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# ASYMPTOTICS OF THE STIRLING NUMBERS OF THE SECOND KIND REVISITED

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Using the Saddle point method and multiseries expansions, we obtain from the generating function of the Stirling numbers of the second kind $\left\{ n \atop m \right\}$ and Cauchy's integral formula, asymptotic results in central and non-central regions. In the central region, we revisit the celebrated Gaussian theorem with more precision. In the region $m = n - n^{\alpha}$, $1 > \alpha > 1/2$, we analyze the dependence of $\left\{ n \atop m \right\}$ on $\alpha$. An extension of some Moser and Wyman's result to full $m$ range is also provided. This paper fits within the framework of Analytic Combinatorics.

## 1. INTRODUCTION

Let $\left\{ n \atop m \right\}$ be the Stirling numbers of the second kind. Their generating function is given by

$$\sum_n \frac{m!}{n!} \left\{ n \atop m \right\} z^n = f(z)^m, \qquad f(z) := e^z - 1,$$

and $f(z)$ is an entire function. In the following all asymptotics are meant for $n \to \infty$.

Without being exhaustive, let us summarize some related literature. If we associate a random variable $J_n$ with $\left\{ n \atop m \right\}$ (see Sec. 2 for details), we denote by $M$ and $\sigma^2$ the corresponding mean an variance. We define the central region by $x = \mathcal{O}(1)$, where $x = (m - M)/\sigma$. The asymptotic Gaussian approximation in the

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central region is proved by Harper [**9**] and Canfield [**3**], see also Bender [**1**], Sachkov [**18**] and Hwang [**12**].

In the non-central region, $m = n - n^\alpha$, $0 < \alpha < 1$, most of the previous papers use the solution of

$$\frac{\rho e^\rho}{e^\rho - 1} = \frac{n}{m}. \tag{1}$$

As shown in the next section, this actually corresponds to a Saddle point.

Let us mention

• Hsu [**10**]:

For $t = o(n^{1/2})$

$$\left\{ \begin{matrix} n \\ n-t \end{matrix} \right\} = \frac{(n-t)^{2t}}{2^t t!} \left[ 1 + \frac{f_1(t)}{n-t} + \frac{f_2(t)}{(n-t)^2} + \cdots \right],$$

with

$$f_1(t) = \frac{1}{3} t(2t+1).$$

• Moser and Wyman [**14**]:

For $t = o(\sqrt{n})$,

$$\left\{ \begin{matrix} n \\ n-t \end{matrix} \right\} = \binom{n}{t} q^{-t} \left[ 1 + \frac{(t)_2}{12} q + \frac{(t)_2}{288} q^2 + \dots \right], \tag{2}$$

where

$$q = \frac{2}{n-t}, \quad (t)_r := t(t-1)\dots(t-r+1).$$

For $n - m \to \infty, n \to \infty$,

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{n!(e^\rho - 1)^m}{2\rho^n m! (\pi m \rho H)^{1/2}} \left[ 1 - \frac{1}{m\rho} \left( \frac{15 C_3^2}{16 \rho^2 H} - \frac{3 C_4}{4 \rho H^2} \right) + \dots \right], \tag{3}$$

where

$$H = \frac{e^\rho (e^\rho - 1 - \rho)}{2(e^\rho - 1)^2},$$

$C_3, C_4$ are functions of $\rho$.

• Good [**8**]:

For $\kappa$ bounded above and below by positive constants,

$$\left\{ \begin{matrix} n \\ n-t \end{matrix} \right\} = \frac{(n)!(e^\rho - 1)^\tau}{\tau! \rho^n \left[ 2\pi\tau \left( 1 + \kappa - (1+\kappa)^2 e^{-\rho} \right) \right]^{1/2}} \left[ 1 + \frac{g_1(\kappa)}{\tau} + \frac{g_2(\kappa)}{\tau^2} + \dots \right],$$

with

$$\tau = n - t, \quad \kappa := \frac{n}{n-t}, \quad g_1(\kappa) = \frac{3\lambda_4 - 5\lambda_3^2}{24},$$

$$\lambda_i = \kappa_i(\rho)/\sigma^i, \lambda_0, \dots, \lambda_2 \text{ are not used here},$$
$$\kappa_i = (\partial/\partial u)^i \left(\ln\left(f\left(\rho e^u\right)\right)\right)\big|_{u=0}, \quad f(x) = (e^x - 1)/x,$$
$$\sigma = \kappa_2(\rho)^{1/2}, \quad \kappa_1 = \kappa, \kappa_2 = (\kappa_1 + 1)(\rho - \kappa_1).$$

• BENDER [1]:
Uniformly for $\varepsilon < m/n < 1 - \varepsilon$, with $\varepsilon > 0$,

$$\left\{ {n \atop m} \right\} \sim \frac{n! e^{-\beta m}}{m! \rho^{n-1} (1 + e^\beta) \sigma \sqrt{2\pi n}},$$

where

$$\frac{n}{m} = (1 + e^\beta) \ln(1 + e^{-\beta}),$$
$$\rho = \ln(1 + e^{-\beta}), \qquad \sigma^2 = \left(\frac{m}{n}\right)^2 \left[1 - e^\beta \ln(1 + e^{-\beta})\right],$$

It is easy to see that $\rho$ here coincides with the solution of (1). BENDER's expression is similar to MOSER and WYMAN's result (3).

• BLEICK and WANG [2]:
Let $\rho_1$ be the solution of

$$\frac{\rho_1 e^{\rho_1}}{e^{\rho_1} - 1} = \frac{n+1}{m}.$$

Then

$$\left\{ {n \atop m} \right\} = \frac{n! (e^{\rho_1} - 1)^m}{(2\pi(n+1))^{1/2} m! \rho_1^n (1 - G)^{1/2}}$$
$$\times \left[1 - \frac{2 + 18G - 20G^2(e^{\rho_1} + 1) + 3G^3(e^{2\rho_1} + 4e^{\rho_1} + 1) + 2G^4(e^{2\rho_1} - e^{\rho_1} + 1)}{24(n+1)(1 - G)^3} + \mathcal{O}\left(\frac{1}{n^2}\right)\right],$$

where $G = \dfrac{\rho_1}{e^{\rho_1} - 1}$. The series is convergent for for $m = o(n^{2/3})$.

