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ASYMPTOTIC EXPANSION OF THE SOLUTION FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM WITH BOUNDARY JUMPS

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Abstract. We consider the two-point integral boundary value problem with boundary jumps for third order linear integro-differential equation with the small parameter at two highest derivatives. An asymptotic expansion of the solution of the integral boundary value problem with any degree of accuracy with respect to a small parameter was constructed. A justification of the asymptotic is provided.

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1. INTRODUCTION

Singularly perturbed equations act as mathematical models in many applied problems related to diffusion, heat and mass transfer, chemical kinetics and combustion, heat propagation in thin bodies, semiconductor theory, gyroscope motion, quantum mechanics, biology and biophysics, and many other branches of science and technology. There are a number of introductory as well as advanced books on asymptotic methods that discuss some aspects of different perturbation techniques and their applications. Among them, let us mention, e.g., the monographs and textbooks by Thikhonov [22], Chang and Howes [3], Kevorkian and Cole [14], Murdock [15], Hinch [9], Wasow [27], Vishik and Lyusternik [26], Bogoliubov and Mitropolskii [2], O'Malley [17], Van Dyke [23], Nayfeh [16], Smith [21], Eckhaus [8], Hoppensteadt [10], Sanders [20], Vasil'eva and Bytyzov [24, 25] and others.

We should mention that the boundary function, although widely used, is not universal. It is applied most successfully to problems whose solutions exhibit boundary

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layer types of behavior. There exist problems where this method does not work. However, for those problems to which this method can be applied, it works in a most effective and easy way, which makes it advantageous as compared with the other methods. The boundary function method makes it possible to construct the asymptotic solution in a form that provides a uniform asymptotic approximation for the solution of the original problem over the domain of interest and allows one to estimate the remainder term, which justifies the algorithm. Conditions that allow one to verify for each particular problem, whether the boundary function method can or cannot be used to obtain the asymptotic solution are discussed in the book [24,25] for quite general classes of ordinary and partial differential equations. Initial and boundary value problems with initial jumps for singularly perturbed ordinary and integro-differential equations were considered [4,5,11–13]. Singularly perturbed differential equations with piece-wise constant argument of generalized type were considered [1]. Boundary-value problems by using parametrization were investigated in [18, 19].

In the articles [6,7] local and integral boundary value problems for singularly perturbed integro-differential equations were considered, showing phenomena of initial jumps at both boundaries of a given interval. For example, the solutions of the boundary value problems have at both points $t = 0$ and $t = 1$ phenomena of initial jumps respectively of the first and zero orders. Such boundary value problems are called boundary value problems with boundary jumps. In these papers asymptotic estimates of solutions were obtained and in the case of a local boundary value problem the passage to the limit from the solution of assumed singularly perturbed problem to the solution of the corresponding degenerate problem was described. An a similar way the passage to the limits can be shown with some changes in the case of integral boundary value problems. However, we note that these passages to the limit are non-uniform and have $O(\varepsilon)$ degree of accuracy with respect to the small parameter.

Therefore, the aim of this work is to construct a uniform asymptotic expansion of the solutions for the integral boundary value problem with any degree of accuracy with respect to the small parameter. The scientific novelty lies in the fact that a modification of the method of boundary functions to boundary value problems with boundary jumps is proposed. Here the orders of the initial jumps at the ends of the considered segment are taken into account.

2. MAIN RESULTS

Consider the singularly perturbed integro-differential equation

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = F(t) + \sum_{i=0}^{1} \int_0^1 H_i(t,x)y^{(i)}(x,\varepsilon)dx, \quad (2.1)$$

with integral boundary conditions

$$y(0,\varepsilon)=\alpha,\quad y'(0,\varepsilon)=\beta,\quad y(1,\varepsilon)=\gamma+\sum_{i=0}^{1}\int_0^1 a_i(x)y^{(i)}(x,\varepsilon)dx, \tag{2.2}$$

where $\varepsilon>0$ is a small parameter, α,β,γ are known constants independent of ε.

We will need the following assumptions:

C1) $A_i(t)$, $F(t)$, $a_j(x)\in C^{N+3}[0,1]$, $i=\overline{0,2}$, $j=0,1$, $H_0(t,x), H_1(t,x)$ are sufficiently smooth functions defined in the domain $D=\{0\le t\le 1, 0\le x\le 1\}$ and also $H_1(t,1)\neq 0$.

C2) The roots $\mu_i(t)$, $i=1,2$ of "the additional characteristic equation" $\mu^2+A_0(t)\mu+A_1(t)=0$ satisfy the inequalities $\mu_1(t)<-\gamma_1<0$, $\mu_2(t)>\gamma_2>0$.

C3) $a_1(1)\neq 1$.

In the work [19] a theorem was proved about asymptotic estimates of the solution for the problem (2.1), (2.2).

In this paper, we will construct uniformly asymptotic expansion of the boundary value problem (2.1), (2.2) on the interval $0\le t\le 1$. We will look for the asymptotic expansion of the solution of the problem (2.1), (2.2):

$$y(t,\varepsilon)=y_\varepsilon(t)+\varepsilon w_\varepsilon(\tau_1)+v_\varepsilon(\tau_2),\quad \tau_1=\frac{t}{\varepsilon},\ \ \tau_2=\frac{t-1}{\varepsilon}, \tag{2.3}$$

where

$$y_\varepsilon(t)=y_0(t)+\varepsilon y_1(t)+\varepsilon^2 y_2(t)+\dots, \tag{2.4}$$

$$w_\varepsilon(\tau_1)=w_0(\tau_1)+\varepsilon w_1(\tau_1)+\varepsilon^2 w_2(\tau_1)+\dots, \tag{2.5}$$

$$v_\varepsilon(\tau_2)=v_0(\tau_2)+\varepsilon v_1(\tau_2)+\varepsilon^2 v_2(\tau_2)+\dots, \tag{2.6}$$

(2.4) is called the regular part of the asymptotic and (2.5), (2.6) are called the boundary layer parts of the asymptotic.

