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HERMITE-HADAMARD TYPE INEQUALITIES FOR UNIFORMLY CONVEX FUNCTIONS WITH RESPECT TO GEODESIC IN HADAMARD SPACE

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Abstract. In this paper, we obtain some Hermite-Hadamard type inequalities for uniformly convex functions with respect to geodesic in Hadamard space. Also, we give some application of these functions in means.

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1. INTRODUCTION AND PRELIMINARIES

Convex analysis plays an important role in many areas of mathematics such inequality, optimization and monotone operator. The Hermite-Hadamard's inequalities give us an estimate of the (integral) mean value of a continuous convex function. Moreover, equality holds in either side only for affine functions (i.e., for functions of the form $ax+b$).

Recently, much attention have given to develop various inequalities for several classes of convex functions and their generalizations using novel ideas (see [1–3,6,8–10] and the references therein). Hermite-Hadamard's inequality is given as follows: Let $f\colon I \to \mathbb{R}$ be a convex function, and let $a,b \in I$ with $a < b$. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where I is an interval (finite or infinite) in $\mathbb{R}$. Throughout the paper we denote by I^o the interior of I.

The Hermite-Hadamard inequality is expressed in terms of the concept of convexity. This concept can be generalized to a general metric space called Hadamard space. So, the inequalities related to the convex or uniformly convex functions are investigated in these spaces. In [5], the Hermite-Hadamard inequality is expressed on the

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Hadamard space for the convex functions. The purpose of this paper is to generalize the Hermite-Hadamard inequality for the uniformly convex functions on this space.

We consider the basic concepts and results, which are needed to obtain our main results. The following definitions can be found in [5, 7].

Definition 1. A geodesic is a rectifiable curve $\gamma\colon [0,1] \to N$ such that the length of $\gamma|_{[t_1,t_2]}$ is $d(\gamma(t_1),\gamma(t_2))$ for all $0 \le t_1 \le t_2 \le 1$.

Definition 2. A complete metric space $(N;d)$ is called a global NPC space (or Hadamard space) if for $x_1, x_2 \in N$ there exists a point $z \in N$ such that for each $x \in N$ we have

$$d(x,z)^2 \le \frac{1}{2}d(x,x_1)^2 + \frac{1}{2}d(x,x_2)^2 - \frac{1}{4}d(x_1,x_2)^2.$$

Remark 1. The point z occuring in the preceding definition plays the role of a midpoint between x_1 and x_2.

Remark 2. If $(N;d)$ is a global *NPC* and α, β are two geodesic arcs starting at $x \in X$, then the distance map $t \to d(\alpha(t), \beta(t))$ is a convex function.

Remark 3. The following classes of spaces are *NPC*: complete Riemannian manifolds with non-positive sectional curvature, Hilbert spaces, Bruhat Tits buildings, in particular metric trees.

Definition 3. A subset $C \subseteq B$ is called convex if for each geodesic $\gamma\colon [0,1] \to B$ joining two arbitrary points in C holds that $\gamma([0,1]) \subset C$.

2. Main results

In this section we give main results.

Let X be a vector space. For any two distinct a, b points in X we define the line segment connecting a to b by

$$\gamma(\lambda) := \{(1-\lambda)a + \lambda b : \lambda \in [0,1]\}.$$

If need to emphasise a, b we shall write $\gamma(\lambda) = \alpha_{a,b}(\lambda)$. We denote by $\alpha_{(a,b)}$ the set of all $(1-\lambda)a + \lambda b$ such that $\lambda \in [0,1]$.

Definition of uniformly convex functions found in [4,5]. In the following definition we extended it in geodesic concept. Here, we refer to more restrictive versions of convexity introduced in reference [4].

Definition 4. Let $g\colon H \longrightarrow (-\infty, +\infty]$ be proper and H be a Hilbert space. Then g is uniformly convex with modulus $\phi\colon [0,+\infty) \to [0,+\infty]$ if ϕ is increasing, ϕ vanishes only at zero and

$$g(tr + (1-t)s) + t(1-t)\phi(|r-s|) \le tg(r) + (1-t)g(s).$$

Using the above definition, we introduce the definition of uniformly convex with geodesic for a function which defined from a subset of a global *NPC* space to $\mathbb{R}$.

