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A HALF-INVERSE PROBLEM FOR THE SINGULAR DIFFUSION OPERATOR WITH JUMP CONDITIONS

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Abstract. In this paper, half inverse spectral problem for diffusion operator with jump conditions dependent on the spectral parameter and discontinuity coefficient is considered. The half inverse problems is studied of determining the coefficient and two potential functions of the boundary value problem its spectrum by Hocstadt-Lieberman and Yang-Zettl methods. We show that two potential functions on the whole interval and the parameters in the boundary and jump conditions can be determined from the spectrum.

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1. Introduction and preliminaries

We consider the boundary value problem of the form

$$l(y) := -y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \delta(x)\, y,\ x \in [0,\pi] / \{a_1, a_2\} \tag{1.1}$$

with the boundary conditions

$$y'(0) = 0, \quad y(\pi) = 0 \tag{1.2}$$

and the jump conditions

$$y(a_1+0) = \alpha_1 y(a_1-0) \tag{1.3}$$

$$y'(a_1+0) = \beta_1 y'(a_1-0) + i\lambda\gamma_1 y(a_1-0) \tag{1.4}$$

$$y(a_2+0) = \alpha_2 y(a_2-0) \tag{1.5}$$

$$y'(a_2+0) = \beta_2 y'(a_2-0) + i\lambda\gamma_2 y(a_2-0), \tag{1.6}$$

where λ is a spectral parameter, $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real valued functions, $a_1 \in \left[0, \frac{\pi}{2}\right]$, $a_2 \in \left[\frac{\pi}{2}, \pi\right]$, $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are real numbers, $|\alpha_i - 1|^2 + \gamma_i^2 \neq 0$ $(\alpha_i > 0;\ i = 1,2)$, $\beta_i = \frac{1}{\alpha_i}\,(i = 1,2)$ and

$$\delta(x) = \begin{cases} \alpha^2, & x \in \left(0, \frac{\pi}{2}\right) \\ \beta^2, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

for $0<\alpha<\beta<1$, $\alpha+\beta>1$.

The inverse problems consist in recovering the coefficients of an operator from their spectral characteristics. A lot of study were done the inverse spectral problem for Sturm-Liouville operators and diffusion operators [1,2,4–26]. The first results on inverse problems theory of Sturm-Liouville operators were given by Ambarzumyan [3]. The half inverse problems for Sturm-Liouville equations; the known potential in half interval is determined by the help of a one spectrum over the interval. New results on the half inverse problem were obtained by Hochstadt and Lieberman [11]. They proved the spectrum of the problem as:

$$\begin{aligned} -y''+q(x)\,y &= \lambda y,\ x\in[0,1] \\ y'(0)-hy(0) &= 0 \\ y'(1)+Hy(1) &= 0 \end{aligned}$$

and potential $q(x)$ on the $\left(\frac{1}{2},1\right)$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$ almost everywhere. Hald [10] proved similar results in the case when there exists a impulsive conditions inside the interval. Many studies have been done by different authors for half inverse problems using these methods [14,19]. In [19] the authors studied the existence of the solution for the half-inverse problem of Sturm-Liouville problems and gave method of reconstructing this solution under same conditions by Sakhnovich [19]. Recently, some new uniqueness results have been given on the inverse or half inverse spectral analysis of differential operators. Koyunbakan and Panakhov [14] proved the half inverse problem for diffusion operator on the finite interval $[0,\pi]$. Ran Zhang, Xiao-Chuan Xu, Chuan-Fu Yang and Natalia Pavlovna Bondarenko proved the determination of the impulsive Sturm-Liouville operator from a set of eigenvalues [26]. The purpose of this study is to prove half inverse problem by using the Hocstadt- Lieberman and Yang-Zettl methods for the following equations

$$\tilde{l}(y):=-y''+[2\lambda\tilde{p}(x)+\tilde{q}(x)]\,y=\lambda^2\tilde{\delta}(x)\,y,\ x\in[0,\pi]/\{a_1,a_2\} \tag{1.7}$$

$$y'(0)=0,\ y(\pi)=0 \tag{1.8}$$

$$y(a_1+0)=\tilde{\alpha}_1 y(a_1-0) \tag{1.9}$$

$$y'(a_1+0)=\tilde{\beta}_1 y'(a_1-0)+i\lambda\tilde{\gamma}_1 y(a_1-0) \tag{1.10}$$

$$y(a_2+0)=\tilde{\alpha}_2 y(a_2-0) \tag{1.11}$$

$$y'(a_2+0)=\tilde{\beta}_2 y'(a_2-0)+i\lambda\tilde{\gamma}_2 y(a_2-0). \tag{1.12}$$

Lemma 1. *Let $p(x)\in W_2^1(0,\pi)$, $q(x)\in L_2(0,\pi)$, $M(x,t)$,$N(x,t)$ are summable functions on $[0,\pi]$ such that the representation for each $x\in[0,\pi]/\{a_1,a_2\}$.* $\varphi(x,\lambda)$

is the solution of the equations (1.1)*, providing boundary conditions* (1.2) *and discontinuity conditions* (1.3)-(1.6)*,*

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x M(x,t)\cos\lambda t dt + \int_0^x N(x,t)\sin\lambda t dt$$

for $0 < x < \frac{\pi}{2}$*, that is given as*

$$\begin{aligned}\varphi_0(x,\lambda) = {} & \left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^+(x) - \frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right] \\ & + \left(\beta_1^- - \frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^-(x) + \frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right],\end{aligned} \tag{1.13}$$

for $\frac{\pi}{2} < x \le \pi$*,*

$$\begin{aligned}\varphi_0(x,\lambda) = {} & \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^+(\pi) - \frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right] \\ & + \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^-(\pi) - \frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right] \\ & + \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^+(\pi) + \frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right] \\ & + \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^-(\pi) + \frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right],\end{aligned} \tag{1.14}$$

where $\xi^\pm(x) = \pm\alpha x \mp \alpha a_1 + a_1, \qquad k^\pm(x) = \xi^+(a_2) \pm \beta x \mp \beta a_2,$

$$s^\pm(x) = \xi^-(a_2) \pm \beta x \mp \beta a_2, \qquad \beta_1^\mp = \frac{1}{2}\left(\alpha_1 \mp \frac{\beta_1}{\alpha}\right), \qquad \beta_2^\mp = \frac{1}{2}\left(\alpha_2 \mp \frac{\alpha\beta_2}{\beta}\right).$$