Substituting (2.3) into the equation (2.1), we obtain the following equality

$$\begin{aligned}
&\varepsilon^2\left(y_\varepsilon'''(t)+\frac{1}{\varepsilon^2}\dddot{w}_\varepsilon(\tau_1)+\frac{1}{\varepsilon^3}\dddot{v}_\varepsilon(\tau_2)\right)+\varepsilon A_0(t)\left(y_\varepsilon''(t)+\frac{1}{\varepsilon}\ddot{w}_\varepsilon(\tau_1)+\frac{1}{\varepsilon^2}\ddot{v}_\varepsilon(\tau_2)\right)\\
&\quad+A_1(t)\left(y_\varepsilon'(t)+\dot{w}_\varepsilon(\tau_1)+\frac{1}{\varepsilon}\dot{v}_\varepsilon(\tau_2)\right)+A_2(t)\left(y_\varepsilon(t)+\varepsilon w_\varepsilon(\tau_1)+v_\varepsilon(\tau_2)\right)\\
&=F(t)+\int_0^1\sum_{i=0}^{1}H_i(t,x)\left(y_\varepsilon^{(i)}(x)+\varepsilon^{1-i}\overset{(i)}{w}_\varepsilon\left(\frac{x}{\varepsilon}\right)+\frac{1}{\varepsilon^i}\overset{(i)}{v}_\varepsilon\left(\frac{x-1}{\varepsilon}\right)\right)dx.
\end{aligned} \tag{2.7}$$

We make the substitution $\frac{x}{\varepsilon}=s_1,\ \frac{x-1}{\varepsilon}=s_2$. Then the expression in integral of the right-hand side of (2.7) and the integral expressions have the form

$$J_1(t,\varepsilon)=\int_0^{\frac{1}{\varepsilon}}\sum_{i=0}^{1}\varepsilon^{2-i}H_i(t,\varepsilon s_1)\overset{(i)}{w}_\varepsilon(s_1)ds_1 \tag{2.8}$$
$$=\int_0^{\infty}\sum_{i=0}^{1}\varepsilon^{2-i}H_i(t,\varepsilon s_1)\overset{(i)}{w}_\varepsilon(s_1)ds_1-\int_{\frac{1}{\varepsilon}}^{\infty}\sum_{i=0}^{1}\varepsilon^{2-i}H_i(t,\varepsilon s_1)\overset{(i)}{w}_\varepsilon(s_1)ds_1,$$

$$J_2(t,\varepsilon)=\int_{-\frac{1}{\varepsilon}}^{0}\sum_{i=0}^{1}\varepsilon^{1-i}H_i(t,\varepsilon s_2+1)\overset{(i)}{v}_\varepsilon(s_2)ds_2 \tag{2.9}$$
$$=\int_{-\infty}^{0}\sum_{i=0}^{1}\varepsilon^{1-i}H_i(t,\varepsilon s_2+1)\overset{(i)}{v}_\varepsilon(s_2)ds_2-\int_{-\infty}^{-\frac{1}{\varepsilon}}\sum_{i=0}^{1}\varepsilon^{1-i}H_i(t,\varepsilon s_2+1)\overset{(i)}{v}_\varepsilon(s_2)ds_2.$$

The improper integrals in (2.8), (2.9) converge and the second term in (2.8), (2.9) vanishes, because $O\left(e^{\left(-\gamma_1\frac{t}{\varepsilon}\right)}\right)$, $O\left(e^{\left(-\gamma_2\frac{1-t}{\varepsilon}\right)}\right)$ are less than any power of ε, as $\varepsilon\to 0$. Writing separately for coefficients depending on t and on $\tau_1,\ \tau_2$, we get the equations for functions $y_\varepsilon(t),\ w_\varepsilon(\tau_1),\ v_\varepsilon(\tau_2)$:

$$\begin{aligned}&\varepsilon^2y_\varepsilon'''(t)+\varepsilon A_0(t)y_\varepsilon''(t)+A_1(t)y_\varepsilon'(t)+A_2(t)y_\varepsilon(t)\\&=F(t)+\int_0^1\sum_{i=0}^{1}H_i(t,x)y_\varepsilon^{(i)}(x)dx+\int_0^{\infty}\sum_{i=0}^{1}\varepsilon^{2-i}H_i(t,\varepsilon s_1)\overset{(i)}{w}_\varepsilon(s_1)ds_1\\&\quad+\int_{-\infty}^{0}\sum_{i=0}^{1}\varepsilon^{1-i}H_i(t,\varepsilon s_2+1)\overset{(i)}{v}_\varepsilon(s_2)ds_2,\end{aligned} \tag{2.10}$$

$$\dddot{w}_\varepsilon(\tau_1)+A_0(\varepsilon\tau_1)\ddot{w}_\varepsilon(\tau_1)+A_1(\varepsilon\tau_1)\dot{w}_\varepsilon(\tau_1)+\varepsilon A_2(\varepsilon\tau_1)w_\varepsilon(\tau_1)=0, \tag{2.11}$$

$$\dddot{v}_\varepsilon(\tau_2)+A_0(\varepsilon\tau_2+1)\ddot{v}_\varepsilon(\tau_2)+A_1(\varepsilon\tau_2+1)\dot{v}_\varepsilon(\tau_2)+\varepsilon A_2(\varepsilon\tau_2+1)v_\varepsilon(\tau_2)=0. \tag{2.12}$$

