Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **7** (2013), 302–313. doi:10.2298/AADM130521009F

ON THE METRIC DIMENSION AND FRACTIONAL METRIC DIMENSION OF THE HIERARCHICAL PRODUCT OF GRAPHS

Min Feng, Kaishun Wang

A set of vertices W *resolves* a graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in W. The *metric dimension* for G, denoted by $\dim(G)$, is the minimum cardinality of a resolving set of G. In order to study the metric dimension for the hierarchical product $G_2^{u_2} \sqcap G_1^{u_1}$ of two rooted graphs $G_2^{u_2}$ and $G_1^{u_1}$, we first introduce a new parameter, the *rooted metric dimension* $\mathrm{rdim}(G_1^{u_1})$ for a rooted graph $G_1^{u_1}$. If G_1 is not a path with an end-vertex u_1, we show that $\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \mathrm{rdim}(G_1^{u_1})$, where $|V(G_2)|$ is the order of G_2. If G_1 is a path with an end-vertex u_1, we obtain some tight inequalities for $\dim(G_2^{u_2} \sqcap G_1^{u_1})$. Finally, we show that similar results hold for the fractional metric dimension.

1. INTRODUCTION

All graphs considered in this paper are nontrivial and connected. For a graph G, we often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For any two vertices u and v of G, denote by $d_G(u,v)$ the distance between u and v in G, and write $R_G\{u,v\} = \{w \mid w \in V(G), d_G(u,w) \neq d_G(v,w)\}$. If the graph G is clear from the context, the notations $d_G(u,v)$ and $R_G\{u,v\}$ will be written $d(u,v)$ and $R\{u,v\}$, respectively. A subset W of $V(G)$ is a *resolving set* of G if $W \cap R\{u,v\} \neq \emptyset$ for any two distinct vertices u and v. A *metric basis* of G is a resolving set of G with minimum cardinality. The cardinality of a metric basis of G is the *metric dimension* for G, denoted by $\dim(G)$.

Metric dimension was introduced independently by Harary and Melter [**15**], and by Slater [**24**]. As a graph parameter it has numerous applications,

2010 Mathematics Subject Classification. 05C12.
Keywords and Phrases. Resolving set, metric dimension, resolving function, fractional metric dimension, hierarchical product, binomial tree.

among them are computer science and robotics [**18**], network discovery and verification [**5**], strategies for the Mastermind game [**8**] and combinatorial optimization [**23**]. Metric dimension has been heavily studied, see [**3**] for a number of references on this topic.

The problem of finding the metric dimension for a graph was formulated as an integer programming problem independently by Chartrand et al. [**7**], and by Currie and Oellermann [**10**]. In graph theory, fractionalization of integer-valued graph theoretic concepts is an interesting area of research (see [**22**]). Currie and Oellermann [**10**] and Fehr et al. [**11**] defined fractional metric dimension as the optimal solution of the linear relaxation of the integer programming problem. Arumugam and Mathew [**1**] initiated the study of the fractional metric dimension for graphs. For more information, see [**2, 12, 13**].

Let $g : V(G) \longrightarrow [0,1]$ be a real function. For $W \subseteq V(G)$, denote $g(W) = \sum_{v \in W} g(v)$. The *weight* of g is defined by $|g| = g(V(G))$. We call g a *resolving function* of G if $g(R\{u,v\}) \geq 1$ for any two distinct vertices u and v. The minimum weight of a resolving function of G is called the *fractional metric dimension* for G, denoted by $\dim_f(G)$.

It was noted in [**14**, p.204] and [**18**] that determining the metric dimension for a graph is an NP-complete problem. So it is desirable to reduce the computation for the metric dimension for product graphs to the computation for some parameters of the factor graphs; see [**6**] for cartesian products, [**16**] for lexicographic products, and [**25**] for corona products. Recently, the fractional metric dimension for the above three products was studied in [**2, 12, 13**].

In order to model some real-life complex networks, Barrière et al. [**4**] introduced the hierarchical product of graphs and showed that it is associative. A *rooted graph* G^u is the graph G in which one vertex u, called *root vertex*, is labeled in a special way to distinguish it from other vertices. Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. The *hierarchical product* $G_2^{u_2} \sqcap G_1^{u_1}$ is the rooted graph with the vertex set $\{x_2x_1 \mid x_i \in V(G_i), i = 1, 2\}$, having the root vertex u_2u_1, where x_2x_1 is adjacent to y_2y_1 whenever $x_2 = y_2$ and $\{x_1, y_1\} \in E(G_1)$, or $x_1 = y_1 = u_1$ and $\{x_2, y_2\} \in E(G_2)$. See [**17, 19, 20, 21**] for more information.