Definition 5. Suppose that (N,d) is a global *NPC* space, $C \subset N$ be a convex set. A function $f\colon C \subseteq N \to \mathbb{R}$ is called uniformly convex with modulus $\phi\colon [0,+\infty) \to [0,+\infty]$ with respect to γ (or simply γ-uniformly convex) if ϕ is increasing, ϕ vanishes only at 0, and also, the function $fo\gamma\colon [0,1] \to \mathbb{R}$ is uniformly convex function (concept in 4), defined as

$$f(\gamma(tr+(1-t)s))+t(1-t)\phi(|r-s|) \leq (1-t)f(\gamma(s))+tf\big(\gamma(r)\big), \tag{2.1}$$

for each $t \in [0,1]$. Where $\gamma\colon [0,1] \to C$ is geodesic and $0 \leq r,s \leq 1$.

Theorem 1. *Let (N,d) be a global NPC space, $C \subset N$ be a convex set and $f\colon C \to \mathbb{R}$ be a uniformly convex function. $\gamma\colon [0,1] \to \alpha_{(a,b)}$ is geodesic with $\gamma(0)=a, \gamma(1)=b$, then*

$$f\left(\gamma\left(\frac{1}{2}\right)\right)+\frac{1}{4}\int_0^1 \phi(t)dt \leq \int_0^1 f(\gamma(\lambda))d\lambda \leq \alpha_{(f(a),f(b))}\left(\frac{1}{2}\right)-\frac{1}{6}\phi(1). \tag{2.2}$$

Proof. Since, $\gamma\colon [0,1] \to \alpha_{(a,b)}$ is geodesic also, f is uniformly convex for $0 \leq r,s \leq 1$ we have

$$f\left(\gamma(tr+(1-t)s)\right)+t(1-t)\phi(|r-s|) \leq (1-t)f(\gamma(s))+tf(\gamma(r))$$

Let $t=\frac{1}{2}$ in (2.1), we conclude that

$$f\left(\gamma\left(\frac{r+s}{2}\right)\right)+\frac{1}{4}\phi(|r-s|) \leq \frac{f(\gamma(r))+f(\gamma(s))}{2}.$$

Also, if $s=1-r$ then

$$f\left(\gamma\left(\frac{1}{2}\right)\right)+\frac{1}{4}\phi(2r-1) \leq \frac{f(\gamma(r))+f(\gamma(1-r))}{2}.$$

By integrating with respect to r from $r=0$ to $r=1$ we obtain

$$f\left(\gamma\left(\frac{1}{2}\right)\right)+\frac{1}{4}\int_0^1 \phi(u)du \leq \int_0^1 f(\gamma(r))dr,$$

since

$$\int_0^1 f(\gamma(1-r))dr = \int_0^1 f(\gamma(r))dr,$$
$$\int_0^1 \phi(|2r-1|)dr = \int_0^1 \phi(u)du,$$
$$f\left(\gamma\left(\frac{1}{2}\right)\right)+\frac{1}{4}\int_0^1 \phi(u)du \leq \int_0^1 f(\alpha_{(a,b)}(\lambda))d\lambda.$$

For second inequality from (2.1), if $r=1, s=0$ in (2.1) we have

$$f\left(\gamma(t)\right)+t(1-t)\phi(1) \leq tf(\gamma(1))+(1-t)f(\gamma(0)).$$

Now, by integrating with respect t from $t=0$ to $t=1$ we have

$$\int_0^1 f(\gamma(t))dt + \frac{1}{6}\phi(1) \leq \frac{1}{2}f(\gamma(1)) + \frac{1}{2}f(\gamma(0)).$$

Therefore,

$$\int_0^1 f(\alpha_{(a,b)}(\lambda))d\lambda \leq \alpha_{(f(a),f(b))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1).$$

□

Remark 4. Note that the first inequality, in (2.2), is stronger than the second , i.e.

