$ in Theorem 2.2, we get that for non-negative integers m, n, r , (see, e.g., [15])

$$(2.24) \quad \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} B_{n+k}(y) x^{m+r-k} \\ = (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} B_{m+k}(z) x^{n+r-k}.$$

If we set $x = 1$ and $y = z = 0$ in Theorem 2.2, we obtain that for non-negative integers m, n, r ,

$$(2.25) \quad \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} q^{n+k} \beta_{n+k}(q) \\ = (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} q^{-(m+k)} \beta_{m+k}(q^{-1}),$$

which is a q -analogue of a formula on the classical Bernoulli numbers due to AGOH (see, e.g., [1, 25])

$$(2.26) \quad \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} B_{n+k} \\ = (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} B_{m+k} \quad (m, n, r \geq 0).$$

It is worth mentioning that since $B_n = (-1)^n B_n$ for positive integer $n \geq 2$, the case $r = 1$ in (2.26) gives the following result for non-negative integers m, n ,

$$(2.27) \quad (-1)^m \sum_{k=0}^m \binom{m+1}{k} \binom{n+k+r}{r} B_{n+k} \\ + (-1)^n \sum_{k=0}^n \binom{n+1}{k} \binom{m+k+r}{r} B_{m+k} = 0 \quad (m+n \geq 1),$$

which was obtained by MOMIYAMA [21] who made use of p -adic integral over \mathbb{Z}_p and used (2.27) to give a brief proof of the famous Kummer congruence. The case $m = n$ in (2.26), for non-negative integer n and odd integer $r \geq 1$, gives

$$(2.28) \quad \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{n+k+r}{r} B_{n+k} = 0,$$

which can be derived by applying the extended Zeilberger's algorithm (see, e.g., [7]). In particular, the case $r = 1$ in (2.28) was first discovered by KANEKO [17].

We are now in the position to give the corresponding q -analogue of Gessel's formula presented in [4] on the classical Bernoulli numbers. By setting $x = a, y = 0$ and $z = 1 - a$ in Theorem 2.2, we have

$$\begin{aligned}
 (2.29) \quad & \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} q^{(n+k)a} \beta_{n+k}(q) [a]_q^{m+r-k} \\
 &= (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} \\
 & \quad \times q^{-(m+k)} \beta_{m+k}(1-a, q^{-1}) [a]_q^{n+r-k}.
 \end{aligned}$$

Since Carlitz's q -Bernoulli polynomials satisfy the difference equation (see, e.g., [8]):

$$(2.30) \quad q\beta_n(x+1) - \beta_n(x) = nq^x [x]_q^{n-1} + (q-1)[x]_q^n \quad (n \geq 0),$$

then for non-negative integers a, n ,

$$(2.31) \quad \beta_n(1-a, q) = q^{a-1} \beta_n(q) - \sum_{i=1}^{a-1} q^{i-1} \{nq^{i-a} [i-a]_q^{n-1} + (q-1)[i-a]_q^n\}.$$

Hence, in view of $[x]_q = (-q^x)[-x]_q$, the formula (2.31) can be rewritten as

$$(2.32) \quad \beta_n(1-a, q) = q^{a-1} \beta_n(q) + (-1)^n \sum_{i=1}^{a-1} q^{(i-a)n+i-1} \{n[a-i]_q^{n-1} - (q-1)[a-i]_q^n\}.$$

If we apply (2.32) to (2.29) we get

$$\begin{aligned}
 (2.33) \quad & \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} q^{(n+k)a} \beta_{n+k}(q) [a]_q^{m+r-k} \\
 &= (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} q^{1-(m+k+a)} \beta_{m+k}(q^{-1}) [a]_q^{n+r-k} \\
 & \quad + (-1)^{n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} (-1)^k q^{-(m+k)} \sum_{i=1}^{a-1} q^{(a-i)(m+k)-i+1} \\
 & \quad \times \{(m+k)[a-i]_{q^{-1}}^{m+k-1} - (q^{-1}-1)[a-i]_{q^{-1}}^{m+k}\} [a]_q^{n+r-k}.
 \end{aligned}$$