In this paper, we study the (fractional) metric dimension for the hierarchical product $G_2^{u_2} \sqcap G_1^{u_1}$ of rooted graphs $G_2^{u_2}$ and $G_1^{u_1}$. In Section 2, we introduce a new parameter, the rooted metric dimension $\mathrm{rdim}(G^u)$ for a rooted graph G^u. If G_1 is not a path with an end-vertex u_1, we show that $\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \mathrm{rdim}(G_1^{u_1})$. If G_1 is a path with an end-vertex u_1, we obtain some tight inequalities for $\dim(G_2^{u_2} \sqcap G_1^{u_1})$. In Section 3, we show that similar results hold for the fractional metric dimension.

2. METRIC DIMENSION

In order to study the metric dimension for the hierarchical product of graphs, we first introduce the rooted metric dimension for a rooted graph.

A *rooted resolving set* of a rooted graph G^u is a subset W of $V(G)$ such that $W \cup \{u\}$ is a resolving set of G. A *rooted metric basis* of G^u is a rooted resolving set of G^u with the minimum cardinality. Here the cardinality of a rooted metric basis of G^u is called *rooted metric dimension* of G^u and denoted by $\mathrm{rdim}(G^u)$.

The following observation is obvious.

Observation 2.1. *If there exists a metric basis of G containing u, then* $\mathrm{rdim}(G^u) = \dim(G) - 1$. *If any metric basis of G does not contain u, then* $\mathrm{rdim}(G^u) = \dim(G)$.

For graphs H_1 and H_2 we use $H_1 \cup H_2$ to denote the disjoint union of H_1 and H_2 and $H_1 + H_2$ to denote the graph obtained from the disjoint union of H_1 and H_2 by joining every vertex of H_1 with every vertex of H_2.

Observation 2.2. ([**7**]) *Let G be a graph of order n. Then* $1 \leq \dim(G) \leq n - 1$. *Moreover,*

(i) $\dim(G) = 1$ *if and only if G is the path P_n of length n.*

(ii) $\dim(G) = n - 1$ *if and only if G is the complete graph K_n on n vertices.*

Proposition 2.3. ([**7**, Theorem 4]) *Let G be a graph of order $n \geq 4$. Then* $\dim(G) = n - 2$ *if and only if* $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$, *or* $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$, *where $\overline{K_t}$ is a null graph and $K_{s,t}$ is a complete bipartite graph.*

Proposition 2.4. *Let G^u be a rooted graph of order n. Then* $0 \leq \mathrm{rdim}(G^u) \leq n-2$. *Moreover,*

(i) $\mathrm{rdim}(G^u) = 0$ *if and only if $G = P_n$ and u is one of its end-vertices.*

(ii) $\mathrm{rdim}(G^u) = n - 2$ *if and only if $G = K_n$, or $G = K_{1,n-1}$ and u is the centre.*

Proof. If G is a complete graph, by Observation 2.2 (ii) we have $\dim(G) = n-1$, so Observation 2.1 implies that $\mathrm{rdim}(G^u) = n - 2$. If G is not a complete graph, then $1 \leq \dim(G) \leq n - 2$, which implies that $0 \leq \mathrm{rdim}(G^u) \leq n - 2$ by Observation 2.1.

(i) Since $\mathrm{rdim}(G^u) = 0$ if and only if $\{u\}$ is a metric basis of G, by Observation 2.2 (i), (i) holds.

(ii) Suppose that $\mathrm{rdim}(G^u) = n - 2$. Then $\dim(G) = n - 1$ or $n - 2$. If $\dim(G) = n - 1$, then $G = K_n$. Now we consider $\dim(G) = n - 2$. If $n = 3$, then $\dim(G) = 1$, which implies that $G = K_{1,2}$ and u is the centre. Now suppose that $n \geq 4$. Then G is one of graphs listed in Proposition 2.3. If $s, t \geq 2$ or $G = K_s + (K_1 \cup K_t)$, then there exists a metric basis containing u, implying that $\mathrm{rdim}(G^u) = n-3$, which is a contradiction. Hence $G = K_{1,n-1}$. Since any metric basis of $K_{1,n-1}$ does not contain the centre, the vertex u is the centre of $K_{1,n-1}$. The converse is routine. □

Next, we express the metric dimension for the hierarchical product $G_2^{u_2} \sqcap G_1^{u_1}$ in terms of $\mathrm{rdim}(G_1^{u_1})$.

Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. For any two vertices x_2x_1 and y_2y_1

of $G_2^{u_2} \sqcap G_1^{u_1}$, observe that

$$d(x_2x_1, y_2y_1) = \begin{cases} d_{G_1}(x_1, y_1), & \text{if } x_2 = y_2, \\ d_{G_2}(x_2, y_2) + d_{G_1}(x_1, u_1) + d_{G_1}(y_1, u_1), & \text{if } x_2 \neq y_2. \end{cases} \tag{1}$$

Lemma 2.5. *Let x_2x_1 and y_2y_1 be two distinct vertices of $G_2^{u_2} \sqcap G_1^{u_1}$.*