Let us expand the functions $A_i(\varepsilon\tau_1),\ i=0,1,2$ and $H_j(t,\varepsilon s_1),\ j=0,1$ into Taylor power series in ε in a neighborhood of the point 0:

$$A_i(\varepsilon\tau_1)=A_i(0)+\varepsilon\tau_1A_i'(0)+\frac{(\varepsilon\tau_1)^2}{2!}A_i''(0)+\ldots+\frac{(\varepsilon\tau_1)^k}{k!}A_i^{(k)}(0)\ldots,\quad i=0,1,2,$$

$$H_j(t,\varepsilon s_1)=H_j(t,0)+\varepsilon s_1H_j'(t,0)+\ldots+\frac{(\varepsilon s_1)^k}{k!}H_j^{(k)}(t,0)\ldots,\quad j=0,1. \tag{2.13}$$

Expand the functions $A_i(\varepsilon\tau_2+1),\ i=0,1,2$ and $H_j(t,\varepsilon s_2+1),\ j=0,1$ into Taylor power series in ε in a neighborhood of the point 1:

$$A_i(\varepsilon\tau_2+1)=A_i(1)+\varepsilon\tau_2 A_i'(1)+\frac{(\varepsilon\tau_2)^2}{2!}A_i''(1)+\ldots+\frac{(\varepsilon\tau_2)^k}{k!}A_i^{(k)}(1)\ldots,\ \ i=0,1,2,$$

$$H_j(t,\varepsilon s_2+1)=H_j(t,1)+\varepsilon s_2 H_j'(t,1)+\ldots+\frac{(\varepsilon s_2)^k}{k!}H_j^{(k)}(t,1)\ldots,\ \ j=0,1. \quad (2.14)$$

Substituting the formulas (2.4)-(2.6), (2.13), (2.14) into (2.10)-(2.12) and equating the coefficients at the corresponding powers of ε on both sides of (2.10)-(2.12), we obtain the sequence of equations for the functions $y_k(t),\ w_k(\tau_1),\ v_k(\tau_2)$.

For $y_k(t),\ \ k=0,1,\ldots$ we have the linear integro-differential equations

$$A_1(t)y_k'(t)+A_2(t)y_k(t)=F_k(t)+\int_0^1\sum_{i=0}^{1}H_i(t,x)y_k^{(i)}(x)dx+\Delta_k(t), \quad (2.15)$$

where $F_0(t)=F(t)$,

$$\begin{aligned}F_k(t)=&\int_0^\infty\left(\sum_{i=0}^{k-2}\frac{s_1^i}{i!}H_0^{(i)}(t,0)w_{k-2-i}(s_1)+\sum_{i=0}^{k-1}\frac{s_1^i}{i!}H_1^{(i)}(t,0)\dot w_{k-1-i}(s_1)\right)ds_1\\&+\int_{-\infty}^0\left(\sum_{i=0}^{k-1}\frac{s_2^i}{i!}H_0^{(i)}(t,1)v_{k-1-i}(s_2)+\sum_{i=0}^{k-1}\frac{s_2^{i+1}}{(i+1)!}H_1^{(i+1)}(t,1)\dot v_{k-1-i}(s_2)\right)ds_2\\&-y_{k-2}'''(t)-A_0(t)y_{k-1}''(t),\ \ k=1,2,\ldots\end{aligned}$$

$$\Delta_k(t)=\int_{-\infty}^0 H_1(t,1)\dot v_k(s_2)ds_2,\qquad k=0,1,2,\ldots \quad (2.16)$$

The values $\Delta_k(t),\ \ k=0,1,2,\ldots$ are called *the initial jumps of the integral term.*

For $w_k(\tau_1),\ \ k=0,1,2,\ldots$ we have the linear differential equations with constant coefficients

$$\dddot w_k(\tau_1)+A_0(0)\ddot w_k(\tau_1)+A_1(0)\dot w_k(\tau_1)=\Phi_k(\tau_1), \quad (2.17)$$

where $\Phi_0(\tau_1)=0$,

$$\begin{aligned}\Phi_k(\tau_1)=&-\sum_{i=1}^{k}\frac{\tau_1^{i-1}}{(i-1)!}A_2^{(i-1)}(0)w_{k-i}(\tau_1)-\sum_{i=1}^{k}\frac{\tau_1^{i}}{i!}A_1^{(i)}(0)\dot w_{k-i}(\tau_1)\\&-\sum_{i=1}^{k}\frac{\tau_1^{i}}{i!}A_0^{(i)}(0)\ddot w_{k-i}(\tau_1),\ \ k=1,2,\ldots\end{aligned} \quad (2.18)$$

For $v_k(\tau_2),\ \ k=0,1,2,\ldots$ we have the linear differential equations with constant coefficients

$$\dddot v_k(\tau_2)+A_0(1)\ddot v_k(\tau_2)+A_1(1)\dot v_k(\tau_2)=P_k(\tau_2), \quad (2.19)$$

where $P_0(\tau_2)=0$,

$$P_k(\tau_2)=-\sum_{i=1}^{k}\frac{\tau_2^{i-1}}{(i-1)!}A_2^{(i-1)}(1)v_{k-i}(\tau_2)-\sum_{i=1}^{k}\frac{\tau_2^{i}}{i!}A_1^{(i)}(1)\dot{v}_{k-i}(\tau_2)$$
$$-\sum_{i=1}^{k}\frac{\tau_2^{i}}{i!}A_0^{(i)}(1)\ddot{v}_{k-i}(\tau_2),\ \ k=1,2,\dots \tag{2.20}$$