Thus, the following relations hold:

If $p(x) \in W_2^2(0,\pi)\,,\, q(x) \in W_2^1(0,\pi)$

$$\begin{cases} \frac{\partial^2 M(x,t)}{\partial x^2} - \rho(x)\frac{\partial^2 M(x,t)}{\partial t^2} = 2p(x)\frac{\partial N(x,t)}{\partial t} + q(x)M(x,t) \\ \frac{\partial^2 N(x,t)}{\partial x^2} - \rho(x)\frac{\partial^2 N(x,t)}{\partial t^2} = -2p(x)\frac{\partial M(x,t)}{\partial t} + q(x)N(x,t) \end{cases}$$

$$M(x,\varsigma^+(x))\cos\frac{\beta(x)}{\alpha} + N(x,\varsigma^+(x))\sin\frac{\beta(x)}{\alpha} = \left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right)\int_0^x \left(q(t) + \frac{p^2(t)}{\alpha^2}\right)dt$$

$$M(x,\varsigma^+(x))\sin\frac{\beta(x)}{\alpha} - N(x,\varsigma^+(x))\cos\frac{\beta(x)}{\alpha} = \left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right)(p(x) - p(0))$$

$$\begin{aligned} & M(x,k^+(x)+0) - M(x,k^+(x)-0) \\ & = -\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)(p(x)-p(0))\sin\frac{\omega(x)}{\beta} - \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\int_0^x \left(q(t) + \frac{p^2(t)}{\beta^2}\right)dt\cos\frac{\omega(x)}{\beta} \\ & N(x,k^+(x)+0) - N(x,k^+(x)-0)\end{aligned}$$

$$= \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)(p(x) - p(0)) \cos\frac{\omega(x)}{\beta} - \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\int_0^x \left(q(t) + \frac{p^2(t)}{\beta^2}\right) dt \sin\frac{\omega(x)}{\beta},$$

$$\left.\frac{\partial M(x,t)}{\partial t}\right|_{t=0} = N(x,0) = 0,$$

where $\beta(x) = \int_0^x p(t)\,dt$, $\omega(x) = \int_{a_2}^x p(t)\,dt + \int_0^{a_1} p(t)\,dt$.

The proof is done as in [6].

Definition 1. The function $\Delta(\lambda)$ is called the characteristic function of the eigenvalues $\{\lambda_n\}$ of the problem (1.1)-(1.6). $\tilde{\Delta}(\lambda)$ is called the characteristic function of the eigenvalues $\left\{\tilde{\lambda}_n\right\}$ of the problem (1.7)-(1.12).

Let $\lambda = s^2, s = \sigma + i\tau, \sigma, \tau \in R$. The solution $\varphi(x,\lambda)$ of (1.1)-(1.6) has the following asymptotic formulas hold on for $|\lambda| \to \infty$,
for $0 < x < \frac{\pi}{2}$,

$$\varphi(x,\lambda) = \frac{1}{2}\left(\frac{\alpha_1}{2} \mp \frac{\beta_1}{2\alpha} + \frac{\gamma_1}{2\alpha}\right)\exp\left(-i\left(\lambda\xi^+(x) - \frac{v(x)}{\alpha}\right)\right)\left(1 + O\left(\frac{1}{\lambda}\right)\right),$$

for $\frac{\pi}{2} < x \le \pi$,

$$\varphi(x,\lambda) = \frac{1}{2}\left(\frac{\alpha_2}{2} + \frac{\alpha\beta_2}{2\beta} + \frac{\gamma_2}{2\beta}\right)\exp\left(-i\left(\lambda k^+(x) - \frac{t(x)}{\beta}\right)\right)\left(1 + O\left(\frac{1}{\lambda}\right)\right),$$

where $v(x) = \int_{a_1}^x p(t)\,dt$, $t(x) = \int_{a_2}^x p(t)\,dt$.

In this study, if $q(x)$ and $p(x)$ to be known almost everywhere on $\left(\frac{\pi}{2}, \pi\right)$, it is sufficient to determine uniquely $p(x)$ and $q(x)$ on the whole interval $(0,\pi)$.

2. Main result

If $\varphi_0(x,\lambda)$ is a nontrivial solution of equation (1.1) with conditions (1.2)-(1.6), then λ_0 is called an eigenvalue. Additionally, $\varphi_0(x,\lambda)$ is called the eigenfunction of the problem corresponding to the eigenvalue λ_0. $\{\lambda_n\}$ are the eigenvalues of the problem.

Lemma 2. *If* $\lambda_n = \tilde{\lambda}_n$, $\frac{\alpha}{\tilde{\alpha}} = \frac{\beta}{\tilde{\beta}}$ *then* $\alpha = \tilde{\alpha}$ *and* $\beta = \tilde{\beta}$ *for all* $n \in N$.

Proof. Since $\lambda_n = \tilde{\lambda}_n$ and $\Delta(\lambda), \tilde{\Delta}(\lambda)$are entire functions in λ of order one by Hadamard factorization theorem for $\lambda \in C$

$$\Delta(\lambda) \equiv C\tilde{\Delta}(\lambda).$$

On the other hand, (1.1) can be written as

$$\Delta_0(\lambda) - C\tilde{\Delta}_0(\lambda) = C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right].$$

Hence

$$C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right]$$

$$
\begin{aligned}
&= \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] + \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^-(\pi) - \frac{w(\pi)}{\beta}\right] \\
&\quad + \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^+(\pi) + \frac{w(\pi)}{\beta}\right] + \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^-(\pi) + \frac{w(\pi)}{\beta}\right] \\
&\quad - C\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^+(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^- + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^-(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] \\
&\quad - C\left(\tilde{\beta}_2^- - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^+(\pi) + \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^+ - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^-(\pi) + \frac{\tilde{w}(\pi)}{\beta}\right].
\end{aligned} \tag{2.1}
$$