(i) *If $x_2 = y_2$, then*

$$R\{x_2x_1, y_2y_1\} = \begin{cases} \{x_2z \mid z \in R_{G_1}\{x_1, y_1\}\}, & \text{if } u_1 \notin R_{G_1}\{x_1, y_1\}, \\ V(G_2^{u_2} \sqcap G_1^{u_1}) \setminus \{x_2z \mid z \notin R_{G_1}\{x_1, y_1\}\}, & \text{if } u_1 \in R_{G_1}\{x_1, y_1\}. \end{cases}$$

(ii) *If $x_2 \neq y_2$, then $\{x_2z, y_2z\} \cap R\{x_2x_1, y_2y_1\} \neq \emptyset$ for any $z \in V(G_1)$.*

Proof. (i) If $u_1 \notin R_{G_1}\{x_1, y_1\}$, then $d_{G_1}(x_1, u_1) = d_{G_1}(y_1, u_1)$. By (1), the inequality $d(x_2x_1, z_2z_1) \neq d(y_2y_1, z_2z_1)$ holds if and only if $z_2 = x_2$ and $d_{G_1}(x_1, z_1) \neq d_{G_1}(y_1, z_1)$. It follows that $R\{x_2x_1, y_2y_1\} = \{x_2z \mid z \in R_{G_1}\{x_1, y_1\}\}$. If $u_1 \in R_{G_1}\{x_1, y_1\}$, then $d_{G_1}(x_1, u_1) \neq d_{G_1}(y_1, u_1)$. By (1), the equality $d(x_2x_1, z_2z_1) = d(y_2y_1, z_2z_1)$ holds if and only if $z_2 = x_2$ and $d_{G_1}(x_1, z_1) = d_{G_1}(y_1, z_1)$. It follows that $R\{x_2x_1, y_2y_1\} = V(G_2^{u_2} \sqcap G_1^{u_1}) \setminus \{x_2z \mid z \notin R_{G_1}\{x_1, y_1\}\}$.

(ii) Suppose that $x_2z \notin R\{x_2x_1, y_2y_1\}$. Then $d(x_2x_1, x_2z) = d(y_2y_1, x_2z)$. By (1),

$$d_{G_1}(x_1, z) = d_{G_2}(y_2, x_2) + d_{G_1}(y_1, u_1) + d_{G_1}(z, u_1) \geq d_{G_2}(x_2, y_2) + d_{G_1}(y_1, z),$$

which implies that

$$d_{G_2}(x_2, y_2) + d_{G_1}(x_1, u_1) + d_{G_1}(z, u_1) \geq 2d_{G_2}(x_2, y_2) + d_{G_1}(y_1, z) > d(y_2y_1, y_2z).$$

Hence, $y_2z \in R\{x_2x_1, y_2y_1\}$, as desired. $\square$

Lemma 2.6. *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. Then*

$$\operatorname{rdim}(G_2^{u_2} \sqcap G_1^{u_1}) \geq |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1}).$$

Proof. Let $\overline{W}$ be a rooted metric basis of $G_2^{u_2} \sqcap G_1^{u_1}$. For $v \in V(G_2)$, write $\overline{W}_v = \{z \mid vz \in \overline{W}\}$. For any two distinct vertices x, y of G_1, there exists a vertex wz in $\overline{W} \cup \{u_2u_1\}$ such that $d(vx, wz) \neq d(vy, wz)$. If $w = v$, by (1) we get $d_{G_1}(x, z) \neq d_{G_1}(y, z)$, which implies that $z \in (\overline{W}_v \cup \{u_1\}) \cap R_{G_1}\{x, y\}$. If $w \neq v$, by (1) we have $d_{G_1}(x, u_1) \neq d_{G_1}(y, u_1)$, which implies that $u_1 \in R_{G_1}\{x, y\}$. Therefore, we have $(\overline{W}_v \cup \{u_1\}) \cap R_{G_1}\{x, y\} \neq \emptyset$, which implies that $\overline{W}_v$ is a rooted resolving set of $G_1^{u_1}$. Since $\overline{W}$ is the disjoint union of all $\overline{W}_v$'s, one gets

$$\operatorname{rdim}(G_2^{u_2} \sqcap G_1^{u_1}) = |\overline{W}| = \sum_{v \in V(G_2)} |\overline{W}_v| \geq |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1}),$$

as desired. $\square$

Theorem 2.7. *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. If G_1 is not a path with an end-vertex u_1, then*

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \mathrm{rdim}(G_1^{u_1}).$$

Proof. By Lemma 2.6, we only need to prove that

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) \leq |V(G_2)| \cdot \mathrm{rdim}(G_1^{u_1}). \tag{2}$$

