



ON THE PRACTICAL ψ^γ –EXPONENTIAL ASYMPTOTIC STABILITY OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a new type of stability for nonlinear systems of differential equations called practical ψ^γ –exponential asymptotic stability, is presented. Some sufficient conditions for practical ψ^γ –exponential asymptotic stability are provided by using Lyapunov theory. These results generalize fundamental well known results for practical exponential asymptotic and ψ –exponential asymptotic stability for nonlinear time-varying systems. In addition, these results are using to investigate the practical ψ^γ –exponential asymptotic stability problem of nonlinear perturbed system and cascade systems. The last part is devoted to the study the problem of practical ψ^γ –exponential asymptotic stabilization for some classes of nonlinear systems with delayed perturbation.

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1. INTRODUCTION

The problem of stability analysis and controllability of nonlinear systems has been a topic of numerous papers and has produced a vast body of important results [4, 12, 17] and the references therein. However, in practice, disturbances often prevent the error signals from tending to the origin. Thus, the origin is not a point of equilibrium of the system. Such a situation often arises problems of guarantee the stability of the origin as an equilibrium point. For this reasons, LaSalle and Lefschetz [12] introduced the theory of practical stability when the origin is not necessary an equilibrium point. The notion of input to state practical stability introduced by [16] for to study uncertain dynamical systems. Under an output feedback controller the global uniform practical stability of the closed loop system is proved by [3]. In addition, the notion of ψ –stability of degree k for ordinary differential equations has been introduced by Akinyele [1]. Marchalo [13] gives a real start for the study of ψ –stability, when he introduced the notions of ψ –uniform stability for trivial solution of the nonlinear system and also obtained new sufficient conditions for the linear

system. Recently various types of ψ -stability have been studied for nonlinear Lyapunov matrix differential equations have been given in many papers [6, 7, 14, 15]. In the study of stability the use of the function ψ is interpreted as a weight in the norm which ensures which ensures rate increase solutions that are not stable in the usual sense.

The main aim of the current paper, motivated by problems of generalized exponential asymptotical stability of nonlinear systems investigated by [4, 11]. Combining the notion of practical stability with ψ -stability to ensure the study of a large number of dynamic systems, the notion of practical ψ^γ -exponential asymptotical stability was introduced. We investigate the preservation of this notion when considering a system with a perturbation term. This leads us to study the problem of ψ^γ -exponential asymptotic stability of cascade systems.

The rest of this paper is organized as follows:

First, in Section 2, basic definitions and some preliminary results about practical ψ^γ -exponential asymptotical stability are presented. Then, in Section 3 some sufficient conditions are given, to prove and to guarantee the main theorem about the global uniform practical ψ^γ -exponential asymptotical stability. However, Section 4 establishes practical ψ^γ -exponentially asymptotically stability for nonlinear system with delayed perturbation. finally we study the Practical ψ^γ -exponential asymptotic stabilization.

2. PRELIMINARIES

Consider the nonlinear systems of differential equations

$$\dot{x} = f(t, x). \quad (2.1)$$

where the function $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. Let $\psi_i : \mathbb{R}_+ \rightarrow (0, \infty)$, $i = 1, 2, \dots, n$ be continuous functions and

$$\Psi = \text{diag}[\psi_1, \psi_2, \dots, \psi_n].$$

The matrix $\psi(t)$ is invertible for each $t \geq 0$. If for all $i = 1, 2, \dots, n$, $\psi_i(t) = 1$, then $\psi = I$ is the identity matrix. In addition, suppose

$$\|f(t, 0)\| \leq f_0 \quad \forall t \geq 0 \quad (2.2)$$

where $f_0 = \text{constant} \neq 0$ in general and the symbol $\|x\|$ denotes arbitrary vector norm of a vector $x \in \mathbb{R}^n$.

The condition (2.2) means that the origin $x = 0$ is not required to be an equilibrium point for the system under consideration. Indeed, this fails in many cases when studying the practical stability (see [8]). Throughout this paper, a solution of system (2.1) through a point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ will be denoted by such a form as $x(t, t_0, x_0)$, where $x(t_0, t_0, x_0) = x_0$. In addition, suppose that the solution $x(t, t_0, x_0)$ is defined for all $t \geq t_0$. The purpose of this paper is to prove some sufficient conditions for practical ψ^γ -exponential asymptotic stability of solution of the system (2.1). In the first, some definitions are given below.

Definition 1 ([5]). The solution $x(\cdot, t_0, x_0)$ of system (2.1) is said to be

i): ψ -bounded if there exists positive constant $\delta(t_0) < +\infty$, such that

$$\sup_{t \geq t_0} \|\psi(t)x(t, t_0, x_0)\| \leq \delta(t_0) < +\infty.$$

ii): uniformly ψ -bounded, if it is ψ -bounded with $\delta(t_0)$ is independent of t_0 .

Definition 2. Let $\gamma > 0$ be given. The system (2.1) is called globally practically ψ^γ -exponentially asymptotically stable if there exist some constants $\lambda > 0$, $k \geq 1$ and $R \geq 0$ such that for any solution $x(t, t_0, x_0)$ of (2.1) satisfies

$$\|\psi(t)x(t, t_0, x_0)\| \leq k\|\psi(t_0)x_0\|^\gamma e^{-\lambda(t-t_0)} + R, t \geq t_0 \geq 0, x_0 \in \mathbb{R}^n. \quad (2.3)$$

If $\gamma = 1$, the system (2.1) is said globally practically ψ -exponentially asymptotically stable.

Remark 1. Note that, in the particular case where $\psi = I$, $\gamma \neq 1$ and $R \geq 0$, the notion of practical ψ^γ -exponential asymptotic stability is defined as follows :

- a):** If $R = 0$, then the system (2.1) is called globally γ -exponentially asymptotically stable.
- b):** If $R > 0$, then the system (2.1) is called globally practically γ -exponentially asymptotically stable.

The motivation of this paper is that the notion of practical ψ^γ -exponential asymptotic stability generalizes some known types of stability. In the following remark we cite some types of stability which are considered as special cases of practical ψ^γ -exponential asymptotic stability.

Remark 2. Definition 2 generalizes the notions of ψ -stability. More precisely, when $R = 0$ and $\gamma = 1$ we recover the usual definition of ψ -stability ([9]). Moreover, for $R \geq 0$, $\psi = I$ (identity matrix) and $\gamma = 1$, the practical ψ^γ -stability coincides with a known practical type of stability:

- a):** If $R = 0$, then the system (2.1) is globally exponentially asymptotically stable (see [10]).
- b):** If $R > 0$, then the system (2.1) is globally practically exponentially asymptotically stable (see [8]).