If we multiply both sides of (2.1) by $\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right]$ and integrate with respect to λ in (ε, T), (ε is a sufficiently small positive number) for any positive real number T, then we get

$$
\begin{aligned}
&\int_\varepsilon^T \left(C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right]\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] d\lambda \\
&= \int_\varepsilon^T \left\{\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] + \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^-(\pi) - \frac{w(\pi)}{\beta}\right]\right. \\
&\quad + \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^+(\pi) + \frac{w(\pi)}{\beta}\right] + \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^-(\pi) + \frac{w(\pi)}{\beta}\right] \\
&\quad - C\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^+(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^- + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^-(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] \\
&\quad - C\left(\tilde{\beta}_2^- - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^+(\pi) + \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^+ - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^-(\pi) + \frac{\tilde{w}(\pi)}{\beta}\right] d\lambda.
\end{aligned}
$$

And so

$$
\begin{aligned}
&\int_\varepsilon^T \left(C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right]\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] d\lambda \\
&= \int_\varepsilon^T \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos^2\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] d\lambda \\
&\quad - C\int_\varepsilon^T \left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right]\cos\left[\lambda k^+(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] d\lambda \\
&= \int_\varepsilon^T \frac{1}{2}\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right) + \frac{1}{2}\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos\left[2\lambda k^+(\pi) - \frac{2w(\pi)}{\beta}\right] d\lambda \\
&\quad - C\int_\varepsilon^T \frac{1}{2}\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\left(\cos\left[2\lambda k^+(\pi) - \frac{\tilde{w}(\pi) + w(\pi)}{\beta}\right] + \cos\left[\frac{w(\pi) - \tilde{w}(\pi)}{\tilde{\beta}}\right]\right) d\lambda,
\end{aligned}
$$

$\Delta(\lambda)-\Delta_0(\lambda)=O\left(\frac{1}{|\lambda|}e^{|Im\lambda|k^+(\pi)}\right)$, $\tilde{\Delta}(\lambda)-\tilde{\Delta}_0(\lambda)=O\left(\frac{1}{|\lambda|}e^{|Im\lambda|k^+(\pi)}\right)$ for all λ in (ε,T),

$$\frac{C}{2}\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)-\frac{1}{2}\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)=O\left(\frac{1}{T}\right).$$

By letting T tend to infinity we see that

$$C=\frac{\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}}{\beta_2^+ + \frac{\gamma_2}{2\beta}}.$$

Similarly, if we multiply both side of (2.1) cos $\left[\lambda k^-(\pi)-\frac{w(\pi)}{\beta}\right]$ and integrate again with respect to λ in (ε,T) and by letting T tend to infinity, then we get

$$C=\frac{\tilde{\beta}_2^- + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}}{\beta_2^- + \frac{\gamma_2}{2\beta}}.$$

But since α,β and $\tilde{\alpha},\tilde{\beta}$ are positive, and $w^+(\pi)-\tilde{w}^+(\pi)=w^-(\pi)-\tilde{w}^-(\pi)$ we conclude that $C=1$. Hence $\frac{\tilde{\beta}_2^+}{\beta_2^+}=\frac{\tilde{\beta}_2^-}{\beta_2^-}$ is obtained. We have therefore proved since $\alpha=\tilde{\alpha}$ that $\beta=\tilde{\beta}$.

The proof is completed. □

Lemma 3. *If* $\lambda_n=\tilde{\lambda}_n$ *then* $\alpha_i=\tilde{\alpha}_i$ *and* $\gamma_i=\tilde{\gamma}_i$ $(i=1,2)$ *for all* $n\in\mathbb{N}$.
The proof is done as in [6].

Theorem 1. *Let* $\{\lambda_n\}$ *be the eigenvalues of both problem* (1.1)-(1.6) *and* (1.7)-(1.12). *If* $p(x)=\tilde{p}(x)$ *and* $q(x)=\tilde{q}(x)$ *on* $\left[\frac{\pi}{2},\pi\right]$, *then* $p(x)=\tilde{p}(x)$ *and* $q(x)=\tilde{q}(x)$ *are almost everywhere on* $[0,\pi]$.

Proof of Theorem 1. Let function $\varphi(x,\lambda)$ be the solution of equation (1.1) under the conditions (1.2)-(1.6) and the function $\tilde{\varphi}(x,\lambda)$ the solution of equation (1.7) under the conditions (1.8)-(1.12) on $\left[0,\frac{\pi}{2}\right]$. The integral forms of the functions $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ can be obtained as follows

$$\begin{aligned}\varphi(x,\lambda)=&\left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]\\&+\left(\beta_1^- - \frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]\\&+\int_0^x M(x,t)\cos\lambda t\,dt+\int_0^x N(x,t)\sin\lambda t\,dt\end{aligned}\tag{2.2}$$

and

$$\tilde{\varphi}(x,\lambda)=\left(\tilde{\beta}_1^+ + \frac{\tilde{\gamma}_1}{2\alpha}\right)\cos\left[\lambda\xi^+(x)-\frac{1}{\alpha}\int_{a_1}^x \tilde{p}(t)\,dt\right]$$

$$+\left(\tilde{\beta}_1^- - \frac{\tilde{\gamma}_1}{2\alpha}\right)\cos\left[\lambda\xi^-(x) + \frac{1}{\alpha}\int_{a_1}^{x}\tilde{p}(t)\,dt\right]$$
$$+\int_0^x \tilde{M}(x,t)\cos\lambda t\,dt + \int_0^x \tilde{N}(x,t)\sin\lambda t\,dt. \tag{2.3}$$