Let W be a rooted metric basis of $G_1^{u_1}$. Then $W \neq \emptyset$. Write $\overline{W} = \{vw \mid v \in V(G_2), w \in W\}$. Note that $|\overline{W}| = |V(G_2)| \cdot \mathrm{rdim}(G_1^{u_1})$. In order to prove (2), we only need to show that $\overline{W}$ is a resolving set of $G_2^{u_2} \sqcap G_1^{u_1}$. It suffices to show that, for any two distinct vertices x_2x_1 and y_2y_1 of $G_2^{u_2} \sqcap G_1^{u_1}$,

$$\overline{W} \cap R\{x_2x_1, y_2y_1\} \neq \emptyset. \tag{3}$$

If $x_2 = y_2$ and $u_1 \notin R_{G_1}\{x_1, y_1\}$, then $W \cap R_{G_1}\{x_1, y_1\} \neq \emptyset$, by Lemma 2.5 (i) we obtain (3). If $x_2 = y_2$ and $u_1 \in R_{G_1}\{x_1, y_1\}$, by Lemma 2.5 (i) we have $vw \in \overline{W} \cap R\{x_2x_1, y_2y_1\}$ for any $v \neq x_2$ and any $w \in W$, which implies that (3) holds. If $x_2 \neq y_2$, since $\{x_2w, y_2w\} \subseteq \overline{W}$ for any $w \in W$, the inequality (3) holds by Lemma 2.5 (ii). □

Combining Observation 2.1 and Theorem 2.7, we have the following result.

Corollary 2.8. *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs.*

(i) *If there exists a metric basis of G_1 containing u_1 and G_1 is not a path, then*

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)|(\dim(G_1) - 1).$$

(ii) *If any metric basis of G_1 does not contain u_1, then*

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \dim(G_1).$$

The *binomial tree T_n* is the hierarchical product of n copies of the complete graph on two vertices, which is a useful data structure in the context of algorithm analysis and designs [**9**]. It was proved that the metric dimension for a tree can be expressed in terms of its parameters in [**7, 15, 24**].

Corollary 2.9. *Let $n \geq 2$. Then $\dim(T_n) = 2^{n-2}$.*

Proof. Note that $\dim(T_2) = 1$. Now suppose $n \geq 3$. Since $T_n = (K_2^0 \sqcap \cdots \sqcap K_2^0) \sqcap (K_2^0 \sqcap K_2^0)$ and $\mathrm{rdim}(K_2^0 \sqcap K_2^0) = 1$, the desired result follows by Theorem 2.7. □

We always assume that 0 is one end-vertex of P_n. In the remaining of this section, we prove some tight inequalities for $\dim(G^u \sqcap P_n^0)$.

Proposition 2.10. *Let G^u be a rooted graph with diameter d. Then*

$$\dim(G^u \sqcap P_n^0) \leq \dim(G^u \sqcap P_{n+1}^0) \text{ for } 1 \leq n \leq d-1, \tag{4}$$

$$\dim(G^u \sqcap P_n^0) = \dim(G^u \sqcap P_{n+1}^0) \text{ for } n \geq d. \tag{5}$$

Proof. If $G = K_2$, then $G^u \sqcap P_n^0$ is the path, which implies that (5) holds. Now we only consider $|V(G)| \geq 3$. Suppose that $\overline{W}_{n+1}$ is a metric basis of $G^u \sqcap P_{n+1}^0$. Let $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$. Define $\pi_n : V(G^u \sqcap P_{n+1}^0) \longrightarrow V(G^u \sqcap P_n^0)$ by

$$\pi_n(vz_i) = \begin{cases} vz_{n-1}, & \text{if } i = n, \\ vz_i, & \text{if } i \leq n-1. \end{cases}$$

Then $\pi_n(\overline{W}_{n+1})$ is a resolving set of $G^u \sqcap P_n^0$, which implies that $\dim(G^u \sqcap P_n^0) \leq \dim(G^u \sqcap P_{n+1}^0)$ for any positive integer n. So (4) holds.

In order to prove (5), we only need to show that $\overline{W}_n$ is a resolving set of $G^u \sqcap P_{n+1}^0$ for $n \geq d$. Pick any two distinct vertices v_1z_i and v_2z_j of $G^u \sqcap P_{n+1}^0$. It suffices to prove that

$$\overline{W}_n \cap R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_2z_j\} \neq \emptyset. \tag{6}$$

Without loss of generality, we may assume that $0 \leq i \leq j \leq n$. If $j \leq n-1$, then $R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_2z_j\} \supseteq R_{G^u \sqcap P_n^0}\{v_1z_i, v_2z_j\}$; and so (6) holds. Now suppose $j = n$.

Claim. There exist two distinct vertices w_1 and w_2 of G such that

$$\overline{W}_n \cap \{w_1z_k \mid 0 \leq k \leq n-1\} \neq \emptyset \text{ and } \overline{W}_n \cap \{w_2z_k \mid 0 \leq k \leq n-1\} \neq \emptyset. \tag{7}$$

Suppose for the contradiction that there exists a vertex $w \in V(G)$ such that $\overline{W}_n \subseteq \{wz_k \mid 0 \leq k \leq n-1\}$. If the degree of w in G is one, then there exists an induced path (w, x, y) in G. For any $wz_k \in \overline{W}_n$, we have $d(xz_1, wz_k) = k + 2 = d(yz_0, wz_k)$, contrary to the fact that $\overline{W}_n$ is a metric basis of $G^u \sqcap P_n^0$. If the degree of w in G is at least two, pick two distinct neighbors x and y of w in G. Then $d(xz_0, wz_k) = k + 1 = d(yz_0, wz_k)$ for any $wz_k \in \overline{W}_n$, a contradiction. Hence our claim is valid.