Note that $[x]_q = q^{x-1}[x]_{q^{-1}}$ and $[x+y]_q = [x]_q + q^x[y]_q$, then

$$[a]_q^{n+r-k} = q^{(a-1)(n+r-k)} ([i]_{q^{-1}} + q^{-i}[a-i]_{q^{-1}})^{n+r-k}$$

$$\begin{aligned}
&= q^{(a-1)(n+r-k)} \sum_{j=0}^{n+r-k} \binom{n+r-k}{j} [i]_{q^{-1}}^j (q^{-i}[a-i]_{q^{-1}})^{n+r-k-j} \\
(2.34) \quad &= q^{(a-1)(n+r-k)} \sum_{j=1-n}^{r+1-k} \binom{n+r-k}{n+j-1} [i]_{q^{-1}}^{n+j-1} (q^{-i}[a-i]_{q^{-1}})^{r+1-k-j}.
\end{aligned}$$

By applying (2.34) to the second sum of the right hand side of (2.33) and changing the order of the summation, we obtain

$$\begin{aligned}
(2.35) \quad &\sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} q^{(n+k)a} \beta_{n+k}(q) [a]_q^{m+r-k} \\
&= (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} q^{1-(m+k+a)} \beta_{m+k}(q^{-1}) [a]_q^{n+r-k} \\
&\quad + (-1)^{n+r} \sum_{j=1-n}^{r+1} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} \binom{n+r-k}{n+j-1} (-1)^k \\
&\quad \times \sum_{i=1}^{a-1} q^{(a-1)(m+n+r)-i(m+r+2-j)+1} \{ (m+k) [i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+r-j} \\
&\quad \quad - (q^{-1}-1) [i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+r+1-j} \}.
\end{aligned}$$

Observe that for $1-n \leq j \leq r+1$,

$$\begin{aligned}
(2.36) \quad &\sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} \binom{n+r-k}{n+j-1} (-1)^k (m+k) \\
&= \frac{(r+1) \cdot (n+r)! \cdot (m+r)!}{(m-1)! \cdot (r+1)! \cdot (n+j-1)! \cdot (r+1-j)!} \\
&\quad \times \sum_{k=0}^{n+r} \frac{(-(r+1-j))^{(k)} \cdot (m+r+1)^{(k)}}{k! \cdot m^{(k)}},
\end{aligned}$$

which together with (2.19) yields that for $1-n \leq j \leq r+1$,

$$\begin{aligned}
(2.37) \quad &\sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} \binom{n+r-k}{n+j-1} (-1)^k (m+k) \\
&= \frac{(r+1) \cdot (n+r)! \cdot (m+r)!}{(r+1)! \cdot (n+j-1)! \cdot (r+1-j)!} \cdot \frac{(-1)^{r+1-j} (r+1)r \cdots (j+1)}{(m+r-j)!}.
\end{aligned}$$

In the same way, for $1-n \leq j \leq r+1$, we have

$$(2.38) \quad \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} \binom{n+r-k}{n+j-1} (-1)^k$$

$$= \frac{(n+r)! \cdot (m+r)!}{r! \cdot (n+j-1)! \cdot (r+1-j)!} \cdot \frac{(-1)^{r+1-j} r(r-1) \cdots j}{(m+r+1-j)!}.$$

Thus, combining (2.35), (2.37) and (2.38) gives the following result.

Theorem 2.3. *Let m, n, r, a be non-negative integers. Then*

$$\begin{aligned} (2.39) \quad & \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} q^{(n+k)a} \beta_{n+k}(q) [a]_q^{m+r-k} \\ &= (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} q^{1-(m+k+a)} \beta_{m+k}(q^{-1}) [a]_q^{n+r-k} \\ & \quad + \sum_{i=1}^{a-1} q^{(a-1)(m+n+r)-i(m+r+2)+1} \\ & \quad \times \left\{ (r+1) \sum_{j=0}^{r+1} \binom{m+r}{j} \binom{n+r}{r+1-j} q^{ij} [i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+r-j} \right. \\ & \quad \left. - (q^{-1}-1) \sum_{j=1}^{r+1} \binom{m+r}{j-1} \binom{n+r}{r+1-j} q^{ij} [i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+r+1-j} \right\}. \end{aligned}$$

It becomes obvious that the Theorem 2.3 can be regarded as a generalization of the formula (2.25). And the case $q \rightarrow 1$ in Theorem 2.3 gives that for non-negative integers m, n, r, a ,

$$\begin{aligned} (2.40) \quad & \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} B_{n+k} a^{m+r-k} \\ & \quad + (-1)^{m+n+r-1} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} B_{m+k} a^{n+r-k} \\ & \quad = (r+1) \sum_{i=1}^{a-1} \sum_{j=0}^{r+1} \binom{m+r}{j} \binom{n+r}{r+1-j} i^{n+j-1} (a-i)^{m+r-j}, \end{aligned}$$

which was discovered by GESSEL [4] who made use of the methods presented in [13].