• TEMME [20]:
For all $m$,

$$\left\{ {n \atop m} \right\} = e^A m^{n-m} \binom{n}{m} \sum_{k=0}^{\infty} (-1)^k f_k(t_0) m^{-k},$$

where

$$f_0(t_0) = \left(\frac{t_0}{(1 + t_0)(\rho - t_0)}\right)^{1/2},$$
$$t_0 = \frac{n}{m} - 1,$$
$$A = -n \ln(\rho) + m \ln(e^\rho - 1) - m t_0 + (n - m) \ln(t_0).$$

• TSYLOVA [**21**]:
Let $m = tn + o(n^{2/3})$.

$$\left\{ {n \atop m} \right\} = \frac{(\gamma n)^n}{\sqrt{2\pi\delta n}(\gamma n)^m} \exp\left[-(m-tn)^2/(2\delta n)\right](1+o(1)),$$

with

$$\gamma(1-e^{-1/\gamma}) = \gamma, \qquad \delta = e^{-1/\gamma}(t - e^{-1/\gamma}).$$

After some algebra, this coincides with MOSER and WYMAN's result (with a smaller range).

• Chelluri, Richmond and Temme [**4**]:
They prove, with other techniques, that MOSER and WYMAN's expression (3) is valid if $n - m = \Omega(n^{1/3})$ and that HSU's formula is valid for $n - m = o(n^{1/3})$.

• ERDŐS and SZEKERES, see SACHKOV [**18**], p.164:
Let $m < n/\ln n$,

$$\left\{ {n \atop m} \right\} = \frac{m^n}{m!} \exp\left[\left(\frac{n}{m} - m\right) e^{-n/m}\right](1+o(1)).$$

Let us finally note that HSU and SHIUE [**11**] consider some generalized Stirling numbers. Let us also mention PEMANTLE and WILSON [**15**] and RAICHEV and WILSON [**17**] where they provide methods for deriving asymptotics of coefficients of algebraic and generating functions. Many examples are given in the survey: PEMANTLE and WILSON [**16**].

Let us summarize the motivation of this paper:

- The choice of $m = n - t$ by most authors is usually such that $t = o(n^\beta)$. We want to be more precise by using $m = n - n^\alpha$ for our non-central range. $\alpha$ is chosen such that $n^\alpha$ is integer.
- Previous papers simply use $\rho$ as the solution of (1). They don't compute the detailed dependence of $\rho$ on $\alpha$ , neither the precise behaviour of functions of $\rho$ they use. Moreover, most results are related to the case $\alpha < 1/2$.
- We use multiseries expansions: multiseries are in effect power series (in which the powers may be non-integral but must tend to infinity) and the variables are elements of a scale: details can be found in SALVY and SHACKELL [**19**]. The scale is a set of variables of increasing order. The series is computed in terms of the variable of maximum order, the coefficients of which are given in terms of the next-to-maximum order, etc. This is more precise than mixing different terms. Actually we implicitly used multiseries in our analysis of Stirling numbers of the first kind in [**13**]. There, we analyzed the central region around the mean $H_n$ (the variance is also of order $\ln(n)$): the result is similar to Theorem 2.1. We also considered the large deviation $m = n^\alpha, \alpha > 1/2$, the multiseries' scale is the same as in Sec. 3.

One of our referees suggested an extension of Moser and Wyman's expression (3) to a full $m$ range. With his/her permission, we include this result in this paper.

Our work fits within the framework of Analytic Combinatorics. A preliminary version of this paper was presented at ALEA 2012.

In Sec. 2, we revisit the asymptotic expansion in the central region and in Sec. 3, we analyse the non-central region $m = n - n^{\alpha}, \quad \alpha > 1/2$. We use Cauchy's integral formula and the Saddle point method. Sec. 4 is devoted to Moser and Wyman's expression full $m$ range extension. Sec. 5 provides a justification of the Saddle point technique we use here.

## 2. CENTRAL REGION

Consider the random variable $J_n$, with probability distribution

$$\mathbb{P}[J_n = m] = Z_n(m), \qquad Z_n(m) := \frac{\left\{ {n \atop m} \right\}}{B_n},$$

where $B_n$ is the $n$-th Bell number. The mean and variance of $J_n$ are given by

$$M := \mathbb{E}(J_n) = \frac{B_{n+1}}{B_n} - 1,$$
$$\sigma^2 := \mathbb{V}(J_n) = \frac{B_{n+2}}{B_n} - \frac{B_{n+1}}{B_n} - 1.$$

Let $\zeta$ be the positive solution of

$$\zeta e^{\zeta} = n.$$

This immediately leads to

$$\zeta = W(n),$$

where $W$ is the Lambert's $W$ function, see Corless and al. [**5**] (we use the principal branch, which is analytic at 0). Let us set $L := \ln(n)$. We have the well-known asymptotic expression

$$\zeta = L - \ln L + \frac{\ln L}{L} + \mathcal{O}\left(\frac{(\ln L)^2}{L^2}\right). \tag{4}$$

To simplify our expressions in the following, let

$$G := e^{\zeta/2} \sim \sqrt{n/L}$$

The multiseries' scale is here $\{\zeta, G\}$.