Now we prove (6) for $j = n$. By the claim, we may pick two distinct vertices w_1 and w_2 satisfying (7).

Case 1. $v_1 = v_2$. Since $\{w_1z_k \mid 0 \leq k \leq n-1\}$ or $\{w_2z_k \mid 0 \leq k \leq n-1\}$ is a subset of $R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_1z_n\}$, the inequality (6) holds.

Case 2. $v_1 \neq v_2$.

Case 2.1. $i = 0$. Without loss of generality, we may assume that $w_1 \neq v_2$. Pick z_k satisfying $w_1z_k \in \overline{W}_n$. Then

$$d(v_1z_0, w_1z_k) = d_G(v_1, w_1) + k \leq d + k \leq n + k < d_G(v_2, w_1) + n + k = d(v_2z_n, w_1z_k),$$

which implies that $w_1z_k \in R_{G^u \sqcap P_{n+1}^0}\{v_1z_0, v_2z_n\}$. So (6) holds.

Case 2.2. $i \geq 1$. Note that

$$R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_2z_n\} = R_{G^u \sqcap P_{n+1}^0}\{v_1z_{i-1}, v_2z_{n-1}\} \supseteq R_{G^u \sqcap P_n^0}\{v_1z_{i-1}, v_2z_{n-1}\}.$$

Then (6) holds. $\square$

Proposition 2.11. *For any rooted graph G^u, we have*

$$\dim(G) \leq \dim(G^u \sqcap P_n^0) \leq |V(G)| - 1. \tag{8}$$

Proof. Let z be the other end-vertex of P_n. Fix a vertex $v_0 \in V(G)$ and write $\overline{S} = \{vz \mid v \in V(G) \setminus \{v_0\}\}$. Since $\{z\}$ is a resolving set of P_n, the set $\overline{S}$ resolves $G \sqcap P_n$ by (1). Hence $\dim(G^u \sqcap P_n^0) \leq |\overline{S}| = |V(G)| - 1$. Since G^u is isomorphic to $G^u \sqcap P_1^0$, Proposition 2.10 implies that $\dim(G) \leq \dim(G^u \sqcap P_n^0)$. □

For $m \geq 2$, we have $\dim(K_m^u \sqcap P_n^0) = m - 1$. This shows that the inequalities (4) and (8) are tight.

Example 2.12. For $m \geq 3$ and $n \geq 2$, we have $\dim(P_m^u \sqcap P_n^0) = 2$. In fact, write $P_k = (z_0 = 0, z_1, \ldots, z_{k-1})$, then $\{z_0 z_{n-1}, z_{m-1} z_{n-1}\}$ is a resolving set of $P_m^u \sqcap P_n^0$.

Example 2.13. Let C_m be the cycle with length m. Then $\dim(C_m^u \sqcap P_n^0) = 2$. In fact, write $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$ and $C_m = (c_0, c_1, \ldots, c_{m-1}, c_0)$, then $\{c_0 z_{n-1}, c_1 z_{n-1}\}$ is a resolving set of $C_m^u \sqcap P_n^0$.

3. FRACTIONAL METRIC DIMENSION

In order to study the fractional metric dimension for the hierarchical product of graphs, we first introduce the fractional rooted metric dimension for a rooted graph.

Similar to the fractionalization of metric dimension, we give a fractional version of the rooted metric dimension for a rooted graph. Let G^u be a rooted graph of order n. Write

$$\mathcal{P}^u = \{\{v, w\} \mid v, w \in V(G), v \neq w, d(v, u) = d(w, u)\}.$$

Suppose that $\mathcal{P}^u \neq \emptyset$. Write $V(G) \setminus \{u\} = \{v_1, \ldots, v_{n-1}\}$ and $\mathcal{P}^u = \{\alpha_1, \ldots, \alpha_m\}$. Let A^u be the $m \times (n-1)$ matrix with

$$(A^u)_{ij} = \begin{cases} 1, & \text{if } v_j \text{ resolves } \alpha_i, \\ 0, & \text{otherwise.} \end{cases}$$

The integer programming formulation of the rooted metric dimension for G^u is given by

$$\begin{aligned} &\text{Minimize } f(x_1, \ldots, x_{n-1}) = x_1 + \cdots + x_{n-1} \\ &\text{Subject to } A^u \mathbf{x} \geq \mathbf{1} \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_{n-1})^{\mathrm{T}}, x_i \in \{0, 1\}$ and $\mathbf{1}$ is the $m \times 1$ column vector all of whose entries are 1. The optimal solution of the linear programming relaxation of the above integer programming problem, where we replace $x_i \in \{0, 1\}$ by $x_i \in [0, 1]$, gives the *fractional rooted metric dimension* for G^u, which we denote by $\mathrm{rdim}_f(G^u)$.