We next give another generalization of Theorem 2.1. Similarly to (2.6), we have

$$\begin{aligned} (2.41) \quad & \sum_{k=0}^m \binom{m}{k} q^{(n+k)x} (n+k+1)^{(r)} \beta_{n+k+r}(y, q) [x]_q^{m-k} \\ & \quad = \sum_{i=0}^{m+n+r} q^{(m+n-i)x} \beta_i(x+y, q) [-x]_q^{m+n+r-i} \end{aligned}$$

$$\times \sum_{k=0}^m \binom{m}{k} \binom{n+k+r}{i} (-1)^{m-k} (n+k+1)^{(r)}.$$

Observe that for non-negative integers n, k, r ,

$$(2.42) \quad (n+k+1)^{(r)} = (n+k+1) \cdots (n+k+r) = \frac{1}{r! \cdot \binom{n+k+r}{r}},$$

which together with (2.41) yields

$$(2.43) \quad \begin{aligned} & \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+k+r}{r}} q^{(n+k)x} \beta_{n+k+r}(y, q) [x]_q^{m-k} \\ &= \sum_{i=0}^{m+n+r} q^{(m+n-i)x} \beta_i(x+y, q) [-x]_q^{m+n+r-i} \\ & \quad \times \sum_{k=0}^m \binom{m}{k} \binom{n+k+r}{i} \frac{(-1)^{m-k}}{\binom{n+k+r}{r}}. \end{aligned}$$

Hence, in light of (2.19), we obtain

$$(2.44) \quad \begin{aligned} & \sum_{k=0}^m \binom{m}{k} \binom{n+k+r}{i} \frac{(-1)^{m-k}}{\binom{n+k+r}{r}} \\ &= \frac{(-1)^m n! \cdot r!}{i! \cdot (n+r-i)!} \sum_{k=0}^m \frac{(-m)^{(k)} \cdot (n+1)^{(k)}}{(n+r+1-i)^{(k)}} \\ &= \frac{(-1)^m n! \cdot r! \cdot (r-i)(r-i+1) \cdots (r-i+m-1)}{i! \cdot (m+n+r-i)!}. \end{aligned}$$

Combining (2.43) and (2.44) gives

$$(2.45) \quad \begin{aligned} & \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+k+r}{r}} q^{(n+k)x} \beta_{n+k+r}(y, q) [x]_q^{m-k} \\ &= \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{m+k+r}{r}} q^{(n-r-k)x} \beta_{m+k+r}(x+y, q) [-x]_q^{n-k} + (-1)^m r q^{(m+n+1-r)x} \\ & \quad \times [-x]_q^{m+n+1} \sum_{i=0}^{r-1} \frac{\binom{r-1}{i}}{\binom{m+n+i}{n}} q^{ix} \beta_{r-1-i}(x+y, q) \frac{[-x]_q^i}{m+n+i+1}. \end{aligned}$$

Thus, by setting $x + y + z = 1$ in (2.45), in light of (2.11) and $[x]_q = (-q^x)[-x]_q$, we state the following result.

Theorem 2.4. *Let m, n, r be non-negative integers. Then for $x + y + z = 1$,*

$$\begin{aligned}
 (2.46) \quad & (-1)^m \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+k+r}{r}} q^{(n+k+r)x} \beta_{n+k+r}(y, q) [x]_q^{m-k} \\
 &= (-1)^{n+r} \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{m+k+r}{r}} q^{-(m+k+r)} \beta_{m+k+r}(z, q^{-1}) [x]_q^{n-k} \\
 &\quad + (-1)^{m+n+r} r [x]_q^{m+n+1} \sum_{i=0}^{r-1} \frac{\binom{r-1}{i}}{\binom{m+n+r-1-i}{n}} q^{-i} \beta_i(z, q^{-1}) \frac{[x]_q^{r-1-i}}{m+n+r-i}.
 \end{aligned}$$