The relation between practical ψ^γ -exponential asymptotic stability and ψ -boundedness is given by the following remark.

Remark 3. By combining definition of practical ψ^γ -exponential asymptotic stability with that of ψ -boundedness, it is clear that if any solution of system (2.1) is uniformly ψ -bounded, then it is practically ψ^γ -exponentially asymptotically stable.

For some systems the insurance of practical exponential asymptotic stability might be more difficult. That is why it is sometimes more useful to look for another way

of stability, among which the practical ψ^{γ} -exponential asymptotic stability which requires appropriate sufficient conditions.

Next, we shall define a class of Lyapunov functions that will be used in the qualitative investigations of the system (2.1).

Definition 3. Consider a continuous function $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$.

- (1) The function V is said to be globally Lipschitz with respect to x , if there a constant $L > 0$ such that

$$|V(t, x) - V(t, y)| \leq L\|x - y\|, \text{ for all } t \in \mathbb{R}_+ \text{ and } x, y \in \mathbb{R}^n.$$

- (2) The total derivative of V with respect to system (2.1), denoted by $D^+V_{(2.1)}$, is given by

$$D^+V_{(2.1)}(t, x) = \limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} [V(t + \delta, x + \delta f(t, x)) - V(t, x)] \quad (2.4)$$

- (3) If $x(t)$ is a solution of (2.1), the upper right-hand derivative of $V(t, x(t))$, denoted by $D^+V(t, x(t))$, is given by

$$D^+V(t, x(t)) = \limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} [V(t + \delta, x(t + \delta)) - V(t, x(t))]. \quad (2.5)$$

Remark 4. Let V be a continuous function from $\mathbb{R}_+ \times \mathbb{R}^n$ to \mathbb{R}_+ .

- If V is Lipschitz with respect to x , Yoshizawa [17] has proved that

$$D^+V_{(2.1)}(t, x) = D^+V(t, x(t)). \quad (2.6)$$

- If $V \in C^1[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, then the equalities are satisfied

$$D^+V_{(2.1)}(t, x) = D^+V(t, x(t)) = \dot{V}_{(2.1)}(t, x) := \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) \cdot f(t, x). \quad (2.7)$$

Definition 4. A continuous function $V(t, x)$ is called a practical Lyapunov- ψ function for system (2.1), if there exist positive numbers $\lambda_1, \lambda_2, \lambda_3 > 0, a, b \geq 0, p, q, r > 0$ such that

$$\text{i): } \lambda_1 \|\psi(t)x\|^p \leq V(t, x) \leq \lambda_2 \|\psi(t)x\|^q + a, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

$$\text{ii): } D^+V_{(2.1)}(t, x) \leq -\lambda_3 \|\psi(t)x\|^r + b, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

where $a, b \geq 0, \lambda_1, \lambda_2, \lambda_3 > 0$ and $p, q, r > 0$.

Without loss of generality, throughout the rest of the paper, if $V(t, x)$ is a practical Lyapunov- ψ function for system (2.1), we assume that $\lambda_1 \leq \lambda_3$ and $\lambda_1 \lambda_3 < b \lambda_2$ which we need in the rest of this work.

Remark 5. Notes that, in the above definition,

- i): if $V(t, x)$ is a Lyapunov- ψ function for system (2.1) ([9]), then $V(t, x)$ is also a practical Lyapunov- ψ function for system (2.1).

- ii): if $p = q = r > 0$ and $\psi \equiv I$ then, the practical Lyapunov– ψ function $V(t, x)$ ensure the global uniform practical exponential asymptotic stability of system (2.1)(see [8]).

The following lemmas will also be required in the investigations of our results of the paper.

Lemma 1. *Let $V(t)$ a function whose derivative $V'(t)$ exists for all $t \in \mathbb{R}_+$ and satisfies the differential inequality*

$$V'(t) \leq p(t)V(t) + q(t)$$

where $p(t)$ and $q(t)$ are two continuous functions. Then, the function V satisfies

$$V(t) \leq V(t_0)e^{\int_{t_0}^t p(s)ds} + \int_{t_0}^t e^{\int_s^t p(z)dz} q(s)ds, \forall t \geq t_0 \geq 0.$$

Lemma 2. *Let $\alpha, \beta \geq 0$. Then, we have the following inequalities:*

- a): For all $p \geq 1$, $(\alpha + \beta)^{\frac{1}{p}} \leq \alpha^{\frac{1}{p}} + \beta^{\frac{1}{p}}$.
- b): For all $p \geq 1$, $(\alpha + \beta)^p \leq 2^{p-1}(\alpha^p + \beta^p)$.
- c): If $(\alpha, \beta) \neq (0, 0)$, then $0 \leq \frac{\alpha\beta}{\alpha + \beta} \leq \beta$.

3. MAIN RESULTS

In this section, we consider practical ψ^γ -exponential asymptotic stability problems of solutions of system (2.1) will be discussed by the Lyapunov’s second method. In the begining of this section, we give some sufficient conditions of practical ψ^γ -exponential asymptotic stability if the system (2.1) admits a practical Lyapunov– ψ function candidate with p, q, r and a, b are arbitrary constants.

Theorem 1. *Assume that there exists a continuously differentiable function $V(t, x)$ satisfying the following properties: For all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$*

- i): $\lambda_1 \|\psi(t)x\|^p \leq V(t, x) \leq \lambda_2 \|\psi(t)x\|^q + a$,
- ii): $\dot{V}_{(2.1)}(t, x) \leq -\lambda_3 \|\psi(t)x\|^r + b$

where $\lambda_1, \lambda_2, \lambda_3 > 0$, $a, b \geq 0$, $p > 0$ and $r \geq q > 0$. Then, the system (2.1) is globally practically $\psi^{\frac{q}{p}}$ -exponentially asymptotically stable.

Proof. The details of the proof are given in appendix. □

Now, the following corollaries can be easily obtained. First, suppose that the system (2.1) admits a practical Lyapunov– ψ function candidate such that $q = r > 0$, $p > 0$ and $a, b \geq 0$. The following corollary shows the existence of $\gamma > 0$ such that the system (2.1) is practically ψ^γ -exponentially asymptotically stable. In this case the practical ψ -stability result is obtained by the following Corollary.

Corollary 1. *Suppose there exists a continuously differentiable function $V(t, x)$ satisfying the following properties :*

- i):** $\lambda_1 \|\Psi(t)x\|^p \leq V(t, x) \leq \lambda_2 \|\Psi(t)x\|^q + a, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$
- ii):** $\dot{V}_{(2.1)}(t, x) \leq -\lambda_3 \|\Psi(t)x\|^q + b, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$

where $\lambda_1, \lambda_2, \lambda_3 > 0, a, b \geq 0$ and $p, q > 0$. Then, the system (2.1) is globally practically $\Psi^{\frac{q}{p}}$ -exponentially asymptotically stable.