Substituting the boundary conditions (2.2) into the asymptotic expansion (2.3), we get

$$\sum_{i=0}^{\infty}\varepsilon^i y_i(0)+\varepsilon\sum_{i=0}^{\infty}\varepsilon^i w_i(0)+\sum_{i=0}^{\infty}\varepsilon^i v_i\left(-\frac{1}{\varepsilon}\right)=\alpha, \tag{2.21}$$

$$\sum_{i=0}^{\infty}\varepsilon^i y_i'(0)+\sum_{i=0}^{\infty}\varepsilon^i \dot{w}_i(0)+\frac{1}{\varepsilon}\sum_{i=0}^{\infty}\varepsilon^i \dot{v}_i\left(-\frac{1}{\varepsilon}\right)=\beta, \tag{2.22}$$

$$\sum_{i=0}^{\infty}\varepsilon^i y_i(1)+\varepsilon\sum_{i=0}^{\infty}\varepsilon^i w_i\left(\frac{1}{\varepsilon}\right)+\sum_{i=0}^{\infty}\varepsilon^i v_i(0)=\gamma+\int_0^1\sum_{i=0}^{1}a_i(x)\left(\sum_{j=0}^{\infty}\varepsilon^j y_j^{(i)}(x)\right)dx$$
$$+\int_0^1\sum_{i=0}^{1}a_i(x)\left(\varepsilon^{1-i}\sum_{j=0}^{\infty}\varepsilon^j \overset{(i)}{w_j}\left(\frac{x}{\varepsilon}\right)\right)dx+\int_0^1\sum_{i=0}^{1}a_i(x)\left(\frac{1}{\varepsilon^i}\sum_{j=0}^{\infty}\varepsilon^j \overset{(i)}{v_j}\left(\frac{x-1}{\varepsilon}\right)\right)dx. \tag{2.23}$$

We make the substitution $\frac{x}{\varepsilon}=s_1$, $\frac{x-1}{\varepsilon}=s_2$. Then the expression in integral of the right-hand side of (2.23) and the integral expressions have the form

$$I_1(\varepsilon)=\int_0^{\frac{1}{\varepsilon}}\sum_{i=0}^{1}\varepsilon^{2-i}a_i(\varepsilon s_1)\overset{(i)}{w_\varepsilon}(s_1)ds_1 \tag{2.24}$$
$$=\int_0^{\infty}\sum_{i=0}^{1}\varepsilon^{2-i}a_i(\varepsilon s_1)\overset{(i)}{w_\varepsilon}(s_1)ds_1-\int_{\frac{1}{\varepsilon}}^{\infty}\sum_{i=0}^{1}\varepsilon^{2-i}a_i(\varepsilon s_1)\overset{(i)}{w_\varepsilon}(s_1)ds_1,$$

$$I_2(\varepsilon)=\int_{-\frac{1}{\varepsilon}}^{0}\sum_{i=0}^{1}\varepsilon^{1-i}a_i(\varepsilon s_2+1)\overset{(i)}{v_\varepsilon}(s_2)ds_2 \tag{2.25}$$
$$=\int_{-\infty}^{0}\sum_{i=0}^{1}\varepsilon^{1-i}a_i(\varepsilon s_2+1)\overset{(i)}{v_\varepsilon}(s_2)ds_2-\int_{-\infty}^{-\frac{1}{\varepsilon}}\sum_{i=0}^{1}\varepsilon^{1-i}a_i(\varepsilon s_2+1)\overset{(i)}{v_\varepsilon}(s_2)ds_2.$$

The improper integrals in (2.24), (2.25) converges and the second term in (2.24), (2.25) vanishes, because $O\left(e^{\left(-\gamma_1\frac{t}{\varepsilon}\right)}\right)$, $O\left(e^{\left(-\gamma_2\frac{1-t}{\varepsilon}\right)}\right)$ are less than any power of ε, as

$\varepsilon \to 0$. Since $\overset{(i)}{w}_k\left(\frac{1}{\varepsilon}\right)$, $\overset{(i)}{v}_k\left(-\frac{1}{\varepsilon}\right)$, $i=0,1$ are boundary layer functions, hence $\overset{(i)}{w}_k\left(\frac{1}{\varepsilon}\right) \to 0$, $\overset{(i)}{v}_k\left(-\frac{1}{\varepsilon}\right) \to 0$, $i=0,1$, $k=0,1,2\dots$ as $\varepsilon \to 0$.

Expand the functions $a_i(\varepsilon s_1)$, $i=0,1$ into Taylor power series in ε in a neighborhood of the point 0:

$$a_i(\varepsilon s_1) = a_i(0) + \varepsilon s_1 a_i'(0) + \frac{(\varepsilon s_1)^2}{2!} a_i''(0) + \ldots + \frac{(\varepsilon s_1)^k}{k!} a_i^{(k)}(0)\ldots, \quad i=0,1. \tag{2.26}$$

Expand the functions $a_i(\varepsilon s_2+1)$, $i=0,1$ into Taylor power series in ε in a neighborhood of the point 1:

$$a_i(\varepsilon s_2+1) = a_i(1) + \varepsilon s_2 a_i'(1) + \frac{(\varepsilon s_2)^2}{2!} a_i''(1) + \ldots + \frac{(\varepsilon s_2)^k}{k!} a_i^{(k)}(1)\ldots, \quad i=0,1. \tag{2.27}$$

In view of (2.24)-(2.27), equating the coefficients at the corresponding powers of ε on both sides of (2.21)-(2.23), we obtain the following conditions

$$y_0(0) = \alpha, \quad y_k(0) + w_{k-1}(0) = 0, \qquad k=1,2,\ldots \tag{2.28}$$

$$y_0'(0) + \dot{w}_0(0) = \beta, \; y_k'(0) + \dot{w}_k(0) = 0, \; k=1,2,\ldots \tag{2.29}$$

$$y_0(1) = \gamma + (a_1(1)-1)v_0(0) + \int_0^1 \sum_{i=0}^{1} a_i(x) y_0^{(i)}(x)dx; \tag{2.30}$$

$$y_k(1) = \gamma_k + (a_1(1)-1)v_k(0) + \int_0^1 \sum_{i=0}^{1} a_i(x) y_k^{(i)}(x)dx; \quad k=1,2,\ldots, \tag{2.31}$$

where

$$\gamma_k = \int_0^\infty \left(\sum_{i=0}^{k-2} \frac{s_1^i}{i!} a_0^{(i)}(0) w_{k-2-i}(s_1) + \sum_{i=0}^{k-1} \frac{s_1^i}{i!} a_1^{(i)}(0) \dot{w}_{k-1-i}(s_1) \right) ds_1 \tag{2.32}$$

$$+ \int_{-\infty}^0 \left(\sum_{i=0}^{k-1} \frac{s_2^i}{i!} a_0^{(i)}(1) v_{k-1-i}(s_2) + \sum_{i=0}^{k-1} \frac{s_2^{i+1}}{(i+1)!} a_1^{(i+1)}(1) \dot{v}_{k-1-i}(s_2) \right) ds_2, \; k=1,2,\ldots$$