We next discuss some special cases of Theorem 2.4. Clearly, the case $r = 0$ in Theorem 2.4 gives the Theorem 2.1. If we set $r = 1$ in Theorem 2.4, we obtain that for non-negative integers m, n ,

$$\begin{aligned}
 (2.47) \quad & (-1)^m \sum_{k=0}^m \binom{m}{k} q^{(n+k+1)x} \frac{\beta_{n+k+1}(y, q)}{n+k+1} [x]_q^{m-k} \\
 &\quad + (-1)^n \sum_{k=0}^n \binom{n}{k} q^{-(m+k+1)} \frac{\beta_{m+k+1}(z, q^{-1})}{m+k+1} [x]_q^{n-k} = \frac{(-[x]_q)^{m+n+1}}{(m+n+1) \binom{m+n}{m}},
 \end{aligned}$$

which is a q -analogue of Sun's formula on the classical Bernoulli polynomials (see, e.g., [7, 15, 32])

$$\begin{aligned}
 (2.48) \quad & (-1)^m \sum_{k=0}^m \binom{m}{k} x^{m-k} \frac{B_{n+k+1}(y)}{n+k+1} + (-1)^n \sum_{k=0}^n \binom{n}{k} x^{n-k} \frac{B_{m+k+1}(z)}{m+k+1} \\
 &= \frac{(-x)^{m+n+1}}{(m+n+1) \binom{m+n}{m}} \quad (m, n \geq 0).
 \end{aligned}$$

In fact, the formula (2.47) has other applications. For example, since Carlitz's q -Bernoulli polynomials can be expressed by the closed formula (see, e.g., [8]):

$$(2.49) \quad \beta_n(x, q) = \frac{1}{(1-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kx} \frac{k+1}{[k+1]_q} \quad (n \geq 0),$$

by applying the derivative operation $\partial/\partial x$ to both sides of (2.49), with the help of

$$(2.50) \quad k \binom{n}{k} = n \binom{n-1}{k-1} = n \left\{ \binom{n}{k} - \binom{n-1}{k} \right\} \quad (k, n \geq 0),$$

one can easily derive that for non-negative integer n ,

$$(2.51) \quad \frac{\partial}{\partial x} \beta_n(x, q) = \ln q \left(n \beta_n(x, q) - \frac{n}{1-q} \beta_{n-1}(x, q) \right).$$

Hence, replacing z with $1 - x - y$ and applying the derivative operation $\partial/\partial y$ to both sides of (2.47), in view of (2.51), we obtain that for non-negative integers m, n ,

$$(2.52) \quad \begin{aligned} & (-1)^m \sum_{k=0}^m \binom{m}{k} q^{(n+k+1)x} [x]_q^{m-k} \{ \beta_{n+k}(y, q) + (q-1) \beta_{n+k+1}(y, q) \} \\ &= (-1)^n \sum_{k=0}^n \binom{n}{k} q^{-(m+k+1)} [x]_q^{n-k} \{ -q \beta_{m+k}(z, q^{-1}) \\ & \quad + (q-1) \beta_{m+k+1}(z, q^{-1}) \} \quad (x + y + z = 1), \end{aligned}$$

which is another q -analogue of Sun's formula (2.13). On the other hand, if we set $x = a, y = 0, z = 1 - a$ in (2.47), by (2.32) and (2.34) we get

$$(2.53) \quad \begin{aligned} & (-1)^m \sum_{k=0}^m \binom{m}{k} q^{(n+k+1)a} \frac{\beta_{n+k+1}(q)}{n+k+1} [a]_q^{m-k} \\ &+ (-1)^n \sum_{k=0}^n \binom{n}{k} q^{-(m+k+a)} \frac{\beta_{m+k+1}(q^{-1})}{m+k+1} [a]_q^{n-k} \\ &+ (-1)^{m+n+1} \sum_{j=1-n}^1 \sum_{k=0}^n \binom{n}{k} \binom{n-k}{n+j-1} \frac{(-1)^k}{m+k+1} \\ &\times \sum_{i=1}^{a-1} q^{(a-1)(m+n+1)-i(m+3-j)+1} \{ (m+k+1) [i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+1-j} \\ & \quad - (q^{-1}-1) [i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+2-j} \} = \frac{(-1)^{m+n+1} \cdot m! \cdot n!}{(m+n+1)!} [a]_q^{m+n+1}. \end{aligned}$$