Remark 6. In Corollary 1, if $V(t, x)$ is a practical Lyapunov- Ψ function candidate for the system (2.1) such that $p = q > 0$ and $a, b \geq 0$, then we obtain sufficient condition for practical Ψ -exponential asymptotic stability of system (2.1). In addition, if $\Psi(t) = I$ we obtain sufficient conditions of practical exponential stability of system (2.1) (see [8]).

In the Theorem 1, we will end up taking $a = 0$ and p, q are chosen arbitrarily and $r \geq q$. The practical Ψ^r -exponential asymptotic stability of system (2.1) is given by the following corollary.

Corollary 2. *Suppose that there exists a continuously differentiable function $V(t, x)$ satisfying the following properties*

- i):** $\lambda_1 \|\Psi(t)x\|^p \leq V(t, x) \leq \lambda_2 \|\Psi(t)x\|^q, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$
- ii):** $\dot{V}_{(2.1)}(t, x) \leq -\lambda_3 \|\Psi(t)x\|^r + b, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$

where $\lambda_1, \lambda_2, \lambda_3 > 0, b \geq 0$ and $p, q > 0$ such that $r \geq q$. Then, the system (2.1) is globally practically $\Psi^{\frac{q}{p}}$ -exponentially asymptotically stable.

Remark 7. The practical $\Psi^{\frac{q}{p}}$ -exponential asymptotic stability given by Corollary 2, generalizes that of practical stability given by Theorem 3.4 [2].

4. PRACTICAL Ψ^r -EXPONENTIAL ASYMPTOTIC STABILITY OF PERTURBED AND CASCADED SYSTEMS

The purpose of this section is to investigate the practical Ψ^r -exponential asymptotic stability problem for perturbed and cascaded nonlinear systems. By using practical Lyapunov- Ψ function, some criteria which guarantee the practical Ψ^r -exponential asymptotic stability of the addressed systems are provided.

4.1. Practical Ψ^r -exponential asymptotic stability of perturbed systems

A great interest is attached to the relations between the solutions of the unperturbed system (2.1) and the solutions of the perturbed system having the following form

$$\dot{y} = f(t, y) + g(t, y) \quad (4.1)$$

where $f, g : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous functions. We suppose that functions f and g satisfy all required conditions for existence and uniqueness of the solutions of system (4.1) on the interval $[t_0, +\infty)$ for all suitable initial data $y_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$.

The unique solutions of (4.1) is denoted by $y(t, t_0, y_0)$, satisfying the initial conditions $y(t_0, t_0, y_0) = y_0 \in \mathbb{R}^n$. Suppose the following assumption hold

(A1): There exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfying the following properties:

i): $|V(t, x) - V(t, y)| \leq L\|x - y\|$, $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

ii): $\lambda_1 \|\psi(t)x\|^p \leq V(t, x) \leq \lambda_2 \|\psi(t)x\|^q + a$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

iii): $D^+V_{(2.1)}(t, x) \leq -\lambda_3 \|\psi(t)x\|^r + b$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$

where $L > 0$, $\lambda_1, \lambda_2, \lambda_3 > 0$, $a, b \geq 0$ and $p > 0$, $r \geq q > 0$.

The practical ψ^l -exponential asymptotic stability of perturbed system (4.1) is given by the following theorem.

Theorem 2. Suppose that the assumption (A1) is satisfied and the perturbation term $g(t, y)$ satisfies

$$\|g(t, y)\| \leq \lambda \|\psi(t)y\|^r + \beta \quad (4.2)$$

where $\lambda, \beta > 0$ and $r \geq q > 0$ such that $\lambda_3 - L\lambda > 0$ and . Then, the perturbed system (4.1) is globally practically ψ^q -exponentially asymptotically stable.

Proof. The total derivative $D^+V_{(4.1)}(t, y)$ of the function $V(t, y)$ with respect to system (4.1) satisfies

$$\begin{aligned} D^+V_{(4.1)}(t, y) &= \limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} [V(t + \delta, y + \delta(f(t, y) + g(t, y))) - V(t, y)] \\ &\leq \limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} [V(t + \delta, y + \delta(f(t, y) + g(t, y))) - V(t + \delta, y + \delta f(t, y))] \\ &\quad + \limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} [V(t + \delta, y + \delta f(t, y)) - V(t, y)] \\ &\leq D^+V_{(2.1)}(t, y) + L\|g(t, y)\|, \\ &\leq -\lambda_3 \|\psi(t)y\|^r + b + L\lambda \|\psi(t)y\|^r + L\beta, \\ &\leq -(\lambda_3 - L\lambda) \|\psi(t)y\|^r + b + L\beta. \end{aligned}$$

Therefore, $V(t, y)$ is a practical Lyapunov- ψ function for the perturbed system (4.1). Then, by Theorem 1, the perturbed system (4.1) is practically ψ^q -exponentially asymptotically stable. \square

4.2. Practical ψ^l -exponential asymptotic stability of cascaded systems

Consider now cascaded system of the form:

$$(\Sigma_1) : \dot{x}_1 = f_1(t, x_1) + g(t, x_1, x_2) \quad (4.3)$$

$$(\Sigma_2) : \dot{x}_2 = f_2(t, x_2) \quad (4.4)$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ with $n_1, n_2 \in \mathbb{N}^*$ and $x := [x_1, x_2]^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, denotes the state of the closed-loop system. We assume that functions f_1 , f_2 and g satisfy

all required conditions for existence and uniqueness of the solutions of system (4.3)-(4.4) on the interval $[t_0, +\infty)$ for all suitable initial data $x_0 \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $t_0 \in \mathbb{R}_+$.

The main goal is to give some sufficient conditions which guarantee that if the system (4.4) and the system

$$\dot{x}_1 = f_1(t, x_1) \quad (4.5)$$

are practically ψ^γ -exponentially stable where $\gamma > 0$, then the cascaded system (4.3)-(4.4) is also practically ψ^γ -exponentially asymptotically stable, if the interconnection term $g(t, x_1, x_2)$ satisfies some condition of boundedness. The unique solution of cascaded system (4.3)-(4.4) is denoted by $x(\cdot, t_0, x_0)$, which satisfies $x(t_0, t_0, x_0) = x(t_0) = x_0$.