Calculating the improper integrals (2.16) and using the condition $v_k(-\infty)=0$, $k=0,1,2,\ldots$, we get

$$\Delta_k(t) = \int_{-\infty}^0 H_1(t,1)\dot{v}_k(s_2)ds_2 = H_1(t,1)\left(v_k(0) - v_k(-\infty)\right) = H_1(t,1)v_k(0).$$

The function $y_0(t)$ is a solution of the following problem

$$A_1(t)y_0'(t) + A_2(t)y_0(t) = F(t) + \int_0^1 \sum_{i=0}^{1} H_i(t,x) y_0^{(i)}(x)dx + H_1(t,1)v_0(0),$$

$$y_0(0)=\alpha,\ y_0(1)=\gamma+(a_1(1)-1)v_0(0)+\int_0^1\sum_{i=0}^{1}a_i(x)y_0^{(i)}(x)dx. \tag{2.33}$$

We can find $y_0(t)$ and $v_0(0)$ from problem (2.33).

Thus, for determining the boundary functions $w_0(\tau_1)$, $v_0(\tau_1)$, we consider the case $k=0$. The differential equation (2.19) is of the third order; this equation requires three conditions. We construct the characteristic equation of

$$\mu^3+A_0(1)\mu^2+A_1(1)\mu=0. \tag{2.34}$$

The numbers $\mu_1(1)$, $\mu_2(1)$ are the roots of $\mu^2+A_0(1)\mu+A_1(1)=0$ and satisfy $\mu_1(1)<0$, $\mu_2(1)>0$, $\mu_3=0$. Then the general solution of (2.19) is

$$v_0(\tau_2)=C_1e^{\mu_1(1)\tau_2}+C_2e^{\mu_2(1)\tau_2}+C_3. \tag{2.35}$$

Using the initial condition (2.30) in (2.35) and the equality $v_0(-\infty)=0$, we get

$$C_1=C_3=0,\ \ C_2=\frac{1}{1-a_1(1)}\left(\gamma-y_0(1)+\int_0^1\sum_{i=0}^{1}a_i(x)y_0^{(i)}(x)dx\right).$$

As a result,

$$v_0(\tau_2)=\frac{1}{1-a_1(1)}\left(\gamma-y_0(1)+\int_0^1\sum_{i=0}^{1}a_i(x)y_0^{(i)}(x)dx\right)e^{\mu_2(1)\tau_2}. \tag{2.36}$$

Considering the derivative of the solution (2.36) taken twice with respect to τ_2 and estimating the value at the point $\tau_2=0$, we obtain

$$\dot v_0(0)=\frac{\mu_2(1)}{1-a_1(1)}\left(\gamma-y_0(1)+\int_0^1\sum_{i=0}^{1}a_i(x)y_0^{(i)}(x)dx\right),$$

$$\ddot v_0(0)=\frac{\mu_2^2(1)}{1-a_1(1)}\left(\gamma-y_0(1)+\int_0^1\sum_{i=0}^{1}a_i(x)y_0^{(i)}(x)dx\right). \tag{2.37}$$

The boundary layer function $v_0(\tau_2)$ is a solution of the problem

$$\dddot v_0(\tau_2)+A_0(1)\ddot v_0(\tau_2)+A_1(1)\dot v_0(\tau_2)=0,$$

$$v_0(0)=\frac{1}{1-a_1(1)}\left(\gamma-y_0(1)+\int_0^1\sum_{i=0}^{1}a_i(x)y_0^{(i)}(x)dx\right),$$

$$\dot v_0(0)=\frac{\mu_2(1)}{1-a_1(1)}\left(\gamma-y_0(1)+\int_0^1\sum_{i=0}^{1}a_i(x)y_0^{(i)}(x)dx\right),$$

$$\ddot{v}_0(0) = \frac{\mu_2^2(1)}{1-a_1(1)}\left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x) y_0^{(i)}(x)dx\right).$$

By this means, we construct the characteristic equation of (2.17)

$$\mu^3 + A_0(0)\mu^2 + A_1(0)\mu = 0. \tag{2.38}$$

The numbers $\mu_1(0)$, $\mu_2(0)$ are the roots of $\mu^2 + A_0(0)\mu + A_1(0) = 0$ and satisfy $\mu_1(0) < 0$, $\mu_2(0) > 0$, $\mu_3 = 0$. Then the general solution of (2.17) is

$$w_0(\tau_1) = C_1 e^{\mu_1(0)\tau_1} + C_2 e^{\mu_2(0)\tau_1} + C_3. \tag{2.39}$$

Using the initial condition (2.29) in (2.39) and the equality $w_0(\infty) = 0$, we get

$$C_1 = \frac{\beta - y_0'(0)}{\mu_1(0)} \quad C_2 = C_3 = 0,$$

and as a result,

$$w_0(\tau_1) = \frac{\beta - y_0'(0)}{\mu_1(0)} e^{\mu_1(0)\tau_1}. \tag{2.40}$$

Considering the derivative of the solution (2.40) taken twice with respect to τ_1 and estimating the value at the point $\tau_1 = 0$, we obtain

$$w_0(0) = \frac{\beta - y_0'(0)}{\mu_1(0)}, \quad \ddot{w}_0(0) = \mu_1(0)(\beta - y_0'(0)). \tag{2.41}$$

The boundary layer function $w_0(\tau_1)$ is a solution of the problem

$$\dddot{w}_0(\tau_1) + A_0(0)\ddot{w}_0(\tau_1) + A_1(0)\dot{w}_0(\tau_1) = 0,$$

$$w_0(0) = \frac{\beta - y_0'(0)}{\mu_1(0)}, \quad \dot{w}_0(0) = \beta - y_0'(0), \quad \ddot{w}_0(0) = \mu_1(0)(\beta - y_0'(0)).$$

Thus, the zeroth approximation of the asymptotic expansion is completely constructed.