$$\begin{aligned}0 &\leq \int_0^1 f(\gamma(\lambda))d\lambda - f\left(\gamma\left(\frac{1}{2}\right)\right) - \frac{1}{4}\int_0^1 \phi(t)dt \\ &\leq \alpha_{(f(a),f(b))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1) - \int_0^1 f(\gamma(\lambda))d\lambda.\end{aligned}$$

Proof. Let $m = m(a,b)$ be midpoint a and b then

$$\begin{aligned}2\int_0^1 f(\gamma_{a,b}(\lambda))d\lambda &= 2\int_0^{\frac{1}{2}} f(\gamma_{a,b}(\lambda))d\lambda + 2\int_{\frac{1}{2}}^1 f(\gamma_{a,b}(\lambda))d\lambda \\ &= \int_0^1 f(\gamma_{a,m}(\lambda))d\lambda + \int_0^1 f(\gamma_{m,b}(\lambda))d\lambda \\ &\leq \frac{f(\gamma_{a,m}(0)) + f(\gamma_{a,m}(1))}{2} - \frac{1}{6}\phi(1) \\ &\quad + \frac{f(\gamma_{m,b}(0)) + f(\gamma_{m,b}(1))}{2} - \frac{1}{6}\phi(1) \\ &= f\left(\gamma_{a,b}\left(\frac{1}{2}\right)\right) + \frac{f(\gamma_{a,b}(0)) + f(\gamma_{a,b}(1))}{2} - \frac{1}{3}\phi(1) \\ &\leq f\left(\gamma_{a,b}\left(\frac{1}{2}\right)\right) + \frac{f(\gamma_{a,b}(0)) + f(\gamma_{a,b}(1))}{2} \\ &\quad - \frac{1}{6}\phi(1) + \frac{1}{4}\int_0^1 \phi(t)dt,\end{aligned}$$

which completes the proof. □

Proposition 1. *Let (N,d) be a global NPC space, $C \subset N$ be a convex set and $f: C \to \mathbb{R}$ a uniformly convex function, then*

$$\begin{aligned}f\left(\gamma\left(\frac{1}{2}\right)\right) &\leq \frac{1}{2}\left(f\left(\gamma\left(\frac{1}{3}\right)\right) + f\left(\gamma\left(\frac{2}{3}\right)\right)\right) - \frac{1}{4}\phi\left(\frac{1}{3}\right) \\ &\leq \frac{1}{2}\left(f\left(\gamma\left(\frac{1}{4}\right)\right) + f\left(\gamma\left(\frac{3}{4}\right)\right)\right) - \frac{7}{18}\phi\left(\frac{1}{3}\right) \\ &\leq \int_0^1 f(\gamma(t))dt - \frac{1}{4}\int_0^1 \phi(t)dt - \frac{7}{18}\phi\left(\frac{1}{3}\right)\end{aligned}$$

$$\leq \gamma_{(f(\gamma(0)),f(\gamma(1)))}\left(\frac{1}{2}\right) - \frac{1}{8}\int_0^1 \phi(t)dt - \frac{43}{72}\phi\left(\frac{1}{2}\right).$$

Proof. In veiw of the following relation

$$f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4}\int_0^1 \phi(t)dt \leq \int_0^1 f(\gamma_{(a,b)}(\lambda))d\lambda \leq \gamma_{(f(a),f(b))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1)$$