Note that from (2.38) we have

$$(2.54) \quad \sum_{k=0}^n \binom{n}{k} \binom{n-k}{n+j-1} (-1)^k = \begin{cases} 1, & j = 1, \\ 0, & 1-n \leq j \leq 0, \end{cases}$$

and from (2.19), we obtain that for $1-n \leq j \leq 1$,

$$(2.55) \quad \begin{aligned} & \sum_{k=0}^n \binom{n}{k} \binom{n-k}{n+j-1} \frac{(-1)^k}{m+k+1} = \frac{m! \cdot n!}{(m+1)! \cdot (n+j-1)! \cdot (1-j)!} \\ & \times \sum_{k=0}^n \frac{(-(1-j))^{(k)} \cdot (m+1)^{(k)}}{k! \cdot (m+2)^{(k)}} = \frac{m! \cdot n!}{(n+j-1)! \cdot (m+2-j)!}. \end{aligned}$$

Hence, combining (2.53), (2.54) and (2.55) gives the following result.

Theorem 2.5. *Let m, n, a be non-negative integers. Then*

$$\begin{aligned}
 (2.56) \quad & (-1)^{n+1} \sum_{k=0}^m \binom{m}{k} q^{(n+k+1)a} \frac{\beta_{n+k+1}(q)}{n+k+1} [a]_q^{m-k} \\
 & + (-1)^{m+1} \sum_{k=0}^n \binom{n}{k} q^{-(m+k+a)} \frac{\beta_{m+k+1}(q^{-1})}{m+k+1} [a]_q^{n-k} \\
 & = \frac{m! \cdot n!}{(m+n+1)!} [a]_q^{m+n+1} - \sum_{i=1}^{a-1} q^{(a-1)(m+n+1)-i(m+2)+1} [i]_{q^{-1}}^{n-1} \cdot [a-i]_{q^{-1}}^m \\
 & \quad + (q^{-1} - 1) \cdot m! \cdot n! \sum_{i=1}^{a-1} q^{(a-1)(m+n+1)-i(m+3)+1} \\
 & \quad \times \sum_{j=1-n}^1 \frac{q^{ij}}{(n+j-1)! \cdot (m+2-j)!} [i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+2-j}.
 \end{aligned}$$

It is obvious that the case $a = 1$ in Theorem 2.5 gives that for non-negative integers m, n ,

$$\begin{aligned}
 (2.57) \quad & (-1)^{n+1} \sum_{k=0}^m \binom{m}{k} q^{n+k+1} \frac{\beta_{n+k+1}(q)}{n+k+1} \\
 & + (-1)^{m+1} \sum_{k=0}^n \binom{n}{k} q^{-(m+k+1)} \frac{\beta_{m+k+1}(q^{-1})}{m+k+1} = \frac{m! \cdot n!}{(m+n+1)!},
 \end{aligned}$$

which is a q -analogue of a formula of SAALSCHÜTZ [26], later rediscovered by GELFAND [12], namely

$$\begin{aligned}
 (2.58) \quad & (-1)^{n+1} \sum_{k=0}^m \binom{m}{k} \frac{B_{n+k+1}}{n+k+1} + (-1)^{m+1} \sum_{k=0}^n \binom{n}{k} \frac{B_{m+k+1}}{m+k+1} \\
 & = \frac{m! \cdot n!}{(m+n+1)!} \quad (m, n \geq 0).
 \end{aligned}$$

And the case $q \rightarrow 1$ in Theorem 2.5 gives that for non-negative integers m, n, a ,

$$\begin{aligned}
 (2.59) \quad & (-1)^{n+1} \sum_{k=0}^m \binom{m}{k} \frac{B_{n+k+1}}{n+k+1} a^{m-k} + (-1)^{m+1} \sum_{k=0}^n \binom{n}{k} \frac{B_{m+k+1}}{m+k+1} a^{n-k} \\
 & = \frac{m! \cdot n!}{(m+n+1)!} a^{m+n+1} - \sum_{i=1}^{a-1} i^n (a-i)^m.
 \end{aligned}$$

which was considered by NEUMAN and SCHONBACH [22] from the point of view of numerical analysis. See also [2] for a different proof and a detailed introduction to the formula (2.59).

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Faculty of Science,
Kunming University of Science and Technology,
Kunming, Yunnan 650500
People's Republic of China
E-mail: hyyhe@aliyun.com, hyyhe@yahoo.com.cn

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