Our result can be summarized in the following local limit theorem

**Theorem 2.1.** *Let $x = (m - M)/\sigma$. Then for $x = \mathcal{O}(1)$,*

$$Z_n(m) = \frac{\left\{ {n \atop m} \right\}}{B_n} = e^{-x^2/2} \frac{(1+\zeta)^{1/2}}{\sqrt{2\pi} G} \left[ 1 + \frac{x(-6\zeta + 2x^2\zeta + x^2 - 3)}{6G(1+\zeta)^{3/2}} + \mathcal{O}(1/G^2) \right].$$

**Proof.** By Salvy and Shackell [**19**], we have

$$M = G^2 + A_1 + \mathcal{O}(1/G^2) \sim \frac{n}{L}, \qquad \sigma^2 = \frac{G^2}{1+\zeta} + A_2 + \mathcal{O}(1/G^2) \sim \frac{n}{L^2},$$

$$\frac{B_n}{n!} = \exp(T_1) H_0, \tag{5}$$

$$T_1 = -\ln(\zeta)\zeta G^2 + G^2 - \zeta/2 - \ln(\zeta) - 1 - \ln(2\pi)/2, \tag{6}$$

$$A_1 = -\frac{2 + 3/\zeta + 2/\zeta^2}{2(1+1/\zeta)^2}, \qquad A_2 = -\frac{2 + 8/\zeta + 11/\zeta^2 + 9/\zeta^3 + 2/\zeta^4}{2(1+1/\zeta)^4},$$

$$H_0 = \frac{1}{(1+1/\zeta)^{1/2}} \left[1 + A_5/G^2 + \mathcal{O}(1/G^4)\right],$$

$$A_5 = -\frac{2 + 9/\zeta + 16/\zeta^2 + 6/\zeta^3 + 2/\zeta^4}{24(1+1/\zeta)^3}.$$

This leads to (from now on, we only provide a few terms in our expansions, but of course we use more terms in our computations), using expansions in $G$,

$$\sigma = \frac{G}{(1+\zeta)^{1/2}} + \frac{A_2(1+\zeta)^{1/2}}{2G} + \mathcal{O}(\zeta^{3/2}/G^3), \qquad \sigma \sim \frac{G}{\sqrt{\zeta}} \sim \frac{\sqrt{n}}{L}.$$

We now use the Saddle point technique (for a good introduction to this method, see Flajolet and Sedgewick [**6**, ch. VIII]). Let $\rho$ be the saddle point and $\Omega$ the circle $\rho e^{\mathbf{i}\theta}$. By Cauchy's theorem,

$$Z_n(m) = \frac{n!}{m! B_n} \frac{1}{2\pi\mathbf{i}} \int_\Omega \frac{f(z)^m}{z^{n+1}} \mathrm{d}z = \frac{n!}{m! B_n \rho^n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{\mathbf{i}\theta})^m e^{-n\mathbf{i}\theta} \mathrm{d}\theta$$

$$x = \frac{n!}{m! B_n \rho^n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{m \ln(f(\rho e^{\mathbf{i}\theta})) - n\mathbf{i}\theta} \mathrm{d}\theta$$

$$= \frac{n!}{m! B_n \rho^n} \frac{f(\rho)^m}{2\pi} \int_{-\pi}^{\pi} \exp\left[m\left\{-\frac{1}{2}\kappa_2\theta^2 - \frac{\mathbf{i}}{6}\kappa_3\theta^3 + \dots\right\}\right] \mathrm{d}\theta, \tag{7}$$

$$\kappa_i(\rho) = \left(\frac{\partial}{\partial u}\right)^i \ln(f(\rho e^u))\big|_{u=0}. \tag{8}$$

See Good [**7**] for an ancient but neat description of this technique. Note that $e^z - 1 = 0$ for $z_k = 2k\mathbf{i}\pi, k$ integer. So our path is chosen such that to avoid these values.

Let us now turn to the saddle point computation. $\rho$ is the root (of smallest modulus) of

$$m\rho f'(\rho) - n f(\rho) = 0 \quad \text{i. e.} \quad \frac{\rho e^\rho}{e^\rho - 1} = \frac{n}{m},$$

which is, of course identical to (1). After some algebra, this gives

$$\rho = \frac{n}{m} + W\left(-\frac{n}{m} e^{-n/m}\right).$$

In the central region, we choose

$$m = M + \sigma x = G^2 + \frac{x}{(1+\zeta)^{1/2}} G + A_1 + \frac{x A_2 (1+\zeta)^{1/2}}{2G} + \mathcal{O}(\zeta/G^2).$$

This leads to

$$\begin{aligned}
\ln(m) &= \zeta + \frac{x}{(1+\zeta)^{1/2} G} + \mathcal{O}(1/G^2),\\
\frac{n}{m} &= \zeta - \frac{\zeta x}{(1+\zeta)^{1/2} G} + \frac{-A_1\zeta + \zeta x^2/(1+\zeta)}{G^2} + \mathcal{O}(\zeta^{3/2}/G^3),\\
\rho &= \zeta - \frac{\zeta x}{(1+\zeta)^{1/2} G} + \frac{\zeta(-A_1 + x^2/(1+\zeta) - 1)}{G^2} + \mathcal{O}(\zeta^{3/2}/G^3),\\
\ln(\rho) &= \ln(\zeta) - \frac{x}{(1+\zeta)^{1/2} G} + \mathcal{O}(1/G^2).
\end{aligned}$$

Let us remark that the coefficients of $G$ powers are rational expressions in $\zeta$.

Now we note that

$$\begin{aligned}
e^{\rho} - 1 &= \rho e^{\rho} \frac{m}{n},\\
\ln\left(e^{\rho} - 1\right) &= \rho + \ln(\rho) + \ln(m) - L,\\
Ł &= \zeta + \ln(\zeta),
\end{aligned} \tag{9}$$

so, by Stirling's formula, with (6), the first part of (7) leads to

$$\begin{aligned}
&\frac{n!}{m! B_n \rho^n} f(\rho)^m = \exp\left[T_2\right] H_1 H_2,\\
&T_2 = m(\rho + \ln(\rho) - \zeta - \ln(\zeta)) - (-m + \ln(2\pi)/2 + \ln(m)/2) - \zeta F \ln(\rho) - T_1,\\
&H_1 = 1/H_0 = (1 + 1/\zeta)^{1/2} - \frac{A_5(1+1/\zeta)^{1/2}}{G^2} + \mathcal{O}(1/G^4),\\
&H_2 = \frac{1}{1 + \frac{1}{12m} + \frac{1}{288m^2} + \mathcal{O}\left(\frac{1}{m^3}\right)} = 1 - \frac{1}{12G^2} + \frac{x}{12G^3(1+\zeta)^{1/2}} + \mathcal{O}\left(\frac{1}{\zeta G^4}\right).
\end{aligned}$$