The function $y_k(t)$ is a solution of the problem

$$A_1(t)y_k'(t) + A_2(t)y_k(t) = F_k(t) + \int_0^1 \sum_{i=0}^1 H_i(t,x) y_k^{(i)}(x)dx + H_1(t,1)v_k(0),$$

$$y_k(0) = -w_{k-1}(0),\ y_k(1) = \gamma_k + (a_1(1)-1)v_k(0) + \int_0^1 \sum_{i=0}^1 a_i(x) y_k^{(i)}(x)dx, \tag{2.42}$$

where the functions $F_k(t)$ and constants γ_k are defined by the formulas (2.16) and (2.32). We can find $y_k(t)$ and $v_k(0)$ from the problem (2.42).

Let us consider $k = 1,2,\ldots$. The general solution of (2.19) has the form

$$v_k(\tau_2) = C_1 e^{\mu_1(1)\tau_2} + C_2 e^{\mu_2(1)\tau_2} + C_3 + \int_{-\infty}^{\tau_2} K_3(\tau_2,s)P_k(s)ds, \tag{2.43}$$

where $K_3(\tau_2,s)$ is the Cauchy function, which has the form

$$K_3(\tau_2,s) = \frac{1}{A_1(1)} - \frac{1}{\mu_1(1)\,(\mu_2(1)-\mu_1(1))} e^{\mu_1(1)(\tau_2-s)} + \frac{1}{\mu_2(1)\,(\mu_2(1)-\mu_1(1))} e^{\mu_2(1)(\tau_2-s)}$$

and $P_k(s)$ is given by the formula (2.20). Using the initial condition (2.32) in (2.43) and the equality $v_k(-\infty) = 0$, we get $C_1 = C_3 = 0$,

$$C_2 = \frac{1}{1-a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^{1} a_i(x) y_k^{(i)}(x)dx \right) - \int_{-\infty}^{0} K_3(0,s)P_k(s)ds.$$

As a result,

$$v_k(\tau_2) = \left(\frac{1}{1-a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^{1} a_i(x) y_k^{(i)}(x)dx \right) - \int_{-\infty}^{0} K_3(0,s)P_k(s)ds \right) e^{\mu_2(1)\tau_2} + \int_{-\infty}^{\tau_2} K_3(\tau_2,s)P_k(s)ds. \tag{2.44}$$

Considering the derivative of the solution (2.44) taken twice with respect to τ_2 and estimating the value at the point $\tau_2 = 0$, we obtain

$$\dot{v}_k(0) = \left(\frac{1}{1-a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^{1} a_i(x) y_k^{(i)}(x)dx \right) - \int_{-\infty}^{0} K_3(0,s)P_k(s)ds \right) \mu_2(1) + \int_{-\infty}^{0} K_3'(0,s)P_k(s)ds, \tag{2.45}$$

$$\ddot{v}_k(0) = \left(\frac{1}{1-a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^{1} a_i(x) y_k^{(i)}(x)dx \right) - \int_{-\infty}^{0} K_3(0,s)P_k(s)ds \right) \mu_2^2(1) + \int_{-\infty}^{0} K_3''(0,s)P_k(s)ds. \tag{2.46}$$

From the initial value problem (2.19), (2.32), (2.45), (2.46), we can define the boundary layer functions $v_k(\tau_2)$, $k = 1,2,\dots$.

In this way, the general solution of (2.17) has the form

$$w_k(\tau_1) = C_1 e^{\mu_1(0)\tau_1} + C_2 e^{\mu_2(0)\tau_1} + C_3 + \int_{\tau_1}^{\infty} K_3(\tau_1,s)\Phi_k(s)ds, \tag{2.47}$$

where $K_3(\tau_1,s)$ is the Cauchy function of the form

$$K_3(\tau_1,s) = \frac{1}{A_1(0)} - \frac{1}{\mu_1(0)\,(\mu_2(0)-\mu_1(0))} e^{\mu_1(0)(\tau_1-s)} + \frac{1}{\mu_2(0)\,(\mu_2(0)-\mu_1(0))} e^{\mu_2(0)(\tau_1-s)}$$

and $\Phi_k(s)$ is given by the formula (2.18). Using the initial condition (2.29) in (2.47) and the equality $w_k(\infty) = 0$, we get $C_2 = C_3 = 0$,

$$C_1 = -\frac{1}{\mu_1(0)}\left(y_k'(0) - \int_0^{\infty} K_3'(0,s)\Phi_k(s)ds\right).$$

As a result,

$$w_k(\tau_1) = -\frac{1}{\mu_1(0)}\left(y_k'(0) - \int_0^{\infty} K_3'(0,s)\Phi_k(s)ds\right) e^{\mu_1(0)\tau_1} - \int_{\tau_1}^{\infty} K_3(\tau_1,s)\Phi_k(s)ds. \tag{2.48}$$

Considering the derivative of the solution (2.48) taken twice with respect to τ_1 and estimating the value at the point $\tau_1 = 0$, we obtain

$$w_k(0) = -\frac{1}{\mu_1(0)}\left(y_k'(0) - \int_0^{\infty} K_3'(0,s)\Phi_k(s)ds\right) - \int_0^{\infty} K_3(0,s)\Phi_k(s)ds, \tag{2.49}$$

$$\ddot{w}_k(0) = -\mu_1(0)\left(y_k'(0) - \int_0^{\infty} K_3'(0,s)\Phi_k(s)ds\right) - \int_0^{\infty} K_3''(0,s)\Phi_k(s)ds. \tag{2.50}$$

From the initial value problem (2.17), (2.31), (2.49), (2.50), we can define the boundary layer functions $w_k(\tau_1)$, $k = 1,2,\dots$.