In this part, we consider $\Psi_1 = \text{diag}[\psi_1^1, \psi_2^1, \dots, \psi_{n_1}^1]$ and $\Psi_2 = \text{diag}[\psi_1^2, \psi_2^2, \dots, \psi_{n_2}^2]$ with $\psi_i^1 : \mathbb{R}_+ \rightarrow (0, \infty), i = 1, 2, \dots, n_1$ and $\psi_i^2 : \mathbb{R}_+ \rightarrow (0, \infty), i = 1, 2, \dots, n_2$ be continuous functions. Before proposing our theorem, we introduce the following assumptions:

(A2): There exist two functions $V_1(t, x_1)$ and $V_2(t, x_2)$ having the following properties: For $i = 1, 2$

i): $V_i \in C^1(\mathbb{R}_+ \times \mathbb{R}^{n_i}, \mathbb{R}_+)$ and $|V_i(t, x_i) - V_i(t, y_i)| \leq L_i \|x_i - y_i\|$ for all $x_i, y_i \in \mathbb{R}^{n_i}$ and for each $t \in \mathbb{R}_+$,

ii): $\lambda_i \|\Psi_i(t)x_i\|^p \leq V_i(t, x_i) \leq \beta_i \|\Psi_i(t)x_i\|^q + a_i, (t, x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i}$,

iii): $D^+ V_{i(\Sigma_i)}(t, x_i) \leq -c_i \|\Psi_i(t)x_i\|^r + b_i, (t, x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i}$

where $a_i, b_i \geq 0, L_i, \lambda_i, \beta_i, c_i > 0$ for $i = 1, 2$ and $p > 0, r \geq q > 0$.

(A3): The interconnection term $g(t, x_1, x_2)$ satisfies the following boundedness condition

$$\|g(t, x_1, x_2)\| \leq \delta (\|\Psi_1(t)x_1\|^r + \|\Psi_2(t)x_2\|^r + 1), t \in \mathbb{R}_+, x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} \quad (4.6)$$

where $\delta > 0$ and $r \geq q > 0$.

Theorem 3. *We assume assumptions (A2) and (A3) are satisfied. Then, under the condition $\min(c_1, c_2) > 2\delta L_1 \max\{2^{\frac{r-2}{2}}, 2^{\frac{2-r}{2}}\}$, the cascaded system (4.3)-(4.4) is globally practically $\psi^{\frac{q}{p}}$ -exponentially asymptotically stable.*

Proof. Let define the function $W(t, x)$ by

$$W(t, x) = V_1(t, x_1) + V_2(t, x_2),$$

where $x = [x_1, x_2]^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Consider the matrix function $\Psi(t)$ given by

$$\Psi(t) = \begin{bmatrix} \Psi_1(t) & 0 \\ 0 & \Psi_2(t) \end{bmatrix}, t \in \mathbb{R}_+.$$

Then, for all $t \in \mathbb{R}_+$ and $x = [x_1, x_2]^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we obtain $\Psi(t)x = [\Psi_1(t)x_1, \Psi_2(t)x_2]^T$.

By Lemma 2 we have, for all $r > 0$

$$\|\Psi(t)x\|^r = \left[\|\Psi_1(t)x_1\|^2 + \|\Psi_2(t)x_2\|^2 \right]^{\frac{r}{2}} \leq \max\{2^{\frac{r-2}{2}}, 2^{\frac{2-r}{2}}\} (\|\Psi_1(t)x_1\|^r + \|\Psi_2(t)x_2\|^r).$$

The total derivative $D^+W_{(\Sigma_1, \Sigma_2)}(t, x)$ of W with respect to system (4.3)-(4.4) satisfies

$$\begin{aligned} D^+W_{(\Sigma_1, \Sigma_2)}(t, x) &\leq D^+V_{1(4.3)}(t, x_1) + D^+V_{2(4.4)}(t, x_2), \\ &\leq D^+V_{1(4.5)}(t, x_1) + D^+V_{2(4.4)}(t, x_2) + L_1 \|g(t, x_1, x_2)\|, \\ &\leq -c_1 \|\psi_1(t)x_1\|^r + b_1 - c_2 \|\psi_2(t)x_2\|^r + b_2 + L_1 \|g(t, x_1, x_2)\|, \\ &\leq -\min(c_1, c_2) \left(\|\psi_1(t)x_1\|^r + \|\psi_2(t)x_2\|^r \right) \\ &\quad + \delta L_1 (\|\psi_1(t)x_1\|^r + \|\psi_2(t)x_2\|^r + 1) + b_1 + b_2, \\ &\leq -\frac{\min(c_1, c_2)}{\max\{2^{\frac{r-2}{2}}, 2^{\frac{2-r}{2}}\}} \|\psi(t)x\|^r + \delta L_1 (\|\psi_1(t)x_1\|^r + \|\psi_2(t)x_2\|^r) \\ &\quad + b_1 + b_2 + \delta L_1. \end{aligned}$$

Using the inequalities

$$\|\psi_1(t)x_1\| \leq \|\psi(t)x\|, \|\psi_2(t)x_2\| \leq \|\psi(t)x\|, \|\psi(t)x\| \leq \|\psi_1(t)x_1\| + \|\psi_2(t)x_2\| \tag{4.7}$$

yield

$$D^+W(t, x) \leq -c \|\psi(t)x\|^r + b, t \in \mathbb{R}_+, x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \tag{4.8}$$

where $b = b_1 + b_2 + \delta L_1$ and $c = \frac{\min(c_1, c_2)}{\max\{2^{\frac{r-2}{2}}, 2^{\frac{2-r}{2}}\}} - 2\delta L_1 > 0$. By condition ii), the function $W(t, x)$ satisfies

$$\lambda_1 \|\psi_1(t)x_1\|^p + \lambda_2 \|\psi_2(t)x_2\|^p \leq W(t, x) \leq \beta_1 \|\psi_1(t)x_1\|^q + \beta_2 \|\psi_2(t)x_2\|^q + a_1 + a_2.$$

Then, by Lemma 2 and (4.7), we obtain

$$\lambda \|\psi(t)x\|^p \leq W(t, x) \leq \beta \|\psi(t)x\|^q + a, t \in \mathbb{R}_+, x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$$

with $\beta = \beta_1 + \beta_2$, $\lambda = \frac{\min(\lambda_1, \lambda_2)}{2^{p-1}}$ and $a = a_1 + a_2$. Therefore, $W(t, x)$ is a practical Lyapunov- ψ function for the cascaded system (4.3)-(4.4). Then, by Theorem 1, the cascaded system (4.3)-(4.4) is globally practically $\psi^{\frac{q}{p}}$ -exponentially asymptotically stable. \square

5. PRACTICAL ψ^γ -EXPONENTIAL ASYMPTOTIC STABILIZATION

We denote by $\mathcal{M}_{(n,m)}(\mathbb{R})$ the set of all $n \times m$ matrices with real coefficients where $m, n \in \mathbb{N}^*$.