Let G^u be a rooted graph which is not a path with an end-vertex u. A *rooted resolving function* of a rooted graph G^u is a real value function $g : V(G) \longrightarrow [0, 1]$ such that $g(R\{v, w\}) \geq 1$ for each $\{v, w\} \in \mathcal{P}^u$. The *fractional rooted metric dimension* for G^u is the minimum weight of a rooted resolving function of G^u.

Proposition 3.1. *Let G^u be a rooted graph which is not a path with an end-vertex u. Then*

(i) $\mathrm{rdim}_f(G^u) \leq \mathrm{rdim}(G^u)$.

(ii) $\mathrm{rdim}_f(G^u) \leq \frac{|V(G)|-1}{2}$.

(iii) $\dim_f(G) - 1 \leq \mathrm{rdim}_f(G^u) \leq \dim_f(G)$.

Proof. (i) Let W be a rooted metric basis of G^u. Define $g : V(G) \longrightarrow [0, 1]$ by

$$g(v) = \begin{cases} 1, & \text{if } v \in W, \\ 0, & \text{if } v \notin W. \end{cases}$$

For any $\{x, y\} \in \mathcal{P}^u$, there exists a vertex $v \in W$ such that $d(x, v) \neq d(y, v)$. Then $g(R\{x, y\}) \geq g(v) = 1$, which implies that g is a rooted resolving function of G^u. Hence $\mathrm{rdim}_f(G^u) \leq |g| = |W| = \mathrm{rdim}(G^u)$.

(ii) The function $g : V(G) \longrightarrow [0, 1]$ defined by

$$g(v) = \begin{cases} 0, & \text{if } v = u, \\ \frac{1}{2}, & \text{if } v \neq u \end{cases}$$

is a rooted resolving function of G^u. Hence $\mathrm{rdim}_f(G^u) \leq \frac{|V(G)|-1}{2}$.

(iii) It is clear that $\mathrm{rdim}_f(G^u) \leq \dim_f(G)$. Let g be a rooted resolving function of G^u. Then the function $h : V(G) \longrightarrow [0, 1]$ defined by

$$h(v) = \begin{cases} 1, & \text{if } v = u, \\ g(v), & \text{if } v \neq u \end{cases}$$

is a resolving function of G. Hence $\dim_f(G) \leq \mathrm{rdim}_f(G^u) + 1$, as desired. $\square$

If u is not an end-vertex of the path P_n, then $\mathrm{rdim}_f(P_n^u) = \mathrm{rdim}(P_n^u) = \dim_f(P_n) = 1$, which implies that the upper bounds in Proposition 3.1 (i) and (iii) are tight. The fact that $\mathrm{rdim}_f(K_n^u) = \frac{n-1}{2}$ shows that the inequality in Proposition 3.1 (ii) is tight.

Next, we study the fractional metric dimension for the hierarchical product of graphs.

For two rooted graphs $G_1^{u_1}$ and $G_2^{u_2}$, write

$$\mathcal{P}^{u_1} = \{\{x, y\} \subseteq V(G_1) | x \neq y, d_{G_1}(x, u_1) = d_{G_1}(y, u_1)\},$$
$$\overline{\mathcal{P}}^{u_2u_1} = \{\{x_2x_1, y_2y_1\} \subseteq V(G_2^{u_2} \sqcap G_1^{u_1}) | x_2x_1 \neq y_2y_1, d(x_2x_1, u_2u_1) = d(y_2y_1, u_2u_1)\}.$$

Lemma 3.2. *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. If G_1 is not a path with an end-vertex u_1, then*

$$\mathrm{rdim}_f(G_2^{u_2} \sqcap G_1^{u_1}) \geq |V(G_2)| \cdot \mathrm{rdim}_f(G_1^{u_1}).$$

Proof. Suppose that $\overline{g}$ is a rooted resolving function of $G_2^{u_2} \sqcap G_1^{u_1}$ with weight $\mathrm{rdim}_f(G_2^{u_2} \sqcap G_1^{u_1})$. For each $z \in V(G_2)$, define

$$\overline{g}_z : V(G_1) \longrightarrow [0,1], \quad x \longmapsto \overline{g}(zx).$$

Write $\overline{\mathcal{P}}^{u_1} = \{\{zx, zy\} \mid z \in V(G_2), \{x,y\} \in \mathcal{P}^{u_1}\}$. By (1), we have $\overline{\mathcal{P}}^{u_1} \subseteq \overline{\mathcal{P}}^{u_2u_1}$. Hence $\overline{g}_z(R_{G_1}\{x,y\}) \geq 1$ for any $\{x,y\} \in \mathcal{P}^{u_1}$, which implies that $|\overline{g}_z| \geq \mathrm{rdim}_f(G_1^{u_1})$. Consequently,

$$\mathrm{rdim}_f(G_2^{u_2} \sqcap G_1^{u_1}) = |\overline{g}| = \sum_{z \in V(G_2)} |\overline{g}_z| \geq |V(G_2)| \cdot \mathrm{rdim}_f(G_1^{u_1}),$$

as desired.