we have

$$\begin{aligned}
f\left(\gamma\left(\frac{1}{2}\right)\right) &= f\left(\gamma\left(\frac{1}{2}\left(\frac{1}{3}+\frac{2}{3}\right)\right)\right) \\
&\leq \frac{1}{2}\left(f(\gamma\left(\frac{1}{3}\right)\right) + f\left(\gamma\left(\frac{2}{3}\right)\right) - \frac{1}{4}\phi\left(\frac{1}{3}\right) \\
&= \frac{1}{2}f\left(\gamma\left(\frac{5}{6}.\frac{1}{4}+\frac{1}{6}.\frac{3}{4}\right)\right) + \frac{1}{2}f\left(\gamma\left(\frac{1}{6}\cdot\frac{1}{4}+\frac{5}{6}\cdot\frac{3}{4}\right)\right) - \frac{1}{4}\phi\left(\frac{1}{3}\right) \\
&\leq \frac{5}{12}f\left(\gamma\left(\frac{1}{4}\right)\right) + \frac{1}{12}f\left(\gamma\left(\frac{3}{4}\right)\right) - \frac{5}{72}\phi\left(\frac{1}{2}\right) + \frac{1}{12}f\left(\gamma\left(\frac{1}{4}\right)\right) \\
&\quad + \frac{5}{12}f\left(\gamma\left(\frac{3}{4}\right)\right) - \frac{5}{72}\phi\left(\frac{1}{2}\right) - \frac{1}{4}\phi\left(\frac{1}{3}\right) \\
&\leq \frac{1}{2}f\left(\gamma\left(\frac{1}{4}\right)\right) + \frac{1}{2}f\left(\gamma\left(\frac{3}{4}\right)\right) - \frac{7}{18}\phi\left(\frac{1}{3}\right) \\
&\leq \int_0^1 f(\gamma(t))\,dt - \frac{1}{4}\int_0^1 \phi(t)dt - \frac{7}{18}\phi\left(\frac{1}{3}\right) \\
&\leq \frac{1}{2}\left(f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{f(\gamma(0))+f(\gamma(1))}{2} - \frac{1}{6}\phi(1) + \frac{1}{4}\int_0^1 \phi(t)dt\right) \\
&\quad - \frac{1}{4}\int_0^1 \phi(t)dt - \frac{7}{18}\phi\left(\frac{1}{3}\right) \\
&\leq \frac{1}{2}\left(\frac{f(\gamma(0))+f(\gamma(1))}{2} - \frac{1}{4}\phi(1)\right) + \frac{f(\gamma(0))+f(\gamma(1))}{4} - \frac{1}{12}\phi(1) \\
&\quad + \frac{1}{8}\int_0^1 \phi(t)dt - \frac{1}{4}\int_0^1 \phi(t)dt - \frac{7}{18}\phi\left(\frac{1}{2}\right) \\
&= \frac{1}{2}\left(f(\gamma(0)) + f(\gamma(1))\right) - \frac{5}{24}\phi(1) - \frac{1}{8}\int_0^1 \phi(t)dt - \frac{7}{18}\phi\left(\frac{1}{3}\right) \\
&\leq \gamma_{f(\gamma(0)),f(\gamma(1))}\left(\frac{1}{2}\right) - \frac{1}{8}\int_0^1 \phi(t)dt - \frac{43}{72}\phi\left(\frac{1}{3}\right).
\end{aligned}$$

□

Lemma 1. *If $t_1,\dots,t_n \geq 0$, $\sum_{k=1}^{n} t_k = 1$, $\{x_i\}_{i=1}^{n}$ is an increasing sequence of $[0,1]$, (N,d) be a global NPC space and $f\colon C \to \mathbb{R}$ be a uniformly convex function then*

$$f\left(\gamma\left(\sum_{k=1}^{n} t_k x_k\right)\right) + \sum_{i=1}^{n-1} t_i t_{i+1}\phi(x_{i+1}-x_i) \leq \sum_{k=1}^{n} t_k f\left(\gamma(x_k)\right).$$

Proof. The proof is by induction. By the use of Definition 5 first observe that the result clearly holds for $n=2$. Assume it holds for n, $x_1 \leq x_2 \leq \cdots \leq x_{n+1}$ and $\sum_{i=1}^{n+1} t_i = 1$, $t_i \geq 0$. So by the inductive hypothesis