Thus, the kth approximation of the asymptotic expansion is completely constructed. The following theorem is true.

Theorem 1. *Let assumptions (C1)-(C3) hold. Then for sufficiently small ε the boundary value problem (2.1), (2.2) has a unique solution on $[0,1]$ which can be expressed by the formula*

$$y(t,\varepsilon) = y_N(t,\varepsilon) + R_N(t,\varepsilon),$$

where $y_N(t,\varepsilon)$ is defined by the relation

$$y_N(t,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i y_i(t) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i(\tau_1) + \sum_{i=0}^{N+2} \varepsilon^i v_i(\tau_2), \tag{2.51}$$

and the following estimates for the remainder term $R_N(t,\varepsilon)$ are true

$$\left| R_N^{(i)}(t,\varepsilon) \right| \leq C\varepsilon^{N+1}, \ i = 0,1,2, \ 0 \leq t \leq 1,$$

where $C > 0$ is a constant independent of ε and $N \geq 0$ is an integer number.

Proof. Let us construct the Nth partial sum (2.51) of the expansion (2.3), here $N \geq 0$ is an integer number, the functions $y_i(t)$, $w_i(\tau_1)$, $v_i(\tau_2)$ are regular and boundary layer parts of the asymptotic.

Substituting (2.51) into equation (2.1), we obtain

$$\varepsilon^2 y_N''' + \varepsilon A_0(t) y_N'' + A_1(t) y_N' + A_2(t) y_N = F(t) + O(\varepsilon^{N+1}) + \int_0^1 \sum_{i=0}^{1} H_i(t,x) y_N^{(i)}(x,\varepsilon)\,dx. \tag{2.52}$$

Hence, the function $y_N(t,\varepsilon)$ satisfies equation (2.1) with the accuracy $O(\varepsilon^{N+1})$. Using (2.51) in (2.2), we obtain the conditions

$$y_N(0,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i y_i(0) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i(0) + \sum_{i=0}^{N+2} \varepsilon^i v_i\left(-\frac{1}{\varepsilon}\right) = \alpha + O(\varepsilon^{N+1}), \tag{2.53}$$

$$y_N'(0,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i y_i'(0) + \sum_{i=0}^{N+1} \varepsilon^i \dot{w}_i(0) + \frac{1}{\varepsilon} \sum_{i=0}^{N+2} \varepsilon^i \dot{v}_i\left(-\frac{1}{\varepsilon}\right) = \beta + O(\varepsilon^{N+1}), \tag{2.54}$$

$$\begin{aligned} y_N(1,\varepsilon) - \int_0^1 \sum_{i=0}^{1} a_i(x) y_N^{(i)}(x)\,dx &= \sum_{i=0}^{N} \varepsilon^i y_i(1) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i\left(\frac{1}{\varepsilon}\right) + \sum_{i=0}^{N+2} \varepsilon^i v_i(0) \\ &- \int_0^1 \Bigg[a_0(x) \left(\sum_{i=0}^{N} \varepsilon^i y_i(x) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i\left(\frac{x}{\varepsilon}\right) + \sum_{i=0}^{N+2} \varepsilon^i v_i\left(\frac{x-1}{\varepsilon}\right) \right) \\ &+ a_1(x) \left(\sum_{i=0}^{N} \varepsilon^i y_i'(x) + \sum_{i=0}^{N+1} \varepsilon^i \dot{w}_i\left(\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} \sum_{i=0}^{N+2} \varepsilon^i \dot{v}_i\left(\frac{x-1}{\varepsilon}\right) \right) \Bigg] dx = \gamma + O(\varepsilon^{N+1}), \end{aligned} \tag{2.55}$$

Since $\overset{(i)}{w}_k\left(\frac{1}{\varepsilon}\right)$, $\overset{(i)}{v}_k\left(-\frac{1}{\varepsilon}\right)$, $i = 0,1$ are boundary layer functions in (2.53)-(2.55), it follows that $\overset{(i)}{w}_k\left(\frac{1}{\varepsilon}\right) \to 0$, $\overset{(i)}{v}_k\left(-\frac{1}{\varepsilon}\right) \to 0$, $i = 0,1$, $k = 0,1,2\ldots$ as $\varepsilon \to 0$.

Let us set

$$R_N(t,\varepsilon) = y(t,\varepsilon) - y_N(t,\varepsilon) \Longrightarrow y(t,\varepsilon) = R_N(t,\varepsilon) + y_N(t,\varepsilon), \tag{2.56}$$

here $R_N(t,\varepsilon)$ is called *the remainder term of the asymptotic.*

Substituting (2.56) into (2.1), considering that the function $y_N(t,\varepsilon)$ satisfies the equation (2.52) and the boundary conditions (2.53)-(2.55), we get the problem for the remainder term $R_N(t,\varepsilon)$:

$$\varepsilon^2 R_N''' + \varepsilon A_0(t) R_N'' + A_1(t) R_N' + A_2(t) R_N = O(\varepsilon^{N+1}) + \int_0^1 \sum_{i=0}^{1} H_i(t,x) R_N^{(i)}(x,\varepsilon)\,dx,$$
$$R_N(0,\varepsilon) = O(\varepsilon^{N+1}), \quad ; R_N'(0,\varepsilon) = O(\varepsilon^{N+1}), \tag{2.57}$$
$$R_N(1,\varepsilon) = \int_0^1 \sum_{i=0}^{1} a_i(x) R_N^{(i)}(x,\varepsilon)\,dx + O(\varepsilon^{N+1}).$$