Note carefully that there is a cancellation of the term $m\ln(m)$ in $T_2$. Using all previous expansions, we obtain

$$\exp(T_2) = e^{-x^2/2 + \ln(\zeta)} H_3, \tag{10}$$

$$H_3 = 1 + \frac{x(-15\zeta - 6\zeta^2 - 6A_1 + x^2 - 12A_1\zeta - 6A_1\zeta^2 + 2x^2\zeta - 9}{6(1+\zeta)^{3/2} G} + \mathcal{O}(\zeta/G^2).$$

We now turn to the integral in (7). We compute

$$\kappa_2 = -\frac{\rho e^{\rho}(-e^{\rho} + 1 + \rho)}{(e^{\rho} - 1)^2} = \zeta - \frac{\zeta x}{(1+\zeta)^{1/2} G} + \mathcal{O}(\zeta^2/G^2) \sim L, \tag{11}$$

and similar expressions for the next $\kappa_i$ that we don't detail here. Note that $\kappa_3, \kappa_5, \ldots$ are useless for the precision we attain here. We will only use the fact that $\kappa_3 = \mathcal{O}(L^4/n)$. We proceed as in FLAJOLET and SEDGEWICK [**6**, ch. VIII]. Let us choose a splitting value $\theta_0$ such that $m\kappa_2\theta_0^2 \to \infty, m\kappa_3\theta_0^3 \to 0, n \to \infty$. For instance, we can use $\theta_0 = L/\sqrt{n}$. We must prove that the integral

$$K_n = \int_{\theta_0}^{2\pi-\theta_0} e^{m\ln(f(\rho e^{\mathrm{i}\theta}))-n\mathrm{i}\theta}\mathrm{d}\theta$$

is such that $|K_n|$ is exponentially small. This is done in Appendix (Sec. 5).

Now we use the classical trick of setting

$$m\Big[-\kappa_2\theta^2/2! + \sum_{l=3}^{\infty}\kappa_\ell(\mathbf{i}\theta)^\ell/\ell!\Big] = -u^2/2.$$

Computing $\theta$ as a series in $u$, this gives, by Lagrange's inversion,

$$\theta = \frac{1}{G}\sum_1^\infty a_i u^i,$$

with, for instance

$$a_1 = \frac{1}{\zeta^{1/2}} + \frac{\zeta^{1/2}}{2G^2} + \mathcal{O}\bigg(\frac{\zeta^{3/2}}{G^3}\bigg).$$

This expansion n is valid in the dominant integration domain

$$|u| \le \frac{G}{a_1}\theta_0 = L.$$

Setting $\mathrm{d}\theta = \dfrac{\mathrm{d}\theta}{\mathrm{d}u}\mathrm{d}u$, we integrate on $u = (-\infty, \infty)$: this extension of the range is justified as in FLAJOLET and SEDGEWICK [**6**, ch. VIII]. Now, inserting the term $\zeta$ coming in (10) as $e^{\ln(\zeta)}$, this gives

$$H_4 = \frac{\zeta^{1/2}}{\sqrt{2\pi}G}\left(1 + \frac{\zeta}{2G^2} + \mathcal{O}(\zeta^2/G^3)\right).$$

Finally, combining all expansions, we obtain the multiseries expression

$$Z_n(m) = \frac{\left\{ n \atop m \right\}}{B_n} = e^{-x^2/2}H_1H_2H_3H_4 = R_1, \tag{12}$$

$$R_1 = e^{-x^2/2}\frac{(1+\zeta)^{1/2}}{\sqrt{2\pi}G}\left[1 + \frac{x(-6\zeta + 2x^2\zeta + x^2 - 3)}{6G(1+\zeta)^{3/2}} + \mathcal{O}(\zeta/G^2)\right].$$

Note that the coefficient of the exponential term is asymptotically equivalent to the dominant term of $\dfrac{1}{\sqrt{2\pi\sigma}}$, as expected. More terms in this expression can be obtained if we compute $M, \sigma^2, B_n/n!$ with more precision. Also, using (4), our result can be put into expansions depending on $n, L, \dots$. $\square$

To check the quality of our asymptotic, we have chosen $n = 3000$. This leads to

$$\begin{aligned}
\zeta &= 6.184346264\dots,\\
G &= 22.02488900\dots,\\
M &= 484.1556441\dots,\\
\sigma &= 8.156422315\dots,\\
B_n &= 0.2574879583\dots 10^{6965},\\
B_n^{as} &= 0.2574880457\dots 10^{6965},
\end{aligned}$$

where $B_n^{as}$ is given by (5). Figure 1 shows $Z_n(m)$ and $\dfrac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\cdot\left(\frac{m-M}{\sigma}\right)^2\right]$.

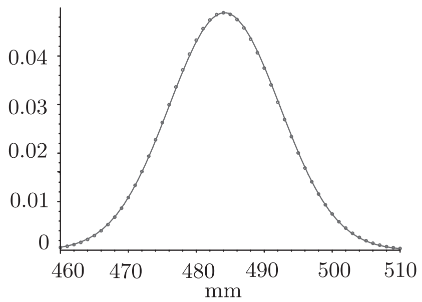


Figure 1. $Z_n(m)$ and $\dfrac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\cdot\left(\frac{m-M}{\sigma}\right)^2\right]$

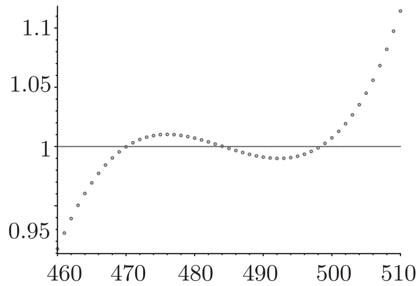


Figure 2. $Z_n(m)\Big/\dfrac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\cdot\left(\frac{m-M}{\sigma}\right)^2\right]$

The fit seems quite good, but to have more precise information, we show in Figure 2 the quotient $Z_n(m)\Big/\dfrac{1}{\sqrt{2\pi}\sigma}\exp\left[-\left(\frac{m-M}{\sigma}\right)^2\Big/2\right]$. The relative error is between 0.05 and 0.10.