If we multiply equations (2.2) and (2.3):

$$\begin{aligned}
\varphi(x,\lambda)\cdot\tilde{\varphi}(x,\lambda) &= \frac{S^+\tilde{S}^+}{2}\left[\cos\left(2\lambda\xi^+(x) - K(x)\right) + \cos L(x)\right] \\
&+ \frac{S^+\tilde{S}^-}{2}\left[\cos\left(2\lambda a_1 t - L(x)\right) + \cos\left(2\lambda\alpha(x-a_1) - K(x)\right)\right] \\
&+ \frac{S^-\tilde{S}^+}{2}\left[\cos\left(2\lambda a_1 + L(x)\right) + \cos\left(2\lambda\alpha(x-a_1) + K(x)\right)\right] \\
&+ \frac{S^-\tilde{S}^-}{2}\left[\cos\left(2\lambda\xi^-(x) + L(x)\right) + \cos K(x)\right] \\
&+ S^+\int_0^x \tilde{M}(x,t)\cos\left[\lambda\xi^+(x) - \frac{t(x)}{\alpha}\right]\cos\lambda t\,dt \\
&+ S^+\int_0^x \tilde{N}(x,t)\cos\left[\lambda\xi^+(x) - \frac{t(x)}{\alpha}\right]\sin\lambda t\,dt \\
&+ S^-\int_0^x \tilde{M}(x,t)\cos\left[\lambda\xi^-(x) + \frac{t(x)}{\alpha}\right]\cos\lambda t\,dt \\
&+ S^-\int_0^x \tilde{N}(x,t)\cos\left[\lambda\xi^-(x) + \frac{t(x)}{\alpha}\right]\sin\lambda t\,dt \\
&+ \tilde{S}^+\int_0^x M(x,t)\cos\left[\lambda\xi^+(x) - \frac{\tilde{t}(x)}{\alpha}\right]\cos\lambda t\,dt \\
&+ \tilde{S}^+\int_0^x N(x,t)\cos\left[\lambda\xi^+(x) - \frac{\tilde{t}(x)}{\alpha}\right]\sin\lambda t\,dt \\
&+ \tilde{S}^-\int_0^x M(x,t)\cos\left[\lambda\xi^-(x) + \frac{\tilde{t}(x)}{\alpha}\right]\cos\lambda t\,dt \\
&+ \tilde{S}^-\int_0^x N(x,t)\cos\left[\lambda\xi^-(x) + \frac{\tilde{t}(x)}{\alpha}\right]\sin\lambda t\,dt \\
&+ \left(\int_0^x M(x,t)\cos\lambda t\,dt\right)\left(\int_0^x \tilde{M}(x,t)\cos\lambda t\,dt\right) \\
&+ \left(\int_0^x N(x,t)\sin\lambda t\,dt\right)\left(\int_0^x \tilde{N}(x,t)\sin\lambda t\,dt\right) \\
&+ \left(\int_0^x M(x,t)\cos\lambda t\,dt\right)\left(\int_0^x \tilde{N}(x,t)\sin\lambda t\,dt\right)
\end{aligned}$$

$$+\left(\int_0^x \tilde{M}(x,t)\cos\lambda t\,dt\right)\left(\int_0^x N(x,t)\sin\lambda t\,dt\right),$$

$$\begin{aligned}\varphi(x,\lambda)\cdot\tilde{\varphi}(x,\lambda) = {} & \frac{S^+\tilde{S}^+}{2}\left[\cos\left(2\lambda\xi^+(x)-K(x)\right)+\cos L(x)\right] \\ & +\frac{S^+\tilde{S}^-}{2}\left[\cos\left(2\lambda a_1 t - L(x)\right)+\cos\left(2\lambda\alpha(x-a_1)-K(x)\right)\right] \\ & +\frac{S^-\tilde{S}^+}{2}\left[\cos\left(2\lambda a_1 + L(x)\right)+\cos\left(2\lambda\alpha(x-a_1)+K(x)\right)\right] \\ & +\frac{S^-\tilde{S}^-}{2}\left[\cos\left(2\lambda\xi^-(x)+L(x)\right)+\cos K(x)\right] \\ & +\frac{1}{2}\left\{\int_0^x U_c(x,t)\cos\left(2\lambda t-K(t)\right)dt-\int_0^x U_s(x,t)\sin\left(2\lambda t-K(t)\right)dt\right\}\end{aligned} \tag{2.4}$$

is obtained, being

$$S^{\pm}=\left(\beta_1^{\pm}\mp\frac{\gamma_1}{2\alpha}\right),\ \tilde{S}^{\pm}=\left(\tilde{\beta}_1^{\pm}\mp\frac{\tilde{\gamma}_1}{2\alpha}\right),\ K(x)=\frac{t(x)+\tilde{t}(x)}{2},\ L(x)=\frac{t(x)-\tilde{t}(x)}{2},$$

$$\begin{aligned} & U_c(x,t) \\ & = S^+\tilde{M}\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right)+S^-\tilde{M}\left(x,\xi^-(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad +\tilde{S}^+M\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)+\tilde{S}^-M\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad -S^-\tilde{N}\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right)-S^-\tilde{N}\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad -\tilde{S}^+N\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)-\tilde{S}^-N\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad +K_1(x,t)\cos K(t)+K_2(x,t)\cos K(t)+M_1(x,t)\sin K(t)+M_2(x,t)\sin K(t),\end{aligned}$$

$$\begin{aligned} & U_s(x,t) \\ & = S^+\tilde{M}\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right)+S^-\tilde{M}\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad +\tilde{S}^+M\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)+\tilde{S}^-M\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad +S^+\tilde{N}\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right)+S^-\tilde{N}\left(x,\xi^-(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad +\tilde{S}^+N\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)+\tilde{S}^-N\left(x,\xi^-(x)-2t\right)\cos\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad +K_1(x,t)\sin K(t)+K_2(x,t)\sin K(t)),\end{aligned}$$

$$K_1(x,t) = \int_{-x}^{x-2t} M(x,s)\tilde{M}(x,s+2t)\,ds + \int_{2t-x}^{x} M(x,s)\tilde{M}(x,s+2t)\,ds$$
$$K_2(x,t) = \int_{-x}^{x-2t} N(x,s)\tilde{N}(x,s+2t)\,ds + \int_{2t-x}^{x} n(x,s)\tilde{N}(x,s+2t)\,ds$$
$$M_1(x,t) = \int_{-x}^{x-2t} M(x,s)\tilde{N}(x,s+2t)\,ds - \int_{2t-x}^{x} M(x,s)\tilde{N}(x,s+2t)\,ds$$
$$M_2(x,t) = -\int_{-x}^{x-2t} N(x,s)\tilde{M}(x,s+2t)\,ds + \int_{2t-x}^{x} N(x,s)\tilde{M}(x,s+2t)\,ds.$$