Theorem 3.3. *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. If G_1 is not a path with an end-vertex u_1, then*

$$\dim_f(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \mathrm{rdim}_f(G_1^{u_1}).$$

Proof. Combining Proposition 3.1 and Lemma 3.2, we only need to prove that

$$\dim_f(G_2^{u_2} \sqcap G_1^{u_1}) \leq |V(G_2)| \cdot \mathrm{rdim}_f(G_1^{u_1}). \tag{9}$$

By Proposition 2.4 we have $\mathcal{P}^{u_1} \neq \emptyset$. Let g be a rooted resolving function of G_1 with weight $\mathrm{rdim}_f(G_1^{u_1})$. Define

$$\overline{g} : V(G_2^{u_2} \sqcap G_1^{u_1}) \longrightarrow [0,1], \quad x_2x_1 \longmapsto g(x_1).$$

We shall show that, for any two distinct vertices x_2x_1 and y_2y_1 of $G_2^{u_2} \sqcap G_1^{u_1}$,

$$\overline{g}(R\{x_2x_1, y_2y_1\}) \geq 1. \tag{10}$$

Case 1. $x_2 = y_2$. If $u_1 \notin R_{G_1}\{x_1,y_1\}$, by Lemma 2.5 we get $R\{x_2x_1, y_2y_1\} = \{x_2z \mid z \in R_{G_1}\{x_1,y_1\}\}$, which implies that $\overline{g}(R\{x_2x_1,y_2y_1\}) = g(R_{G_1}\{x_1,y_1\})$. Since $\{x_1,y_1\} \in \mathcal{P}^{u_1}$, we obtain (10). If $u_1 \in R_{G_1}\{x_1,y_1\}$, by Lemma 2.5 we have $R\{x_2x_1,y_2y_1\} \supseteq \{vz \mid z \in V(G_1)\}$ for any $v \in V(G_2) \setminus \{x_2\}$, which implies that $\overline{g}(R\{x_2x_1,y_2y_1\}) \geq |g|$, so (10) holds.

Case 2. $x_2 \neq y_2$. Write $W = \{z \mid x_2z \in R\{x_2x_1,y_2y_1\}\}$ and $S = \{z \mid y_2z \in R\{x_2x_1,y_2y_1\}\}$. By Lemma 2.5 we have $W \cup S = V(G_1)$. Then

$$\overline{g}(R\{x_2x_1,y_2y_1\}) \geq \sum_{z \in W} \overline{g}(x_2z) + \sum_{z \in S} = g(W) + g(S)\overline{g}(y_2z) \geq |g|,$$

which implies that (10) holds.

Therefore, $\overline{g}$ is a resolving function of $G_2^{u_2} \sqcap G_1^{u_1}$, which implies that $\dim_f(G_2^{u_2} \sqcap G_1^{u_1}) \leq |\overline{g}|$. Since $|\overline{g}| = |V(G_2)| \cdot \mathrm{rdim}_f(G_1^{u_1})$, we obtain (9). Our proof is accomplished. □

By Theorem 3.3, we obtain the following corollary immediately.

Corollary 3.4. *Let* $n \geq 2$. *Then* $\dim_f(T_n) = 2^{n-2}$.

Arumugam and Mathew [**1**] proposed a natural problem: Characterize graphs for which $\dim_f(G) = \dim(G)$. By Corollaries 2.9 and 3.4, the binomial tree T_n satisfies $\dim_f(T_n) = \dim(T_n)$. But this problem is still open.

It seems that there is a gap to determine $\dim_f(G^u \sqcap P_n^0)$. We conclude this paper by giving some tight inequalities involving it.

Proposition 3.5. *For any rooted graph* G^u, *we have*

$$\dim_f(G) \leq \dim_f(G^u \sqcap P_n^0) \leq \dim_f(G^u \sqcap P_{n+1}^0) \leq \frac{|V(G)|}{2}.$$

Proof. Write $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$. For a resolving function $\overline{g}_{n+1}$ of $G^u \sqcap P_{n+1}^0$, we define $\overline{g}'_{n+1} : V(G^u \sqcap P_n^0) \longrightarrow [0,1]$ by

$$\overline{g}'_{n+1}(x_2x_1) = \begin{cases} \overline{g}_{n+1}(x_2z_{n-1}) + \overline{g}_{n+1}(x_2z_n), & \text{if } x_1 = z_{n-1}, \\ \overline{g}_{n+1}(x_2x_1), & \text{if } x_1 \neq z_{n-1}. \end{cases}$$

Then $\overline{g}'_{n+1}$ is a resolving function of $G^u \sqcap P_n^0$. Since $|\overline{g}'_{n+1}| = |\overline{g}_{n+1}|$, we have

$$\dim_f(G) = \dim_f(G^u \sqcap P_1^0) \leq \dim_f(G^u \sqcap P_n^0) \leq \dim_f(G^u \sqcap P_{n+1}^0).$$