In this section, we conclude to some practical ψ^γ -stabilization problem of a class nonlinear control system with delayed perturbations using feedback controls.

Consider the following nonlinear time-varying differential equation with delay.

$$\dot{x} = f(t, x) + g(t, x) \left(\sum_{i=0}^r q_i(t, x(t - \tau_i(t))) + u \right), r \in \mathbb{N}, t \in \mathbb{R}_+, x \in \mathbb{R}^n \tag{5.1}$$

where

- $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n, g : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathcal{M}_{(n,m)}(\mathbb{R})$ are continuous functions.
- $x \in \mathbb{R}^n$ is the state and $u \in \mathcal{M}_{(n,1)}(\mathbb{R})$ is the control.
- $q_i(t, x(t - \tau_i(t))) \in \mathcal{M}_{(m,1)}(\mathbb{R})$ are the perturbations terms.
- $\tau_i(t), i = 1, 2, \dots, r$ are the time delays positive, continuous and there exist positive constants $\bar{\tau}_i, i = 1, 2, \dots, r$ such that $0 \leq \tau_i(t) \leq \bar{\tau}_i$.

The initial condition function is given by $x(t) = \varphi(t), t \in [t_0 - \bar{\tau}, t_0]$ where $\bar{\tau} = \max_{1 \leq i \leq r} \{\bar{\tau}_i\}$ and $\varphi(t)$ is a continuous function on $[t_0 - \bar{\tau}, t_0]$.

The nominal system and is defined by

$$\dot{x} = f(t, x) + g(t, x)u. \tag{5.2}$$

The unforced nominal system of (5.1) is defined by

$$\dot{x} = f(t, x). \tag{5.3}$$

Our goal is to find a feedback $u = u(t, x)$ which make the closed-loop system

$$\dot{x} = f(t, x) + g(t, x) \left(\sum_{i=0}^r q_i(t, x(t - \tau_i(t))) + u(t, x) \right), r \in \mathbb{N}$$

practically ψ^γ -exponentially asymptotically stable, where $\gamma > 0$.

In order to study the practical ψ^γ -exponential asymptotic stability of closed-loop system, we need the following assumptions

(A4): There exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ which satisfies

$$\lambda_1 \|\psi(t)x\|^p \leq V(t, x) \leq \lambda_2 \|\psi(t)x\|^q + a \tag{5.4}$$

$$\dot{V}_{(5.3)}(t, x) \leq -\lambda_3 \|\psi(t)x\|^r + b \tag{5.5}$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $\lambda_1, \lambda_2, \lambda_3 > 0, a, b \geq 0$ and $p > 0, r \geq q > 0$.

(A5): The perturbation term $q_i(t, x(t - \tau_i(t)))$ checking:

$$\|q_i(t, x(t - \tau_i(t)))\| \leq \beta_i \|x(t - \tau_i(t))\|, \beta_i > 0, i = 1, 2, \dots, r. \tag{5.6}$$

Theorem 4. System (5.1) satisfying assumptions (A4) and (A5) is globally practically $\psi^{\frac{q}{p}}$ -exponentially asymptotically stable under the following controller

$$u(t, x) = \frac{- \left[\sum_{i=0}^r \beta_i \chi_i(t) \right]^2 \left[g^T(t, x) \frac{\partial V}{\partial x}(t, x) \right]}{\left\| \frac{\partial V}{\partial x}^T(t, x) \cdot g(t, x) \right\| \sum_{i=0}^r \beta_i \chi_i(t) + \varepsilon \|\psi(t)x\|^r} \tag{5.7}$$

where $\varepsilon > 0$ and

$$\chi_i(t) = \max_{s \in [t - \bar{\tau}_i, t]} \|x(s)\|, i = 1, 2, \dots, r \tag{5.8}$$

Proof. Applying (5.7) to (5.1) gives a new form of a closed-loop dynamical system:

$$\dot{x} = f(t, x) + g(t, x) \sum_{i=0}^r q_i(t, x(t - \tau_i(t))) + g(t, x)u(t, x). \quad (5.9)$$

By using (5.5) and (5.8), the derivative $\dot{V}_{(5.9)}(t, x)$ of the function V along the trajectories of (5.9), satisfies

$$\begin{aligned} \dot{V}_{(5.9)}(t, x) &= \dot{V}_{(5.3)}(t, x) + \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x) \sum_{i=0}^r q_i(t, x(t - \tau_i(t))) \\ &\quad + \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x)u(t, x), \\ &\leq -\lambda_3 \|\psi(t)x\|^r + b + \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x) \sum_{i=0}^r q_i(t, x(t - \tau_i(t))) \\ &\quad + \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x)u(t, x), \\ &\leq -\lambda_3 \|\psi(t)x\|^r + b + \left\| \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x) \right\| \sum_{i=0}^r \|q_i(t, x(t - \tau_i(t)))\| \\ &\quad + \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x)u(t, x), \\ &\leq -\lambda_3 \|\psi(t)x\|^r + b + \left\| \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x) \right\| \sum_{i=0}^r \beta_i \|x(t - \tau_i(t))\| \\ &\quad + \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x)u(t, x). \end{aligned}$$

By using the feedback (5.7) we obtain:

$$\begin{aligned} \dot{V}_{(5.9)}(t, x) &\leq -\lambda_3 \|\psi(t)x\|^r + b + \beta - \frac{\beta^2}{\alpha + \beta} \\ &= -\lambda_3 \|\psi(t)x\|^r + b + \frac{\alpha\beta}{\alpha + \beta} \end{aligned}$$

with $\alpha = \varepsilon \|\psi(t)x\|^r$ and $\beta = \left\| \frac{\partial V^T}{\partial x}(t, x) \cdot g(t, x) \right\| \sum_{i=0}^r \beta_i \chi_i(t)$. Therefore, it follows from ii) of Lemma 2 that,

$$\dot{V}_{(5.9)}(t, x) \leq -\lambda_3 \|\psi(t)x\|^r + b + \varepsilon \|\psi(t)x\|^r = -(\lambda_3 - \varepsilon) \|\psi(t)x\|^r + b. \quad (5.10)$$

By using (5.4) and (5.10), the function $V(t, x)$ is a practical Lyapunov ψ -function for the closed loop system (5.9). Then, under the controller (5.7), the system (5.1) is globally practically $\psi^{\frac{q}{p}}$ -exponentially asymptotically stable. \square

Next, consider the nonlinear uncertain system with delayed perturbations of the form

$$\dot{x} = f(t, x) + g(t, x) \sum_{i=0}^r q_i(t, x(t - \tau_i(t))) + g(t, x)(1 + \delta(t, x))u, \quad r \in \mathbb{N} \quad (5.11)$$

where $\delta(t, x)$ represents the uncertainties terms. Now, we need the following assumption:

(A6): There exists a continuous function $\phi(t, x)$ such that for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$1 + \delta(t, x) \geq \phi(t, x) > 0. \quad (5.12)$$

Theorem 5. Suppose that assumptions (A4) - (A6) are satisfied. Then, the system (5.11) is globally practically $\Psi^{\frac{q}{p}}$ -exponentially asymptotically stable under the following controller

$$u(t, x) = \frac{-\left[\sum_{i=0}^r \beta_i \chi_i(t)\right]^2 \left[g^T(t, x) \frac{\partial V}{\partial x}(t, x)\right]}{\phi(t, x) \left\| \frac{\partial V}{\partial x}(t, x) \cdot g(t, x) \right\| \sum_{i=0}^r \beta_i \chi_i(t) + \varepsilon^*(t, x) \|\Psi(t)x\|^r} \quad (5.13)$$

where $\varepsilon^*(t, x) = \varepsilon \phi(t, x)$ such that $0 < \varepsilon < \lambda_3$.

Proof. Substituting (5.13) into (5.11) yields the following form:

$$\dot{x} = f(t, x) + g(t, x) \sum_{i=0}^r q_i(t, x(t - \tau_i(t))) + g(t, x)(1 + \delta(t, x))u(t, x). \quad (5.14)$$

From assumption (A5), the derivative $\dot{V}_{(5.14)}(t, x)$ of the practical Lyapunov Ψ -function $V(t, x)$ along the trajectories of (5.14) satisfies

$$\begin{aligned} \dot{V}_{(5.14)}(t, x) &= \dot{V}_{(5.3)}(t, x) + \frac{\partial V}{\partial x}(t, x) \cdot g(t, x) \sum_{i=0}^r q_i(t, x(t - \tau_i(t))) \\ &\quad + \frac{\partial V}{\partial x}(t, x) \cdot g(t, x)(1 + \delta(t, x))u(t, x), \\ &\leq -\lambda_3 \|\Psi(t)x\|^r + b + \left\| \frac{\partial V}{\partial x}(t, x) \cdot g(t, x) \right\| \sum_{i=0}^r \|q_i(t, x(t - \tau_i(t)))\| \\ &\quad + \frac{\partial V}{\partial x}(t, x) \cdot g(t, x)(1 + \delta(t, x))u(t, x). \end{aligned}$$

Then, from (5.8) and (5.13) we get

$$\begin{aligned} \dot{V}_{(5.14)}(t, x) &\leq -\lambda_3 \|\Psi(t)x\|^r + b + \frac{\beta}{\phi(t, x)} - \frac{\beta^2}{\phi(t, x)(\alpha + \beta)} \\ &= -\lambda_3 \|\Psi(t)x\|^r + b + \frac{\alpha\beta}{\phi(t, x)(\alpha + \beta)} \end{aligned}$$

with $\alpha = \varepsilon^*(t, x)\|\psi(t)x\|^r$ and $\beta = \phi(t, x)\|\frac{\partial V}{\partial x}^T(t, x) \cdot g(t, x)\| \sum_{i=0}^r \beta_i \chi_i(t)$. By using ii) of Lemma 2, we obtain

$$\begin{aligned} \dot{V}_{(5.14)}(t, x) &\leq -\lambda_3\|\psi(t)x\|^r + b + \frac{\varepsilon^*(t, x)\|\psi(t)x\|^r}{\phi(t, x)}, \\ &= -(\lambda_3 - \varepsilon)\|\psi(t)x\|^r + b. \end{aligned}$$

Similarly of the proof of Theorem 4, the function $V(t, x)$ is a practical Lyapunov ψ -function for the closed loop system (5.14). Then, under the controller (5.13), the system (5.11) is globally practically $\psi^{\frac{q}{p}}$ -exponentially asymptotically stable. □

6. CONCLUSION

In this paper, we have presented some new conditions for practical ψ^γ -exponential asymptotic stability of nonlinear systems of differential equations. A main theorem for practical ψ^γ -exponential asymptotic stability is established. In addition, one of our main interests is the study of the problem of practical ψ^γ -exponential asymptotic stability of the perturbed systems and cascade systems. Based on this study, we reached a novel result in practical ψ^γ -exponential asymptotic stabilization for dynamical systems with delayed perturbations. To guarantee that the closed-loop system is practically ψ^γ -exponentially asymptotically stable, a continuous controller is provided and sufficient conditions are given. Furthermore, a class of nonlinear systems with delayed perturbations and uncertain control has been considered.

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APPENDIX A. PROOF OF THEOREM 1

Proof. Let $x(t) := x(t, t_0, x_0)$ be the solution of system (2.1) through $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. In the following, the practical $\psi^{\frac{q}{p}}$ -exponential asymptotic stability of the system (2.1) is shown. Four cases are presented

Case 1: : If $q = r$, then it follows from the conditions i) and ii) that, the derivative $\dot{V}_{(2.1)}(t, x)$ of the function $V(t, x)$ along the trajectories of (2.1) satisfies

$$\dot{V}_{(2.1)}(t, x) \leq -\frac{\lambda_3}{\lambda_2}V(t, x) + \frac{a\lambda_3}{\lambda_2} + b, t \in \mathbb{R}_+, x \in \mathbb{R}^n.$$

By using Lemma 1 with $p(t) = -\frac{\lambda_3}{\lambda_2}$ and $q(t) = b + \frac{a\lambda_3}{\lambda_2}$, we obtain

$$V(t, x(t)) \leq V(t_0, x_0) e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \left(a + \frac{b\lambda_2}{\lambda_3}\right) \left(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\right), t \geq t_0, x_0 \in \mathbb{R}^n.$$

Thus, for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$

$$\begin{aligned} V(t, x(t)) &\leq V(t_0, x_0) e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \left(a + \frac{b\lambda_2}{\lambda_3}\right) \left(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\right) \\ &\leq \lambda_2 \|\Psi(t_0)x_0\|^p e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + a e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \left(a + \frac{b\lambda_2}{\lambda_3}\right) \left(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\right) \\ &\leq \lambda_2 \|\Psi(t_0)x_0\|^p e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \left(a + \frac{b\lambda_2}{\lambda_3}\right). \end{aligned}$$

By using the condition i) of Theorem 1 and condition a) of Lemma 2, the solution $x(t, t_0, x_0)$ satisfies

$$\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1} \|\Psi(t_0)x_0\|^{\frac{q}{p}} e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)}} + \sqrt[p]{\frac{a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}, \forall t \geq t_0 \geq 0, \forall x_0 \in \mathbb{R}^n.$$

Case 2 : If $r > q$ and $p = q$, we distinguish two subcases: whether the function $\Psi(t)x(t)$ start from outside or inside the closed ball of \mathbb{R}^n , $\mathcal{B}_1 := \{x \in \mathbb{R}^n; \|x\| \leq 1\}$.