Figure 3 shows the quotient $Z_n(m)/R_1$ (without error term). The relative error is now between 0.004 and 0.01.

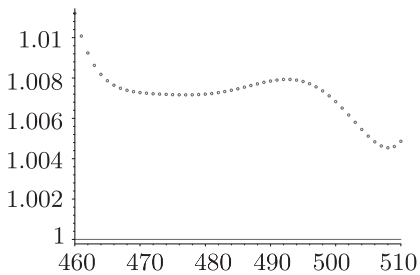


Figure 3. $Z_n(m)/R_1$ (without error term)

## 3. LARGE DEVIATION, $m = n - n^\alpha$, $1 > \alpha > 1/2$, $n^\alpha$ INTEGER

We have $m = n - n^\alpha$, $1 > \alpha > 1/2$, $n^\alpha$ integer. We consider $m$ outside of the central region, i.e.

$$n - n^\alpha \gg \frac{n}{L} \text{ i.e. } \alpha < 1 - \frac{1}{L^2}.$$

We set

$$\varepsilon := n^{\alpha-1}, \qquad \frac{1}{\varepsilon} = n^{1-\alpha} \ll n^{\alpha} \ll n.$$

The multiseries' scale is here $\{n^{1-\alpha}, n^{\alpha}, n\}$.

Our result can be summarized in the following local limit theorem

**Theorem 3.2.** *We have the asymptotic expression*

$$\left\{ {n \atop m} \right\} = e^{T_1} R,$$

*where the exponential term $T_1$ is given by*

$$T_1 = n^{\alpha}(T_{11}L + T_{10}),$$

*and the coefficient $R$ is expressed as*

$$R = \frac{1}{\sqrt{2\pi}n^{\alpha/2}} \left[ R_0 + \frac{R_1}{n} + \frac{R_2}{n^2} + \mathcal{O}(1/n^3) \right], \quad R_0 = R_{00} + \frac{R_{01}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$
$$R_1 = R_{10} + \frac{R_{11}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}), \qquad R_2 = R_{20} + \frac{R_{21}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$

*$T_{i,j}, R_{i,j}$ are power series in $\varepsilon$ given in the proof.*

**Proof.** Using again the Lambert's $W$ function, we derive successively (again we only provide a few terms here, we use 12 terms in our expansions, this is necessary if we want a good precision at the end, as we start from a mixture of numerous different terms)

$$\begin{aligned} m &= n(1-\varepsilon), \qquad \frac{n}{m} = \frac{1}{1-\varepsilon}, \\ \rho &= 2\varepsilon + \frac{4}{3}\varepsilon^2 + \frac{10}{9}\varepsilon^3 + \mathcal{O}(\varepsilon^4), \\ \ln(m) &= L - \varepsilon - \frac{1}{2}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\ \ln(\rho) &= -L(1-\alpha) + \ln(2) + \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{13}$$

For the first part of the Cauchy's integral, we have, noting that $n\varepsilon = n^{\alpha}$, and using (9),

$$\frac{n!}{m!\rho^n} f(\rho)^m = \exp(T)H_2,$$
$$T = m(\rho + \ln(\rho) - L) - (-m + \ln(m)/2) + (-n + nL + L/2) - n\ln(\rho) = T_1 + T_0,$$

the dominant part $T_1$ is given by

$$T_1 = n^{\alpha}(T_{11}L + T_{10}),$$

where

$$T_{11} = 2 - \alpha \quad \text{and} \quad T_{10} = 1 - \ln(2) - \frac{4}{3}\varepsilon - \frac{5}{9}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$

and the $\mathcal{O}(\varepsilon)$ part $T_0$ is given by

$$T_0 = \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

The exponential term $\exp(T_0)$ can itself be expanded:

$$H_1 = \exp(T_0) = 1 + \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

The quotient $n!/m!$ leads to an extra term

$$\begin{aligned} H_2 &= \frac{1 + \frac{1}{12n} + \frac{1}{288n^2} + \mathcal{O}(1/n^3)}{1 + \frac{1}{12m} + \frac{1}{288m^2} + \mathcal{O}(1/m^3)} \\ &= 1 + \frac{\varepsilon}{12(\varepsilon - 1)n} + \frac{\varepsilon^2}{288(\varepsilon - 1)^2 n^2} + \mathcal{O}\Big(\frac{\varepsilon^3}{n^3}\Big). \end{aligned}$$

Note again that there are cancellations, in $T_1$ of the terms $m\ln(m)$ and $\ln(2\pi)/2$.