Let $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ be substituted into (1.1) and (1.7),

$$-\varphi''(x,\lambda) + (2\lambda p(x) + q(x))\,\varphi(x,\lambda) = \lambda^2\rho(x)\,\varphi(x,\lambda) \tag{2.5}$$

$$-\tilde{\varphi}''(x,\lambda) + (2\lambda p(x) + q(x))\,\tilde{\varphi}(x,\lambda) = \lambda^2\rho(x)\,\tilde{\varphi}(x,\lambda) \tag{2.6}$$

The following equations are obtained using (2.5) and (2.6):

$$\int_0^{\frac{\pi}{2}} \varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,[2\lambda(p(x)-\tilde{p}(x)) + (q(x)-\tilde{q}(x))]\,dx$$
$$= \left[\tilde{\varphi}'(x,\lambda)\,\varphi(x,\lambda) - \varphi'(x,\lambda)\,\tilde{\varphi}(x,\lambda)\right]_0^{\frac{\pi}{2}} + |_{\frac{\pi}{2}}^{\pi},$$

$$\int_0^{\frac{\pi}{2}} \varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,[2\lambda(p(x)-\tilde{p}(x)) + (q(x)-\tilde{q}(x))]\,dx$$
$$+\tilde{\varphi}'(\pi,\lambda)\,\varphi(\pi,\lambda) - \varphi'(\pi,\lambda)\,\tilde{\varphi}(\pi,\lambda) = 0. \tag{2.7}$$

Let $Q(x) = q(x) - \tilde{q}(x)$ and $P(x) = p(x) - \tilde{p}(x)$,

$$U(\lambda) = \int_0^{\frac{\pi}{2}} [2\lambda P(x) + Q(x)]\,\varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,dx.$$

It is obvious that the functions $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ are the solutions which satisfy boundary value conditions of (1.2) and (1.8), respectively, then if we consider these facts in equation (2.7), we obtain the following equation

$$U(\lambda_n) = 0 \tag{2.8}$$

for each eigenvalue λ_n. Let us marked

$$U_1(\lambda) = \int_0^{\frac{\pi}{2}} P(x)\,\varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,dx, U_2(\lambda) = \int_0^{\frac{\pi}{2}} Q(x)\,\varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,dx.$$

Then equations (2.7) can be rewritten as

$$2\lambda_n U_1(\lambda_n) + U_2(\lambda_n) = 0.$$

From (2.4) and (2.7) we obtain

$$|U(\lambda)| \leq (C_1 + C_2|\lambda|)\exp(\tau\pi), \tag{2.9}$$

where $C_1, C_2 > 0$ are constants for all complex λ. Since $\lambda_n = \tilde{\lambda}_n$, $\Delta(\lambda) = \varphi(\pi,\lambda) = \tilde{\varphi}(\pi,\lambda)$, thus,

$$U(\lambda) = \int_0^{\frac{\pi}{2}} [2\lambda P(x) + Q(x)]\,\varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,dx = \Delta(\lambda)\left[\varphi(\pi,\lambda) - \tilde{\varphi}(\pi,\lambda)\right].$$

The function $\phi(\lambda) = \frac{U(\lambda)}{\Delta(\lambda)}$ is an entire function with respect to λ.

It follows from $\Delta(\lambda) \geq (|\lambda\beta| - C)\exp(\tau\xi^+(x))$ and (2.9), $\phi(\lambda) = O(1)$ for sufficient large $|\lambda|$. We obtain $\phi(\lambda) = C$, for all λ by Liouville's Theorem.

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,[2\lambda P(x) + Q(x)]\,dx \\
&= C\left[\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right) R_1(a_2)\cos\left[\lambda k^+(\pi) - \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right]\right. \\
&\quad + \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right) R_2(a_2)\cos\left[\lambda k^-(\pi) - \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right] \\
&\quad + \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right) R_1(a_2)\cos\left[\lambda s^+(\pi) + \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right] \\
&\quad \left. + \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right) R_2(a_2)\cos\left[\lambda s^-(\pi) + \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right]\right] + O\left(\exp\left(\tau k^+(\pi)\right)\right).
\end{aligned}$$

By the Riemann-Lebesgue lemma, for $\lambda \to \infty$, $\lambda \in \mathrm{R}$ we get $C = 0$. Then,

$$\begin{aligned}
2U_1(\lambda) = S^+\tilde{S}^+ &\int_0^{\frac{\pi}{2}} P(x)\cos\left(2\lambda\xi^+(x) - K(x)\right)dx \\
&+ S^+\tilde{S}^+ \int_0^{\frac{\pi}{2}} P(x)\cos L(x)\,dx \\
&+ S^+\tilde{S}^- \int_0^{\frac{\pi}{2}} P(x)\cos(2\lambda a_1 t - L(x))\,dx \\
&+ S^+\tilde{S}^- \int_0^{\frac{\pi}{2}} P(x)\cos(2\lambda\alpha(x - a_1) - K(x))\,dx \\
&+ S^-\tilde{S}^+ \int_0^{\frac{\pi}{2}} P(x)\cos(2\lambda a_1 + L(x))\,dx \\
&+ S^-\tilde{S}^+ \int_0^{\frac{\pi}{2}} P(x)\cos\cos(2\lambda\alpha(x - a_1) + K(x))\,dx \\
&+ S^-\tilde{S}^- \int_0^{\frac{\pi}{2}} P(x)\cos\left(2\lambda\xi^-(x) + L(x)\right)dx
\end{aligned}$$

$$+S^{-}\tilde{S}^{-}\int_0^{\frac{\pi}{2}}P(x)\cos K(x)\,dx$$
$$+\int_0^{\frac{\pi}{2}}P(x)\left(\int_0^x U_c(x,t)\cos(2\lambda t-K(t))\,dt\right)dx$$
$$-\int_0^{\frac{\pi}{2}}P(x)\left(\int_0^x U_s(x,t)\sin(2\lambda t-K(t))\,dt\right)dx,$$

where $\xi^{\pm}(x)=\pm\alpha x\mp\alpha a_1+a_1$, $k^{\pm}(x)=\mu^{+}(a_2)\pm\beta x\mp\beta a_2$,

$$s^{\pm}(x)=\mu^{-}(a_2)\pm\beta x\mp\beta a_2,\ \beta_1^{\mp}=\frac{1}{2}\left(\alpha_1\mp\frac{\beta_1}{\alpha}\right),\ \beta_2^{\mp}=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right).$$