For proving the last inequality, define $\overline{h} : V(G^u \sqcap P_{n+1}^0) \longrightarrow [0,1]$ by

$$\overline{h}(x_2x_1) = \begin{cases} \frac{1}{2}, & \text{if } x_1 = z_n, \\ 0, & \text{if } x_1 \neq z_n. \end{cases}$$

Then $\overline{h}$ is a resolving function of $G^u \sqcap P_{n+1}^0$ with weight $\frac{|V(G)|}{2}$. Hence $\dim_f(G^u \sqcap P_{n+1}^0) \leq \frac{|V(G)|}{2}$. □

For $m \geq 2$, we have $\dim_f(K_m^u \sqcap P_n^0) = \frac{m}{2}$. This shows that all the inequalities in Proposition 3.5 are tight.

Acknowledgements. The authors are indebted to the anonymous reviewers for their detailed reports. This research is supported by NSFC(11271047) and the Fundamental Research Funds for the Central University of China.

REFERENCES

1. S. Arumugam, V. Mathew: *The fractional metric dimension of graphs.* Discrete Math., **312** (2012), 1584–1590.
2. S. Arumugam, V. Mathew, J. Shen: *On fractional metric dimension of graphs.* Preprint.
3. R. F. Bailey, P. J. Cameron: *Base size, metric dimension and other invariants of groups and graphs.* Bull. London Math. Soc., **43** (2011), 209–242.

4. L. Barrière, F. Comellas, C. Dalfó, M. A. Fiol: *The hierarchical product of graphs.* Discrete Appl. Math., **157** (2009), 36–48.

5. Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalák, L. S. Ram: *Network discovery and verification.* IEEE J. on Selected Areas in Communications, **24** (2006), 2168–2181.

6. J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood: *On the metric dimension of Cartesian products of graphs.* SIAM J. Discrete Math., **21** (2007), 423–441.

7. G. Chartrand, L. Eroh, M. Johnson, O. R. Oellermann: *Resolvability in graphs and the metric dimension of a graph.* Discrete Appl. Math., **105** (2000), 99–113.

8. V. Chvátal: *Mastermind.* Combinatorica, **3** (1983), 325–329.

9. T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein: *Introduction to Algorithms.* MIT Press, Cambridge, MA, 1990. second ed. 2001.

10. J. Currie, O. R. Oellermann: *The metric dimension and metric independence of a graph.* J. Combin. Math. Combin. Comput., **39** (2001), 157–167.

11. M. Fehr, S. Gosselin, O. R. Oellermann: *The metric dimension of Cayley digraphs.* Discrete Math., **306** (2006), 31–41.

12. M. Feng, B. Lv, K. Wang: *On the fractional metric dimension of graphs.* arXiv:1112.2106v2.

13. M. Feng, K. Wang: *On the fractional metric dimension of corona product graphs and lexicographic product graphs.* arXiv:1206.1906v1.

14. M. R. Garey, D. S. Johnson: *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, New York, 1979.

15. F. Harary, R. A. Melter: *On the metric dimension of a graph.* Ars Combin., **2** (1976), 191–195.

16. M. Jannesari, B. Omoomi: *The metric dimension of the lexicographic product of graphs.* Discrete Math., **312** (2012), 3349–3356.

17. S. Jung, S. Kim, B. Kahng: *Geometric fractal growth model for scale-free networks.* Phys. Rev. E, **65** (2002), 056101.

18. S. Khuller, B. Raghavachari, A. Rosenfeld: *Landmarks in graphs.* Discrete Appl. Math., **70** (1996), 217–229.

19. J. D. Noh: *Exact scaling properties of a hierarchical network model.* Phys. Rev. E, **67** (2003), 045103.

20. E. Ravasz, A. L. Barabási: *Hierarchical organization in complex networks.* Phys. Rev. E, **67** (2003), 026112.

21. E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, A. L. Barabási: *Hierarchical organization of modularity in metabolic networks.* Science, **297** (2002), 1551–1555.

22. E. R. Scheinerman, D. H. Ullman: *Fractional Graph Theory.* In: Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1997.

23. A. Sebő, E. Tannier: *On metric generators of graphs.* Math. Oper. Res., **29** (2004) 383–393.

24. P. J. SLATER: *Leaves of trees.* Conger. Numer., **14** (1975), 549–559.

25. I. G. YERO, D. KUZIAK, J. A. RODRÍGUEZ-VELÁZQUEZ: *On the metric dimension of corona product graphs.* Comput. Math. Appl., **61** (2011), 2793–2798.

School of Mathematical Sciences,
Laboratory of Mathematics and Complex Systems,
Beijing Normal University,
Beijing, 100875
China
E-mail: wangks@bnu.edu.cn

(Received December 5, 2012)
(Revised May 13, 2013)