The problem (2.57) is of the same type as the problem (2.1), (2.2). By applying asymptotic estimates of the solution [19] for the solution of the problem (2.57), we obtain

$$\begin{aligned} |R_N(t,\varepsilon)| &\le C\varepsilon^{N+1} + C\varepsilon^{N+2} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+1} e^{\gamma_2 \frac{t-1}{\varepsilon}} \le C\varepsilon^{N+1}, \\ \left|R_N'(t,\varepsilon)\right| &\le C\varepsilon^{N+1} + C\varepsilon^{N+1} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N} e^{\gamma_2 \frac{t-1}{\varepsilon}}, \\ \left|R_N''(t,\varepsilon)\right| &\le C\varepsilon^{N+1} + C\varepsilon^{N} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N-1} e^{\gamma_2 \frac{t-1}{\varepsilon}}. \end{aligned} \tag{2.58}$$

The estimate of the remainder term $R_{N+2}(t,\varepsilon)$ in the interval $[0,1]$ is similar to (2.58):

$$\begin{aligned} |R_{N+2}(t,\varepsilon)| &\le C\varepsilon^{N+3} + C\varepsilon^{N+4} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+3} e^{\gamma_2 \frac{t-1}{\varepsilon}} \le C\varepsilon^{N+3}, \\ \left|R_{N+2}'(t,\varepsilon)\right| &\le C\varepsilon^{N+3} + C\varepsilon^{N+3} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+2} e^{\gamma_2 \frac{t-1}{\varepsilon}}, \\ \left|R_{N+2}''(t,\varepsilon)\right| &\le C\varepsilon^{N+3} + C\varepsilon^{N+2} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+1} e^{\gamma_2 \frac{t-1}{\varepsilon}}. \end{aligned} \tag{2.59}$$

So, the equality

$$y(t,\varepsilon) = y_N(t,\varepsilon) + R_N(t,\varepsilon) = y_{N+2}(t,\varepsilon) + R_{N+2}(t,\varepsilon) \tag{2.60}$$

holds.

From (2.60) it follows that

$$R_N^{(i)}(t,\varepsilon) = y_{N+2}^{(i)}(t,\varepsilon) - y_N^{(i)}(t,\varepsilon) + R_{N+2}^{(i)}(t,\varepsilon), \;\; i = 0,1,2. \tag{2.61}$$

Using the estimates (2.58), (2.59) for the remainder term (2.61), we obtain

$$\begin{aligned} R_N(t,\varepsilon) &= \varepsilon^{N+1} y_{N+1}(t) + \varepsilon^{N+2} y_{N+2}(t) + \varepsilon^{N+3} w_{N+2}(\tau_1) + \varepsilon^{N+4} w_{N+3}(\tau_1) \\ &\quad + \varepsilon^{N+3} v_{N+3}(\tau_2) + \varepsilon^{N+4} v_{N+4}(\tau_2) + R_{N+2}(t,\varepsilon) \\ &\Rightarrow |R_N(t,\varepsilon)| \le C\varepsilon^{N+1}, \end{aligned} \tag{2.62}$$

$$\begin{aligned} R_N'(t,\varepsilon) &= \varepsilon^{N+1} y_{N+1}'(t) + \varepsilon^{N+2} y_{N+2}'(t) + \varepsilon^{N+2} \dot{w}_{N+2}(\tau_1) + \varepsilon^{N+3} \dot{w}_{N+3}(\tau_1) \\ &\quad + \varepsilon^{N+2} \dot{v}_{N+3}(\tau_2) + \varepsilon^{N+3} \dot{v}_{N+4}(\tau_2) + R_{N+2}'(t,\varepsilon) \end{aligned}$$

$$\Rightarrow \left|R'_N(t,\varepsilon)\right| \leq C\varepsilon^{N+1}, \tag{2.63}$$

$$\begin{aligned} R''_N(t,\varepsilon) &= \varepsilon^{N+1} y''_{N+1}(t) + \varepsilon^{N+2} y''_{N+2}(t) + \varepsilon^{N+1}\ddot{w}_{N+2}(\tau_1) + \varepsilon^{N+2}\ddot{w}_{N+3}(\tau_1) \\ &\quad + \varepsilon^{N+1}\ddot{v}_{N+3}(\tau_2) + \varepsilon^{N+2}\ddot{v}_{N+4}(\tau_2) + R''_{N+2}(t,\varepsilon) \\ &\Rightarrow \left|R''_N(t,\varepsilon)\right| \leq C\varepsilon^{N+1}. \end{aligned} \tag{2.64}$$

The theorem is proved. □

3. Conclusion

The boundary value problem with boundary jumps for singularly perturbed integro-differential equation was considered. To determine the zeroth approximation of the regular part $y_0(t)$, we have problem (2.33). Problem (2.33) is different from the usual degenerate problem obtained from (1) as $\varepsilon = 0$. Here by virtue of the integral term, an additional term occurs, $\Delta(t) = H_1(t,1)v_0(0)$, called the initial jump of the integral term. The integral boundary condition is also changed: $\gamma + (a_1(1)-1)v_0(0)$ is taken instead of γ, where $(a_1(1)-1)v_0(0)$ is called the initial jump of the solution. A similar situation is observed for the kth approximation of the regular part $y_k(t)$ (see (2.42)). An algorithm of asymptotic expansion of the solution with any degree of accuracy with respect to a small parameter was constructed. The problems for determining the regular and boundary layer parts were completely obtained. A theorem about uniformly asymptotic expansion is proved.

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