$$\begin{aligned}
2U_1(\lambda)={}&\frac{S^{+}\tilde{S}^{+}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{-i(K(t))}e^{i\left(2\lambda\xi^{+}(t)\right)}dt+\frac{S^{+}\tilde{S}^{+}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{i(K(t))}e^{-i\left(2\lambda\xi^{+}(t)\right)}dt\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{-i(L(t))}e^{i(2\lambda a_1 t)}dt+\frac{S^{+}\tilde{S}^{-}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{i(L(t))}e^{-i(2\lambda a_1 t)}dt\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{-i(K(t))}e^{i(2\lambda\alpha(t-a_1))}dt\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{i(K(t))}e^{-i(2\lambda\alpha(t-a_1))}dt\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{i(L(t))}e^{i(2\lambda a_1 t)}dt+\frac{S^{-}\tilde{S}^{+}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{-i(L(t))}e^{i(2\lambda a_1 t)}dt\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{i(K(t))}e^{i(2\lambda\alpha(t-a_1))}dt+\frac{S^{-}\tilde{S}^{+}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{-i(K(t))}e^{i(2\lambda\alpha(t-a_1))}dt\\
&+\frac{S^{-}\tilde{S}^{-}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{i(L(t))}e^{-i\left(2\lambda\xi^{-}(t)\right)}dt+\frac{S^{-}\tilde{S}^{-}}{2}\int_0^{\frac{\pi}{2}}P(t)\,e^{-i(L(t))}e^{i\left(2\lambda\xi^{-}(t)\right)}dt\\
&+S^{+}\tilde{S}^{+}\int_0^{\frac{\pi}{2}}P(x)\cos L(x)\,dx+S^{-}\tilde{S}^{-}\int_0^{\frac{\pi}{2}}P(x)\cos K(x)\,dx\\
&+\int_0^{\frac{\pi}{2}}P(x)\left(\int_0^x U_c(x,t)\cos(2\lambda t-K(t))\,dt\right)dx\\
&-\int_0^{\frac{\pi}{2}}P(x)\left(\int_0^x U_s(x,t)\sin(2\lambda t-K(t))\,dt\right)dx
\end{aligned}$$

if necessary operations are performed and integrals are calculated.

$$\begin{aligned}
2U_1(\lambda)={}&\frac{S^{+}\tilde{S}^{+}}{2}\left[\frac{T_1\left(\pi/2\right)}{2i\lambda\alpha}e^{i\left(2\lambda\xi^{+}\left(\frac{\pi}{2}\right)\right)}-\frac{T_1(0)}{2i\lambda\alpha}e^{2i\lambda(\alpha a_1+a_1)}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_1'(t)\,e^{i\left(2\lambda\xi^{+}(t)\right)}dt\right]\\
&+\frac{S^{+}\tilde{S}^{+}}{2}\left[-\frac{T_2\left(\pi/2\right)}{2i\lambda\alpha}e^{-i\left(2\lambda\xi^{+}\left(\frac{\pi}{2}\right)\right)}+\frac{T_2(0)}{2i\lambda\alpha}e^{-2i\lambda(\alpha a_1+a_1)}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_2'(t)\,e^{-i\left(2\lambda\xi^{+}(t)\right)}dt\right]\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\left[\frac{T_3\left(\pi/2\right)}{2i\lambda\alpha}e^{i\lambda a_1}-\frac{T_3(0)}{2i\lambda\alpha}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_3'(t)\,e^{2ia_1 t}dt\right]
\end{aligned}$$

$$+\frac{S^+\tilde{S}^-}{2}\left[-\frac{T_4\left(\pi/2\right)}{2i\lambda\alpha}e^{i\lambda a_1}+\frac{T_4(0)}{2i\lambda\alpha}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_4'(t)\,e^{-2ia_1t}dt\right]$$
$$+\frac{S^+S^-}{2}\left[\frac{T_1\left(\pi/2\right)}{2i\lambda\alpha}e^{2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}-\frac{T_1(0)}{2i\lambda\alpha}e^{-2i\lambda\alpha a_1}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_1'(t)\,e^{2i\lambda\alpha(t-a_1)}dt\right]$$
$$+\frac{S^+S^-}{2}\left[-\frac{T_2\left(\pi/2\right)}{2i\lambda\alpha}e^{-2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}+\frac{T_2(0)}{2i\lambda\alpha}e^{2i\lambda\alpha a_1}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_2'(t)\,e^{-2i\lambda\alpha(t-a_1)}dt\right]$$
$$+\frac{S^-\tilde{S}^+}{2}\left[-\frac{T_3\left(\pi/2\right)}{2i\lambda\alpha}e^{-i\lambda a_1\pi}+\frac{T_3(0)}{2i\lambda\alpha}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_3'(t)\,e^{-2ia_1t}dt\right]$$
$$+\frac{S^-\tilde{S}^+}{2}\left[\frac{T_4\left(\pi/2\right)}{2i\lambda\alpha}e^{i\lambda a_1\pi}-\frac{T_4(0)}{2i\lambda\alpha}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_4'(t)\,e^{2ia_1t}dt\right]$$
$$+\frac{S^-\tilde{S}^+}{2}\left[-\frac{T_1\left(\pi/2\right)}{2i\lambda\alpha}e^{-2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}+\frac{T_1(0)}{2i\lambda\alpha}e^{2i\lambda\alpha a_1}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_1'(t)\,e^{-2i\lambda\alpha(t-a_1)}dt\right]$$
$$+\frac{S^-\tilde{S}^+}{2}\left[\frac{T_2\left(\pi/2\right)}{2i\lambda\alpha}e^{2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}-\frac{T_2(0)}{2i\lambda\alpha}e^{-2i\lambda\alpha a_1}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_2'(t)\,e^{2i\lambda\alpha(t-a_1)}dt\right]$$
$$+\frac{S^-\tilde{S}^-}{2}\left[-\frac{T_4\left(\pi/2\right)}{2i\lambda\alpha}e^{i\left(2\lambda\xi^-\left(\frac{\pi}{2}\right)\right)}+\frac{T_4(0)}{2i\lambda\alpha}e^{2i\lambda(\alpha a_1+a_1)}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_4'(t)\,e^{i\left(2\lambda\xi^-(t)\right)}dt\right]$$
$$+\frac{S^-\tilde{S}^-}{2}\left[\frac{T_3\left(\pi/2\right)}{2i\lambda\alpha}e^{-i\left(2\lambda\xi^-\left(\frac{\pi}{2}\right)\right)}-\frac{T_3(0)}{2i\lambda\alpha}e^{2i\lambda(\alpha a_1-a_1)}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_3'(t)\,e^{-i\left(2\lambda\xi^+(t)\right)}dt\right]$$
$$+S^+\tilde{S}^+\int_0^{\frac{\pi}{2}}P(x)\cos L(x)\,dx+S^-\tilde{S}^-\int_0^{\frac{\pi}{2}}P(x)\cos K(x)\,dx$$
$$+\left[\frac{T_5\left(\pi/2\right)}{2i\lambda}e^{i\pi\lambda}-\frac{T_5(0)}{2i\lambda}-\frac{1}{2i\lambda}\int_0^{\frac{\pi}{2}}T_1'(t)\,e^{2i\lambda t}dt\right]$$
$$+\left[-\frac{T_6\left(\pi/2\right)}{2i\lambda}e^{-i\pi\lambda}+\frac{T_6(0)}{2i\lambda}+\frac{1}{2i\lambda}\int_0^{\frac{\pi}{2}}T_6'(t)\,e^{-2i\lambda t}dt\right],$$