- $\|\Psi(t_0)x_0\| > 1$. In this case, there exists $T_0 \in]0, +\infty]$ such that $\|\Psi(t)x(t)\| > 1$ for all $t \in [t_0, t_0 + T_0[$ and $\|\Psi(t_0 + T_0)x(t_0 + T_0)\| = 1$. Hence, we get that

$$\begin{aligned} \dot{V}_{(2.1)}(t, x) &\leq -\lambda_3 \|\Psi(t)x\|^r + b \\ &= -\lambda_3 \|\Psi(t)x\|^{r-q} \|\Psi(t)x\|^q + b \\ &\leq -\lambda_3 \|\Psi(t)x\|^q + b \\ &\leq -\frac{\lambda_3}{\lambda_2} V(t, x) + \frac{a\lambda_3}{\lambda_2} + b. \end{aligned}$$

By using Lemma 1, the following inequality is obtained

$$V(t, x(t)) \leq V(t_0, x_0) e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \left(a + \frac{b\lambda_2}{\lambda_3}\right) \left(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\right), t \in [t_0, t_0 + T_0[. \quad (\text{A.1})$$

From the condition i), it is easy to see that for all $t \in [t_0, t_0 + T_0[$, we have

$$\|\Psi(t)x(t, t_0, x_0)\|^p \leq \frac{\lambda_2}{\lambda_1} \|\Psi(t_0)x_0\|^p e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \frac{a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}.$$

Then, by condition i) of Lemma 2, the estimation is obtained

$$\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1} \|\Psi(t_0)x_0\|^p e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}} + \sqrt[p]{\frac{a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}, t \in [t_0, t_0 + T_0[.$$

In addition, for each $t \geq t_0 + T_0$, either $\|\psi(t)x(t)\| \leq 1$, or $\|\psi(t)x(t)\| > 1$. In this second case, we can again invoke the continuity of the function $\psi(t)x(t)$ to see that there exists a nonempty time-interval $[\tau, \tau + T]$, with $T \in (0, +\infty]$, containing t and such that $\|\psi(s)x(s)\| > 1$ for all $s \in (\tau, \tau + T]$, with $\|\psi(\tau)x(\tau)\| = 1$. Hence, integrating from τ to $t \in [\tau, \tau + T]$, we obtain, whenever $\|\psi(t)x(t)\| > 1$, it holds that

$$\begin{aligned} V(t, x(t)) &\leq V(\tau, x(\tau))e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)} + (a + \frac{b\lambda_2}{\lambda_3})(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)}) \\ &\leq (\lambda_2\|\psi(\tau)x(\tau)\|^p + a)e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)} + (a + \frac{b\lambda_2}{\lambda_3})(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)}) \\ &\leq \lambda_2 + a + \frac{b\lambda_2}{\lambda_3} \\ &= \frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}. \end{aligned}$$

Then, by condition i) of Theorem 1, the solution $x(t, t_0, x_0)$ satisfies

$$\|\psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Then, the solution $x(t, t_0, x_0)$ is uniformly ψ -bounded.

To sum up, for all $t \geq t_0$, if $\|\psi(t_0)x_0\| > 1$ we have the following

$$\|\psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}\|\psi(t_0)x_0\|^p} e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)} + \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

- $\|\psi(t_0)x_0\| \leq 1$. In this case, as long as $\|\psi(t)x(t)\| \leq 1$, we have trivially that $\|\psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}$. If $\|\psi(t)x(t)\| > 1$ at some instant $t > t_0$, then, again, there exists a nonempty time-interval $[\tau, \tau + T]$, with $T \in (0, +\infty]$ and $\tau > t_0$, containing t and such that $\|\psi(s)x(s)\| > 1$ for all $s \in (\tau, \tau + T]$, with $\|\psi(\tau)x(\tau)\| = 1$. By integration from τ to $t \in [\tau, \tau + T]$, we obtain that, whenever $\|\psi(t)x(t)\| > 1$, it holds that

$$V(t, x(t)) \leq \frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}.$$

By using condition i) of Theorem 1, we have

$$\|\psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Hence, for all $t \geq t_0$ and $x_0 \in \mathbb{R}^n$, we have

$$\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}} \|\Psi(t_0)x_0\| e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)} + \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Then, the system (2.1) is practically Ψ -exponentially asymptotically stable.

Case 3: : If $r > q$ and $q < p$, then from Condition i) it follows that

$$\|\Psi(t)x\|^q \left(\lambda_1 \|\Psi(t)x\|^{p-q} - \lambda_2 \right) \leq a. \quad (\text{A.2})$$

If $a = 0$, then by (A.2), we get

$$\|\Psi(t)x\| \leq \sqrt[p-q]{\frac{\lambda_2}{\lambda_1}}, t \in \mathbb{R}_+, x \in \mathbb{R}^n.$$

In particular, the solution $x(t, t_0, x_0)$ is uniformly Ψ -bounded.

On the other hand, if $a > 0$, we distinguish two subcases: whether the function $\Psi(t)x(t)$ start from outside or inside the ball \mathcal{B}_1

- $\|\Psi(t_0)x_0\| > 1$. In this case, there exists $T_0 \in]0, +\infty]$ such that $\|\Psi(t)x(t)\| > 1$ for all $t \in [t_0, t_0 + T_0[$ and $\|\Psi(t_0 + T_0)x(t_0 + T_0)\| = 1$. Hence, we get that

$$\|\Psi(t)x(t)\|^q \left(\lambda_1 \|\Psi(t)x(t)\|^{p-q} - \lambda_2 \right) \leq a \leq a \|\Psi(t)x(t)\|^p, \forall t \in [t_0, t_0 + T_0[.$$

Then,

$$\|\Psi(t)x(t)\|^q \left(\lambda_1 \|\Psi(t)x(t)\|^{p-q} - \lambda_2 - a \right) \leq 0, \forall t \in [t_0, t_0 + T_0[.$$

Therefore,

$$\|\Psi(t)x(t)\| \leq \sqrt[p-q]{\frac{\lambda_2 + a}{\lambda_1}}, \forall t \in [t_0, t_0 + T_0[.$$

In addition, for each $t \geq t_0 + T_0$, either $\|\Psi(t)x(t)\| \leq 1$, or $\|\Psi(t)x(t)\| > 1$. In this second case, as in the **case 2** we get, whenever if $\|\Psi(t)x(t)\| > 1$, it holds that

$$\|\Psi(t)x(t)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

- $\|\Psi(t_0)x_0\| \leq 1$. In this case, as long as $\|\Psi(t)x(t)\| \leq 1$, we have trivially that

$$\|\Psi(t)x(t)\| \leq \sqrt[p-q]{\frac{\lambda_2 + a}{\lambda_1}} \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