Now we turn to the integral part. We obtain, for instance, using (8),

$$\begin{aligned} \kappa_2 &= \varepsilon + \frac{4}{3}\varepsilon^2 + \frac{13}{9}\varepsilon^3 + \mathcal{O}(\varepsilon^4), \\ \kappa_3 &= \mathcal{O}(\varepsilon), \\ \theta &= \frac{1}{\sqrt{n}} \sum_1^{\infty} a_i u^i, \\ a_1 &= \frac{1}{\sqrt{\varepsilon}} \left[1 - \frac{1}{6}\varepsilon^2 - \frac{1}{72}\varepsilon^4 + \mathcal{O}(\varepsilon^6)\right]. \end{aligned}$$

Again, we choose $\theta_0$ such that $m\kappa_2\theta_0^2 \to \infty, m\kappa_3\theta_0^3 \to 0, n \to \infty$. We choose here $\theta_0 = 1/n^{5\alpha/12}$ and $|u| < \sqrt{n\varepsilon}\theta_0 = n^{\alpha/12}$. Integrating (the justification of the integration procedure is given in Appendix 5.2), this gives

$$\begin{aligned} H_3 &= \frac{1}{\sqrt{2\pi} n^{\alpha/2}} \left[H_{31} + \frac{H_{32}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha})\right], \\ H_{31} &= 1 - \frac{1}{6}\varepsilon - \frac{1}{72}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\ H_{32} &= -\frac{1}{12} + \frac{1}{72}\varepsilon - \frac{71}{864}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Now we compute

$$\left\{ {n \atop m} \right\} = e^{T1} H_1 H_2 H_3 = e^{T_1} R, \tag{14}$$

with multiseries expansions. $R$ is expanded in decreasing powers of $n$:

$$R = \frac{1}{\sqrt{2\pi} n^{\alpha/2}} \left[ R_0 + \frac{R_1}{n} + \frac{R_2}{n^2} + \mathcal{O}(1/n^3) \right].$$

The coefficients are then expanded in decreasing powers of $n^\alpha$:

$$\begin{aligned} R_0 &= R_{00} + \frac{R_{01}}{n^\alpha} + \mathcal{O}(1/n^{2\alpha}), \\ R_1 &= R_{10} + \frac{R_{11}}{n^\alpha} + \mathcal{O}(1/n^{2\alpha}), \\ R_2 &= R_{20} + \frac{R_{21}}{n^\alpha} + \mathcal{O}(1/n^{2\alpha}), \end{aligned}$$

and finally, we have expansions in $\varepsilon$:

$$\begin{aligned} R_{00} &= 1 + \frac{1}{3}\varepsilon + \mathcal{O}(\varepsilon^2), & R_{01} &= -\frac{1}{12} - \frac{1}{36}\varepsilon + \mathcal{O}(\varepsilon^2), \\ R_{10} &= -\frac{1}{12}\varepsilon - \frac{1}{9}\varepsilon^2 + \mathcal{O}(\varepsilon^3), & R_{11} &= \frac{1}{144}\varepsilon + \frac{1}{108}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\ R_{20} &= \frac{1}{288}\varepsilon + \frac{7}{864}\varepsilon^2 + \mathcal{O}(\varepsilon^3), & R_{21} &= -\frac{1}{3456}\varepsilon - \frac{7}{10368}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Given some desired precision, how many terms must we use in our expansions? It depends on $\alpha$. For instance, in $T_1$, $n^\alpha \varepsilon^k \gg 1$ if $k < \alpha/(1-\alpha)$. For instance, if $\alpha = 3/4$, we must use all powers $\varepsilon^k$ in $T_{10}$ such that $k < 3$. Also $\varepsilon^k$ in $R_{00}$ is less than $\varepsilon^\ell/n$ in $R_{10}/n$ if $k - \ell > 1/(1-\alpha)$. For instance, if $\alpha = 4/6$, and if we use $k = 5$ in $R_{00}$, we must use $\ell < 5 - 3 = 2$ in $R_{10}$. Any number of terms can be computed by almost automatic computer algebra. We use Maple in this paper. □

To check the quality of our asymptotic, we have chosen $n = 100$ and a range $\alpha \in [1/2, 9/10]$, i.e. a range $m \in [37, 90]$. We use 5 or 6 terms in our final expansions. Figure 4 shows the quotient $\left\{ {n \atop m} \right\} \Big/ (e^{T_1} R)$ (without error terms). The precision is at least 0.0066. Note that the range $[M - 3\sigma, M + 3\sigma]$, where the Gaussian approximation is useful, is here $m \in [21, 36]$.

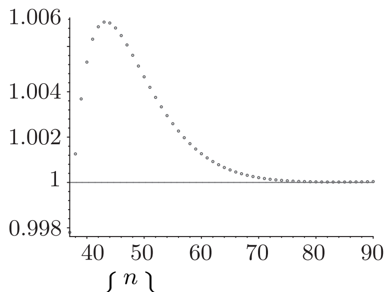


Figure 4. $\dfrac{\left\{ {n \atop m} \right\}}{e^{T_1} R}$ (without error terms)

We finally mention that our non-central range is not sacred: other types of ranges can be analyzed with similar methods.

## 4. A MORE EXPLICIT MOSER AND WYMAN'S ASYMPTOTIC ESTIMATE RELATED TO EXPRESSION (3)

Let us recall MOSER and WYMAN's two asymptotic approximations . The first one is of the form (see (2), only first-term for simplicity)

$$\left\{ {n \atop n-t} \right\} \sim \binom{n}{t} \left( \frac{n-t}{2} \right)^t, \tag{15}$$

uniformly when $1 \le t = o(\sqrt{n})$, and is simple and explicit. The second one (see (3)) is

$$\left\{ {n \atop n-t} \right\} \sim \frac{e^\rho - 1}{\sqrt{2\pi\rho(n-t)e^\rho(e^\rho - 1 - \rho)}} \frac{n!}{(n-t)!} \rho^{-n}(e^\rho - 1)^{n-t}, \tag{16}$$

where $\rho > 0$ solves the Saddle-point equation

$$1 - \frac{1 - e^{-\rho}}{\rho} = \frac{t}{n} =: \beta, \tag{17}$$

which can be written in the standard form

$$\rho = \beta\phi(\rho), \text{ where } \phi(\rho) := \frac{\rho^2}{e^{-\rho} - 1 - \rho} \tag{18}$$