where

$$T_1(t)=P(t)\,e^{-i(K(t))},\ \ T_2(t)=P(t)\,e^{i(K(t))},\ \ T_3(t)=P(t)\,e^{-i(L(t))},$$

$$T_4(t)=P(t)\,e^{i(L(t))},\ \ P_1(t)=\int_t^{\frac{\pi}{2}}P(x)\,U_c(x,t)\,dx,\ \ P_2(t)=\int_t^{\frac{\pi}{2}}P(x)\,U_s(x,t)\,dx,$$

$$T_5(t)=\frac{P_1(t)+iP_2(t)}{2}e^{-iK(t)},\ \ T_6(t)=\frac{P_1(t)-iP_2(t)}{2}e^{iK(t)}.$$

By the Riemann-Lebesgue lemma $\int_0^{\frac{\pi}{2}}P(x)\cos L(x)\,dx=0,\ \ \int_0^{\frac{\pi}{2}}P(x)\cos K(x)\,dx=0$ and $P\left(\frac{\pi}{2}\right)=0$ for $\lambda\to\infty$.

Thus,

$$
\begin{aligned}
2U_1(\lambda) = &-\frac{S^+\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_1'(t)\, e^{i(2\lambda\xi^+(t))}dt + \frac{S^+\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_2'(t)\, e^{-i(2\lambda\xi^+(t))}dt \\
&-\frac{S^+\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_3'(t)\, e^{2ia_1t}dt + \frac{S^+\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_4'(t)\, e^{-2ia_1t}dt \\
&-\frac{S^+S^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_1'(t)\, e^{2i\lambda\alpha(t-a_1)}dt + \frac{S^+S^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_2'(t)\, e^{-2i\lambda\alpha(t-a_1)}dt \\
&+\frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_3'(t)\, e^{-2ia_1t}dt - \frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_4'(t)\, e^{2ia_1t}dt \\
&+\frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_1'(t)\, e^{-2i\lambda\alpha(t-a_1)}dt + \frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_2'(t)\, e^{2i\lambda\alpha(t-a_1)}dt \\
&+\frac{S^-\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_4'(t)\, e^{i(2\lambda\xi^-(t))}dt - \frac{S^-\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_3'(t)\, e^{-i(2\lambda\xi^-(t))}dt \\
&+\frac{i}{2\lambda}\int_0^{\frac{\pi}{2}} T_5'(t)\, e^{2i\lambda t}dt - \frac{i}{2\lambda}\int_0^{\frac{\pi}{2}} T_6'(t)\, e^{-2i\lambda t}dt, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
2U_2(\lambda) = &\, S^+\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\xi^+(x)-K(x))}+e^{-i(2\lambda\xi^+(x)-K(x))}}{2}\right)dx \\
&+S^+\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda a_1t-L(x))}+e^{-i(2\lambda a_1t-L(x))}}{2}\right)dx \\
&+S^+\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\alpha(x-a_1)-K(x))}+e^{-i(2\lambda\alpha(x-a_1)-K(x))}}{2}\right)dx \\
&+S^-\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda a_1t+L(x))}+e^{-i(2\lambda a_1t+L(x))}}{2}\right)dx \\
&+S^-\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\alpha(x-a_1)+K(x))}+e^{-i(2\lambda\alpha(x-a_1)+K(x))}}{2}\right)dx \\
&+S^-\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\xi^-(x)+L(x))}+e^{-i(2\lambda\xi^-(x)+L(x))}}{2}\right)dx \\
&+S^+\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\cos L(x)\,dx + S^-\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\cos K(x)\,dx \\
&+\int_0^{\frac{\pi}{2}} Q(x)\left(\int_0^x U_c(x,t)\cos(2\lambda t-K(t))\,dt\right)dx \\
&-\int_0^{\frac{\pi}{2}} Q(x)\left(\int_0^x U_s(x,t)\sin(2\lambda t-K(t))\,dt\right)dx,
\end{aligned}
$$

where

$$R_1(t)=Q(t)e^{-i(K(t))},\ \ R_2(t)=Q(t)e^{i(K(t))},\ \ R_3(t)=Q(t)e^{-i(L(t))},$$

$$R_4(t)=Q(t)e^{i(L(t))},\ \ Q_1(t)=\int_t^{\frac{\pi}{2}}P(x)U_c(x,t)\,dx,\ \ Q_2(t)=\int_t^{\frac{\pi}{2}}P(x)U_s(x,t)\,dx,$$

$$R_5(t)=\frac{Q_1(t)+iQ_2(t)}{2}e^{-iK(t)},\ \ R_6(t)=\frac{Q_1(t)-iQ_2(t)}{2}e^{iK(t)}.$$