$$\begin{aligned}
f\left(\gamma\left(\sum_{i=1}^{n+1} t_i x_i\right)\right) &\leq \sum_{i=1}^{n-1} t_i f(\gamma(x_i)) + (t_n+t_{n+1}) f\left(\gamma\left(\frac{t_n x_n + t_{n+1}x_{n+1}}{t_n+t_{n+1}}\right)\right) \\
&\quad - t_1t_2\phi(x_2-x_1) - \cdots - t_{n-2}t_{n-1}\phi(x_{n-1}-x_{n-2}) \\
&\quad - t_{n-1}(t_n+t_{n+1})\phi\left(\frac{t_n x_n + t_{n+1}x_{n+1}}{t_n+t_{n+1}} - x_{n-1}\right) \\
&\leq \sum_{i=1}^{n-1} t_i f(\gamma(x_i)) + (t_n+t_{n+1})\left[\frac{t_n f(\gamma(x_n))}{t_n+t_{n+1}}\right. \\
&\quad \left. + \frac{t_{n+1 f(\gamma(x_{n+1}))}}{t_n+t_{n+1}} - \frac{t_n t_{n+1}}{(t_n+t_{n+1})^2}\phi(x_{n+1}-x_n)\right] \\
&\quad - \sum_{i=1}^{n-2} t_i t_{i+1}\phi(x_{i+1}-x_i) - t_{n-1}t_n\phi(x_n-x_{n-1}) \\
&\leq \sum_{i=1}^{n+1} t_i f(\gamma(x_i)) - t_n t_{n+1}\phi(x_{n+1}-x_n) - t_n t_{n-1}\phi(x_n-x_{n-1}) \\
&\quad - \sum_{i=1}^{n-2} t_i t_{i+1}\phi(x_{i+1}-x_i) \\
&= \sum_{i=1}^{n+1} t_i f(\gamma(x_i)) - \sum_{i=1}^{n} t_i t_{i+1}\phi(x_{i+1}-x_i).
\end{aligned}$$

Which completes the proof. □

Theorem 2. *Let (N,d) be a global NPC space, $C \subseteq N$ be a convex set, $f\colon C \to \mathbb{R}$ be a uniformly convex function and k,p be positive integers, then*

$$\begin{aligned}
&f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4}\int_0^1 \phi(t)dt + \frac{1}{k^p}\phi\left(\frac{1}{k^p}\right) \\
&\leq \frac{1}{k^p}\sum_{i=0}^{k^p-1} f\left(\gamma(\frac{2i+1}{2k^p})\right) + \frac{1}{4}\int_0^1 \phi(t)dt \\
&\leq \int_0^1 f(\gamma(t))dt
\end{aligned}$$

$$\leq \frac{1}{2k^p}\sum_{i=0}^{k^p-1}\left[f\left(\gamma\left(\frac{i+1}{k^p}\right)\right)+f\left(\gamma\left(\frac{i}{k^p}\right)\right)\right]-\frac{1}{6}\phi(1)$$
$$\leq \alpha_{(f(\gamma(0)),f(\gamma(1)))}\left(\frac{1}{2}\right)-\frac{1}{6}\phi(1).$$

Proof. Using Hermite-Hadamard inequality we have

$$f\left(\frac{2i+1}{2k^p}\right)+\frac{1}{4}\int_0^1\phi(t)dt \leq \int_0^1 f(\gamma_{\frac{i}{k^p},\frac{i+1}{k^p}}(t))dt$$
$$\leq \frac{f(\gamma(\frac{i}{k^p}))+f(\gamma(\frac{i+1}{k^p}))}{2}-\frac{1}{6}\phi(1),$$

if we add the above inequality from $i=0$ to $i=k^p-1$ we have

$$\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right)+\frac{k^p}{4}\int_0^1\phi(t)dt \leq \sum_{i=0}^{k^p-1}\int_0^1 f\left(\gamma_{(\frac{i}{k^p},\frac{i+1}{k^p})}\right)dt$$
$$\leq \sum_{i=0}^{k^p-1}\frac{f\left(\gamma(\frac{i}{k^p})\right)+f(\gamma(\frac{i+1}{k^p}))}{2}-\frac{k^p}{6}\phi(1).$$

Since,

$$\sum_{i=0}^{k^p-1}\int_0^1 f\left(\gamma|_{(\frac{i}{k^p},\frac{i+1}{k^p})}\right)(t)dt = k^p\int_0^1 f(\gamma(t))dt,$$

$$\frac{1}{k^p}\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right)+\frac{1}{4}\int_0^1\phi(t)dt \leq \int_0^1 f(\gamma(t))dt$$
$$\leq \frac{1}{k^p}\sum_{i=0}^{k^p-1}\frac{f\left(\gamma(\frac{i}{k^p})\right)+f(\frac{i+1}{k^p})}{2}-\frac{1}{6}\phi(1).$$