Note that $\beta$ plays here the role of $\varepsilon$ in Sec. 3

We focus on the range $\beta = o(1)$. In this case, we can use MOSER and WYMAN's second approximation (16) to extend the range of $t$ to that considered in Sec. 3 ($t$ corresponds to $n^\alpha$ there). Since $\beta = t/n = o(1)$, we have, by Lagrange's inversion formula,

$$\rho = \sum_{j \ge 1} c_j \beta^j, \text{ where } c_j = \frac{1}{j}[t^{j-1}]\phi(t)^j \qquad (j \ge 1),$$

and the first few terms are given by (see (13))

$$\rho = 2\beta + \frac{4}{3}\beta^2 + \frac{10}{9}\beta^3 + \frac{136}{135}\beta^4 + \frac{386}{405}\beta^5 + \frac{524}{467}\beta^6 + \cdots \tag{19}$$

With this $\rho$ , we then deduce that

$$\left\{ {n \atop n-t} \right\} \sim \frac{1}{\sqrt{2\pi t}} \frac{n!}{(n-t)!} \rho^{-n}(e^\rho - 1)^{n-t},$$

with an error of the form $\mathcal{O}(1/t)$. Consider now

$$\frac{\left\{ {n \atop n-t} \right\}}{\binom{n}{t}\left(\frac{n-t}{2}\right)^t} \sim \frac{t!2^t}{\sqrt{2\pi t}(n-t)^t} \rho^{-n}(e^\rho - 1)^{n-t} \sim e^{nF(\beta)},$$

where (by Stirling's formula, (17) and (18))

$$\begin{aligned} F(\beta) &= \beta\log\beta - \beta + \beta\log 2 - \beta\log(1-\beta) - \log\rho + (1-\beta)\log(e^\rho - 1) \\ &= -(1-\log 2)\beta + (1-2\beta)\log(1-\beta) - \beta\log\phi(\rho) + (1-\beta)\rho. \end{aligned}$$

Now (with $c_0 := 0$)

$$\begin{aligned} -\beta + (1-2\beta)\log(1-\beta) &= -2\beta + \sum_{j\ge 2}\frac{j+1}{j(j-1)}\beta^j, \\ (1-\beta)\rho &= \sum_{j\ge 1}(c_j - c_{j-1})\beta^j, \\ (\log 2)\beta - \beta\log\phi(\rho) &= -\sum_{j\ge 2} d_{j-1}\beta^j, \end{aligned}$$

where, by another application of Lagrange's inversion formula,

$$d_j := \frac{1}{j}[t^j]\phi(t)^j \qquad (j \ge 2).$$

We thus conclude that

$$F(\beta) = \sum_{j\ge 2} a_j\beta^j, \text{ where } a_j = \frac{j+1}{j(j-1)} + c_j - c_{j-1} - d_{j-1} \quad (j \ge 2).$$

So we obtain

$$\frac{\left\{ {n \atop n-t} \right\}}{\binom{n}{t}\left(\frac{n-t}{2}\right)^t} \sim \exp\left(\sum_{j\ge 2} a_j \frac{t^j}{n^{j-1}}\right) = \exp\left(\frac{t^2}{6n} + \frac{t^3}{9n^2} + \frac{131t^4}{1620n^3} + \frac{5t^5}{81n^4} + \cdots\right)$$

and we are in the typical situation of moderate deviations. This approximation holds uniformly when $t \to \infty$ and $t = o(n)$ because we start with (16). But the restriction that $t \to \infty$ can be dropped due to (15). Also one can use Stirling's formula to derive alternative approximations. For example, if $t = o(n^{1/2})$, then $nF(\beta) = o(1)$ and thus

$$\left\{ {n \atop n-t} \right\} \sim \binom{n}{t}\left(\frac{n-t}{2}\right)^t;$$

if $t = o(n^{2/3})$, then

$$\left\{ {n \atop n-t} \right\} \sim \binom{n}{t}\left(\frac{n-t}{2}\right)^t \exp\left(\frac{t^2}{6n}\right);$$

if $t = o(n^{3/4})$, then

$$\left\{ {n \atop n-t} \right\} \sim \binom{n}{t}\left(\frac{n-t}{2}\right)^t \exp\left(\frac{t^2}{6n} + \frac{t^3}{9n^2}\right);$$

and so on. In general, $t = o(n^{\ell/(\ell+1)})$, where $\ell \geq 1$, then

$$\left\{ {n \atop n-t} \right\} \sim \binom{n}{t} \left(\frac{n-t}{2}\right)^t \exp\left(\sum_{2\leq j\leq \ell} a_j \frac{t^j}{n^{j-1}} + o(1)\right).$$

All error terms can be further refined if needed.

If we compare these results with Theorem 3.2, we have the following differences: here we have a combinatorial term $\binom{n}{t}$ and a power $\left(\frac{n-t}{2}\right)^t$. We think that Theorem 3.2 is more detailed: it depends explicitly on $n^\alpha, \varepsilon = n^{1-\alpha}$ and $\alpha$.

## 5. APPENDIX. JUSTIFICATION OF THE INTEGRATION PROCEDURE

### 5.1. The central region.

Recall that $m \sim n/L, \quad \rho \sim L, \quad \kappa_2 \sim L$. Set

$$G(\theta) = \frac{n}{L} \ln\left[e^{\rho e^{\mathbf{i}\theta}} - 1\right] - n\ln(\rho e^{\mathbf{i}\theta}).$$

We must analyze

$$\begin{aligned} &\operatorname{Re}\left(\ln[e^{\rho e^{\mathbf{i}\theta}} - 1]\right) - \ln\left[e^\rho - 1\right] \\ &= \frac{1}{2}\ln\left[\left[e^{\rho\cos(\theta)}\cos(\rho\sin(\theta)) - 1\right]^2 + \left[e^{\rho\cos(\theta)}\sin(\rho\sin(\theta))\right]^2\right] \\ &\quad - \ln\left[e^\rho - 1\right] = \frac{1}{2}\ln[h(\theta)], \\ h(\theta) &= \frac{e^{2\rho\cos(\theta)} - 2e^{\rho\cos(\theta)}\cos(\rho\sin(\theta)) + 1}{e^{2\rho} - 2e^\rho + 1}. \end{aligned}$$