If $\|\psi(t)x(t)\| > 1$ at some instant $t > t_0$, then, as in the **case 2**, we obtain that, whenever $\|\psi(t)x(t)\| > 1$, it holds that

$$\|\psi(t)x(t)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Therefore, the solution $x(t, t_0, x_0)$ is uniformly ψ -bounded

Case 4: : If $r > q$ and $p < q$, then from Condition i) it follows that

$$\|\psi(t)x\|^p (\lambda_1 - \lambda_2 \|\psi(t)x\|^{q-p}) \leq a. \quad (\text{A.3})$$

If $a = 0$, then from (A.3), we have for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$\|\psi(t)x\| \geq \eta := \sqrt[q-p]{\frac{\lambda_1}{\lambda_2}}.$$

Then, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, we have

$$\begin{aligned} \dot{V}_{(2.1)}(t, x) &\leq -\lambda_3 \|\psi(t)x\|^r + b, \\ &= -\lambda_3 \|\psi(t)x\|^{r-q} \|\psi(t)x\|^q + b, \\ &\leq -\lambda_3 \eta^{r-q} \|\psi(t)x\|^q + b, \\ &\leq -\frac{\lambda_3 \eta^{r-q}}{\lambda_2} V(t, x) + b. \end{aligned}$$

By Lemma 1 with $p(t) = -\frac{\lambda_3 \eta^{r-q}}{\lambda_2}$ and $q(t) = b$, we obtain

$$V(t, x(t)) \leq V(t_0, x_0) e^{-\frac{\lambda_3 \eta^{r-q}}{\lambda_2} (t-t_0)} + \frac{b\lambda_2}{\lambda_3 \eta^{r-q}} (1 - e^{-\frac{\lambda_3 \eta^{r-q}}{\lambda_2} (t-t_0)}), t \geq t_0, x_0 \in \mathbb{R}^n.$$

Therefore, from the condition i) of Theorem 1 and condition a) of Lemma 2, we have

$$\|\psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}} \|\psi(t_0)x_0\|^{\frac{q}{p}} e^{-\frac{\lambda_3 \eta^{r-q}}{p\lambda_2} (t-t_0)} + \sqrt[p]{\frac{b\lambda_2}{\lambda_1 \lambda_3 \eta^{r-q}}}, t \geq t_0, x_0 \in \mathbb{R}^n.$$

If $a > 0$, we distinguish two subcases: whether the function $\psi(t)x(t)$ start from outside or inside the closed ball \mathcal{B}_1 .

- $\|\psi(t_0)x_0\| > 1$: In this case, there exists $T_0 \in]0, +\infty[$ such that

$$\|\psi(t)x(t)\| \geq 1, \forall t \in [t_0, t_0 + T_0[\text{ and } \|\psi(t_0 + T_0)x(t_0 + T_0)\| = 1. \quad (\text{A.4})$$

Then, we have

$$\|\psi(t)x(t)\|^p (\lambda_1 - \lambda_2 \|\psi(t)x(t)\|^{q-p}) \leq a \leq a \|\psi(t)x(t)\|^q, \forall t \in [t_0, t_0 + T_0[.$$

Therefore,

$$\|\Psi(t)x(t)\|^p (\lambda_1 - (\lambda_2 + a)\|\Psi(t)x(t)\|^{q-p}) \leq 0, \forall t \in [t_0, t_0 + T_0[.$$

Then, for all $t \in [t_0, t_0 + T_0[$, we have $\|\Psi(t)x(t)\| \geq \eta := \sqrt[q-p]{\frac{\lambda_1}{\lambda_2 + a}}$.

Hence, for all $t \in [t_0, t_0 + T_0[$, we obtain

$$D^+V(t, x(t)) \leq -\frac{\lambda_3 \eta^{r-q}}{\lambda_2} V(t, x(t)) + b.$$

By Lemma 1 with $p(t) = -\frac{\lambda_3 \eta^{r-q}}{\lambda_2}$ and $q(t) = b$, we obtain

$$V(t, x(t)) \leq V(t_0, x_0) e^{-\frac{\lambda_3 \eta^{r-q}}{\lambda_2} (t-t_0)} + \frac{b \lambda_2}{\lambda_3 \eta^{r-q}} (1 - e^{-\frac{\lambda_3 \eta^{r-q}}{\lambda_2} (t-t_0)}), t \in [t_0, t_0 + T_0[.$$

Therefore, from condition i) of Theorem 1 and condition a) of Lemma 2, we have

$$\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}} \|\Psi(t_0)x_0\|^{\frac{q}{p}} e^{-\frac{\lambda_3 \eta^{r-q}}{p \lambda_2} (t-t_0)} + \sqrt[p]{\frac{b \lambda_2}{\lambda_1 \lambda_3 \eta^{r-q}}}, t \in [t_0, t_0 + T_0[.$$

In addition, for each $t \geq t_0 + T_0$, either $\|\Psi(t)x\| \leq 1$ in which case $\|\Psi(t)x(t)\| \leq \sqrt[p]{\frac{\lambda_2 \lambda_3 + a \lambda_3 + b \lambda_2}{\lambda_1 \lambda_3}}$, or $\|\Psi(t)x\| > 1$. In this second case, as in the case 2, we obtain, whenever $\|\Psi(t)x(t)\| > 1$, it holds that

$$\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2 \lambda_3 + a \lambda_3 + b \lambda_2}{\lambda_1 \lambda_3}}.$$

- $\|\Psi(t_0)x_0\| \leq 1$: As in the previous cases, we get, as long as $\|\Psi(t)x(t)\| \leq 1$, we have trivially that $\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2 \lambda_3 + a \lambda_3 + b \lambda_2}{\lambda_1 \lambda_3}}$. If $\|\Psi(t)x(t)\| > 1$ at some instant $t > t_0$, then, as in the case 2, we obtain, whenever $\|\Psi(t)x(t)\| > 1$, it holds that

$$\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2 \lambda_3 + a \lambda_3 + b \lambda_2}{\lambda_1 \lambda_3}}.$$

To sum up, there exist $\lambda, R > 0$, such that for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$, we have the following:

$$\|\Psi(t)x(t, t_0, x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}} \|\Psi(t_0)x_0\|^{\frac{q}{p}} e^{-\frac{\lambda_3}{p \lambda_2} (t-t_0)} + R.$$

Then, the system (2.1) practically $\Psi^{\frac{q}{p}}$ -exponentially asymptotically stable. □

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