Also, in view of Lemma 1 we have

$$f\left(\gamma\left(\frac{1}{2}\right)\right)=f\left(\gamma\left(\frac{1}{k^p}\sum_{i=0}^{k^p-1}\left(\frac{2i+1}{2k^p}\right)\right)\right)$$
$$\leq \frac{1}{k^p}\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right)-\frac{1}{k^p}\phi\left(\frac{1}{k^p}\right).$$

So,

$$\frac{1}{k^p}\sum_{i=0}^{k^p-1}\frac{f\left(\gamma(\frac{i}{k^p})\right)+f\left(\frac{i+1}{k^p}\right)}{2}=\frac{1}{2k^p}\sum_{k=0}^{k^p-1}\left[f\left(\gamma\left(\frac{i}{k^p}\right)\right)+f\left(\frac{i+1}{k^p}\right)\right]$$
$$=\frac{f(\gamma(0))+f(\gamma(1))}{2}.$$

In view of the above relation the proof is completes. □

Theorem 3. *Let (N,d) be a global NPC space, $C\subset N$ a convex set and $f\colon C\to\mathbb{R}$ be a uniformly convex function, then*

$$\begin{aligned} f\left(\gamma\left(\frac{1}{2}\right)\right)+\lambda(1-\lambda)\phi\left(\frac{1}{2}\right) &\leq \ell(\lambda)\leq \int_0^1 f(\gamma(t))dt-\frac{1}{4}\int_0^1\phi(t)dt\\ &\leq L(\lambda)-\frac{1}{6}\phi(1)-\frac{1}{4}\int_0^1\phi(t)dt\\ &\leq \alpha_{(f(\gamma(0)),f(\gamma(1)))}\left(\frac{1}{2}\right)-\frac{1}{6}\phi(1)-\frac{1}{4}\int_0^1\phi(t)dt, \end{aligned}$$

where

$$\ell(\lambda)=\lambda f\left(\gamma\left(\frac{\lambda}{2}\right)\right)+(1-\lambda)f\left(\gamma\left(\frac{\lambda+1}{2}\right)\right)$$

and

$$L(\lambda)=\frac{1}{2}(f(\gamma(\lambda))+\lambda f(\gamma(0))+(1-\lambda)f(\gamma(1))).$$

Proof. In veiw of Theorem 1 we have

$$f\left(\gamma\left(\frac{1}{2}\right)\right)+\frac{1}{4}\int_0^1\phi(t)dt\leq\int_0^1 f(\alpha_{(a,b)}(\lambda))d\lambda\leq\alpha_{(f(a),f(b))}\left(\frac{1}{2}\right)-\frac{1}{6}\phi(1).$$

Now, assume that f is a uniformly convex function on C. γ is a geodesic from $\gamma(0)$ to $\gamma(\lambda)$ so

$$f\left(\gamma\left(\frac{\lambda}{2}\right)\right)+\frac{1}{4}\int_0^1\phi(t)dt\leq\int_0^1 f\left(\gamma_{[0,\lambda]}(t)\right)dt\leq\frac{f(\gamma(0))+f(\gamma(1))}{2}-\frac{\phi(1)}{6}.$$

Also, if γ is a geodesic from $\gamma(\lambda)$ to $\gamma(1)$ we have

$$f\left(\gamma\left(\frac{1+\lambda}{2}\right)\right)+\frac{1}{4}\int_0^1\phi(t)dt\leq\int_0^1 f\left(\gamma_{[\lambda,1]}(t)\right)dt\leq\frac{f(\gamma(\lambda))+f(\gamma(1))}{2}-\frac{\phi(1)}{6}.$$