But $h(\theta)$ has a peak at $\theta = 0$. This is proved as follows. Recall that $\rho = \Omega(1)$. We have

$$h'(\theta) = -2\rho e^{\rho\cos(\theta)}\left[\sin(\theta)e^{\rho\cos(\theta)} - \sin(\theta + \rho\sin(\theta))\right].$$

We divide the interval $[0, \pi]$ into 4 subintervals:

$$[0, \varepsilon_1], [\varepsilon_1, \pi/2 - \varepsilon_2], [\pi/2 - \varepsilon_2, \pi/2 + \varepsilon_2], [\pi/2 + \varepsilon_2, \pi].$$

- Choose $\varepsilon_1$ small enough such that $\rho\varepsilon_1 = o(1)$. For $\theta \leq \varepsilon_1$, we have

  $$h'(\theta) \sim -2\rho e^\rho\left(e^\rho - 1 - \rho\right)\theta < 0,$$

  and

  $$h'(\varepsilon_1) \sim -2\rho e^{2\rho}\varepsilon_1, \quad |h'(\varepsilon_1)| = \Omega(1).$$

- Choose $\varepsilon_2$ small enough such that $\rho\varepsilon_2 = \Omega(1)$. At $\theta = \pi/2 - \varepsilon_2$, we have

$$h'(\pi/2 - \varepsilon_2) \sim -2\rho e^{\rho\varepsilon_2}\left[e^{\rho\varepsilon_2} + O(1)\right] \gg h'(\varepsilon_1).$$

For $\theta \in [\varepsilon_1, \pi/2 - \varepsilon_2]$, we can approximate $h'(\theta)$ by $-2\rho\sin(\theta)e^{2\rho\cos(\theta)}$ which possesses a minimum at $\theta_1^* \sim \frac{1}{\sqrt{2\rho}}$ and

$$h'(\theta_1^*) \sim -\sqrt{2\rho}e^{2\rho} \ll h'(\varepsilon_1).$$

Finally we have

$$\int_{\varepsilon_1}^{\pi/2-\varepsilon_2} h'(\theta)\mathrm{d}\theta \leq -[\pi/2 - \varepsilon_2]2\rho e^{2\rho\varepsilon_2} =: A \text{ say}.$$

- At $\theta = \pi/2 + \varepsilon_2$, we have

$$h'(\pi/2 + \varepsilon_2) \sim -2\rho e^{-\rho\varepsilon_2}\left[e^{-\rho\varepsilon_2} + \mathcal{O}(1)\right]$$

which of course can be positive. Note that

$$h'(\pi/2) \sim -2\rho[1 - \sin(\pi/2 + \rho)] \leq 0.$$

For $\theta \in [\pi/2 - \varepsilon_2, \pi/2 + \varepsilon_2]$, $h'(\theta)$ possesses a positive maximum at $\pi/2 + \theta_2^*$ that can be computed as follows. We must maximize

$$-2\rho e^{-\rho\varepsilon}\left[e^{-\rho\varepsilon} - 1\right].$$

This gives $\theta_2^* = \frac{\ln(2)}{\rho} \ll \theta_2$ and $h'(\theta_2^*) \sim \rho/2$. Therefore

$$\int_{\pi/2-\varepsilon_2}^{\pi/2+\varepsilon_2} h'(\theta)\mathrm{d}\theta \leq \rho\varepsilon_2 \ll |A|.$$

- For $\theta \in [\pi/2 + \varepsilon_2, \pi]$, we have

$$h'(\theta) \leq 2\rho e^{-\rho\varepsilon_2}$$

and

$$\int_{\pi/2+\varepsilon_2}^{\pi} h'(\theta)\mathrm{d}\theta \leq 2\rho e^{-\rho\varepsilon_2}\pi/2 \ll |A|.$$

So finally $h(\theta)$ has a peak at $\theta = 0$.
Note that $\frac{1}{2}\ln[h(\theta)] \sim -\frac{1}{2}L\theta^2$ which conforms to (7) with $\kappa_2 \sim L$.
So

$$|K_n| = \mathcal{O}\left(\exp[-Cn\theta_0^2]\right) = \mathcal{O}\left(\exp[-CL^2]\right)$$

for some $C > 0$. The tail integral is exponentially small. The tails completion is immediate.

### 5.2. The non-central region

Recall that $\varepsilon = n^{\alpha-1}$, $\quad m = n(1-\varepsilon)$, $\quad \rho \sim 2\varepsilon$. Set

$$\begin{aligned} G(\theta) &= n(1-\varepsilon)\ln\left[e^{\rho e^{\mathbf{i}\theta}} - 1\right] - n\ln(\rho e^{\mathbf{i}\theta}) \\ &= n(1-\varepsilon)\ln\left[\rho e^{\mathbf{i}\theta} + \rho^2 e^{2\mathbf{i}\theta}/2 + \ldots\right] - n\ln(\rho e^{\mathbf{i}\theta}) \\ &= n(1-\varepsilon)\left[\ln\left[\rho e^{\mathbf{i}\theta}\right] + \ln\left[1 + \rho e^{\mathbf{i}\theta}/2 + \ldots\right]\right] - n\ln(\rho e^{\mathbf{i}\theta}) \\ &= -\varepsilon n[\ln(\rho) + \mathbf{i}\theta] + n(1-\varepsilon)\rho e^{\mathbf{i}\theta}/2 + \mathcal{O}(n\rho^2 e^{2\mathbf{i}\theta}). \end{aligned}$$

So

$$\mathrm{Re}[G(\theta) - G(0)] \sim n\rho[\cos(\theta) - 1]/2$$

which is unimodal with peak at 0. Note that $n\rho[\cos(\theta) - 1]/2 \sim -n^\alpha\theta^2/2$ which conformes to (7) with $\kappa_2 \sim \varepsilon$.

Now

$$|K_n| = \mathcal{O}\left(\exp[-Cn^\alpha\theta_0^2]\right) = \mathcal{O}\left(\exp[-Cn^{\alpha/6}]\right)$$

for some $C > 0$. The tail integral is exponentially small. The tails completion is immediate.

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