Multipling above inequality in λ, $1-\lambda$ repectivly and add them we have

$$\begin{aligned} \ell(\lambda)+\frac{1}{4}\int_0^1\phi(t)dt &\leq \lambda\int_0^1 f\left(\gamma_{[0,\lambda]}(t)\right)dt+(1-\lambda)\int_0^1 f(\gamma_{[\lambda,1]})dt\\ &=\int_0^1 f(\gamma(t))dt\leq L(\lambda)-\frac{1}{6}\phi(1), \end{aligned}$$

and

$$\begin{aligned} f\left(\gamma\left(\frac{1}{2}\right)\right) &= f\left(\gamma\left(\lambda\frac{\lambda}{2}+(1-\lambda)\frac{1+\lambda}{2}\right)\right)\\ &\leq \lambda f\left(\gamma\left(\frac{\lambda}{2}\right)\right)+(1-\lambda)f\left(\gamma\left(\frac{1+\lambda}{2}\right)\right)-\lambda(1-\lambda)\phi\left(\frac{1}{2}\right)\\ &=\ell(\lambda)-\lambda(1-\lambda)\phi\left(\frac{1}{2}\right). \end{aligned}$$

Hence,

$$\begin{aligned}
f\left(\gamma\left(\frac{1}{2}\right)\right)+\lambda(1-\lambda)\phi\left(\frac{1}{2}\right) &\leq \ell(\lambda) \leq \int_0^1 f(\gamma(t))dt - \frac{1}{4}\int_0^1 \phi(t)dt \\
&\leq L(\lambda) - \frac{1}{6}\phi(1) - \frac{1}{4}\int_0^1 \phi(t)dt \\
&\leq \alpha_{(f(\gamma(0)),f(\gamma(1)))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1) - \frac{1}{4}\int_0^1 \phi(t)dt.
\end{aligned}$$

□

Theorem 4. *Let $p \in [2,+\infty)$ and γ be a geodesic. Then the following inequality holds.*

$$\begin{aligned}
&\left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{2^{-1-p}}{p+1}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \\
&\leq \left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{1}{4}\int_0^1 \phi(t)dt \\
&\leq \int_0^1 |\alpha_{a,b}(\lambda)|^p d\lambda \\
&\leq \frac{|\gamma(1)|^p + |\gamma(0)|^p}{2} - \frac{2^{-p}}{3}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}.
\end{aligned}$$

Proof. According to ([4], Proposition 10.13), since $|\cdot|^2$ is uniformly convex with modules of convexity $|\cdot|^2$. Hence for $p \in [2,+\infty)$ is uniformly convex with modules of convexity ϕ such that ϕ satisfing

$$\phi \geq 2^{1-p}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|.|^p,$$

by integrating with respect t we have

$$\begin{aligned}
\int_0^1 \phi(t)dt &\geq 2^{1-p}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}\int_0^1 |t|^p dt \\
&= \frac{2^{1-p}}{p+1}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{2^{-1-p}}{p+1}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \\
&\leq \left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{1}{4}\int_0^1 \phi(t)dt \\
&\leq \int_0^1 |\alpha_{a,b}(\lambda)|^p d\lambda \\
&\leq \frac{|\gamma(1)|^p + |\gamma(0)|^p}{2} - \frac{1}{6}\phi(1)
\end{aligned}$$

$$\leq \frac{|\gamma(1)|^p + |\gamma(0)|^p}{2} - \frac{2^{-p}}{3} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\}.$$

$\square$

3. Applications to special means

Consider the following special means for two nonnegative real numbers $\alpha, \beta, \alpha \neq \beta$ as follows:

(1) The arithmetic mean:

$$A = A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \ \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(1) The logarithmic mean:

$$\overline{L} = \overline{L}(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}, \ \ \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(2) The generalized logarithmic mean:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^{\frac{1}{n}}, \ n \in \mathbb{Z} \backslash \{-1, 0\}, \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

Proposition 2. *Let $a, b \in \mathbb{R}$ with $a < b, a \neq 0$ and let p be even number. Then, the following inequality holds:*

$$[A(a,b)]^p + \frac{2^{-1-p}}{p+1} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\} \leq [L_p(a,b)]^p$$
$$\leq A(a^p, b^p) - \frac{2^{-p}}{3} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\}.$$

Proof. Put the geodesic $\alpha(\lambda) = \lambda a + (1-\lambda) b$ in Therorem 4. $\square$

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