By the Riemann-Lebesgue lemma $\int_0^{\frac{\pi}{2}}Q(x)\cos L(x)\,dx=0,\ \ \int_0^{\frac{\pi}{2}}Q(x)\cos K(x)\,dx=0$. Thus,

$$\begin{aligned}2U_2(\lambda)=&\frac{S^+\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_1(t)e^{i(2\lambda\xi^+(t))}dt+\frac{S^+\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_2(t)e^{-i(2\lambda\xi^+(t))}dt\\&+\frac{S^+\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_3(t)e^{2ia_1t}dt+\frac{S^+\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_4(t)e^{-2ia_1t}dt\\&+\frac{S^+S^-}{2}\int_0^{\frac{\pi}{2}}R_1(t)e^{2i\lambda\alpha(t-a_1)}dt+\frac{S^+S^-}{2}\int_0^{\frac{\pi}{2}}R_2(t)e^{-2i\lambda\alpha(t-a_1)}dt\\&+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_3(t)e^{-2ia_1t}dt+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_4(t)e^{2ia_1t}dt\\&+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_1(t)e^{-2i\lambda\alpha(t-a_1)}dt+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_2(t)e^{2i\lambda\alpha(t-a_1)}dt\\&++\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_4(t)e^{i(2\lambda\xi^-(t))}dt+\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_3(t)e^{-i(2\lambda\xi^-(t))}dt\\&+\frac{i}{2\lambda}\int_0^{\frac{\pi}{2}}R_5(t)e^{2i\lambda t}dt+\frac{i}{2\lambda}\int_0^{\frac{\pi}{2}}R_6(t)e^{-2i\lambda t}dt\end{aligned}\tag{2.11}$$

$$2\lambda U_1(\lambda)+U_2(\lambda)=0.\tag{2.12}$$

If (2.10) and (2.11) are substituted into (2.12), we get

$$\begin{aligned}&\frac{S^+\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_1(t)+iT_1'(t)\right)e^{i\left(2\lambda\xi^+(t)\right)}dt+\frac{S^+\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_2(t)-iT_2'(t)\right)e^{-i\left(2\lambda\xi^+(t)\right)}dt\\&+\frac{S^+\tilde{S}^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_3(t)+iT_3'(t)\right)e^{2ia_1t}dt+\frac{S^+\tilde{S}^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_4(t)-iT_4'(t)\right)e^{-2ia_1t}dt\\&+\frac{S^+S^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_1(t)+iT_1'(t)\right)e^{2i\lambda\alpha(t-a_1)}dt+\frac{S^+S^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_2(t)-iT_2'(t)\right)e^{-2i\lambda\alpha(t-a_1)}dt\\&+\frac{S^-\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_4(t)+iT_4'(t)\right)e^{-2ia_1t}dt+\frac{S^-\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_3(t)-iT_3'(t)\right)e^{2ia_1t}dt\\&+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}\left(R_2(t)+iT_2'(t)\right)e^{2i\lambda\alpha(t-a_1)}dt+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}\left(R_1(t)-iT_1'(t)\right)e^{-2i\lambda\alpha(t-a_1)}dt\\&+\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}\left(R_4(t)-iT_4'(t)\right)e^{i\left(2\lambda\xi^-(t)\right)}dt+\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}\left(R_3(t)+iT_3'(t)\right)e^{-i\left(2\lambda\xi^-(t)\right)}dt\end{aligned}$$

$$+\int_0^{\frac{\pi}{2}}\left(R_5(t)+iT_5'(t)\right)e^{2i\lambda t}dt+\int_0^{\frac{\pi}{2}}\left(R_6(t)-iT_6'(t)\right)e^{-2i\lambda t}dt=0.$$

Since the systems $\left\{e^{\pm 2i\lambda\xi^+(t)} : \lambda\in \mathrm{R}\right\}$, $\left\{e^{\pm 2i\lambda a_1 t} : \lambda\in \mathrm{R}\right\}$, $\left\{e^{\pm 2i\lambda\alpha(t-a_1)} : \lambda\in \mathrm{R}\right\}$ and $\left\{e^{\pm 2i\lambda t} : \lambda\in \mathrm{R}\right\}$ are entire in $L_2\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, it follows

$$\begin{aligned}
&R_1(t)+iT_1'(t)=0\,,\ R_2(t)-iT_2'(t)=0\,,\ R_3(t)+iT_3'(t)=0\\
&R_4(t)-iT_4'(t)=0\,,\ R_1(t)+iT_1'(t)=0\,,\ R_2(t)-iT_2'(t)=0\\
&R_4(t)+iT_4'(t)=0\,,\ R_3(t)-iT_3'(t)=0\,,\ R_2(t)+iT_2'(t)=0\\
&R_1(t)-iT_1'(t)=0\,,\ R_4(t)-iT_4'(t)=0\,,\ R_3(t)+iT_3'(t)=0\\
&R_5(t)+iT_5'(t)=0\,,\ R_6(t)-iT_6'(t)=0.
\end{aligned}$$

Then, we get the following system:

$$\begin{aligned}
R_5(t)+iT_5'(t)&=0\\
R_6(t)-iT_6'(t)&=0
\end{aligned}$$

and hence

$$\begin{cases}
\left[Q_1(t)+P_1(t)K'(t)-P_2'(t)\right]+i\left[Q_2(t)+P_2(t)K'(t)+P_1'(t)\right]=0\\
\left[Q_1(t)+P_1(t)K'(t)-P_2'(t)\right]-i\left[Q_2(t)+P_2(t)K'(t)+P_1'(t)\right]=0
\end{cases}$$

and

$$\begin{cases}
Q_1(t)+P_1(t)K'(t)-P_2'(t)=0\\
Q_2(t)+P_2(t)K'(t)+P_1'(t)=0
\end{cases}$$

$$\begin{cases}
P'(t)=U_c(t,t)P(t)\\
-\int_t^{\frac{\pi}{2}}U_s(x,t)Q(x)dx-\int_t^{\frac{\pi}{2}}\left(K'(t)U_s(x,t)+\frac{\partial H_s(x,t)}{\partial t}\right)P(x)dx\\
P(t)=-\int_t^{\frac{\pi}{2}}P'(x)dx\\
Q(t)=-\left(K'(t)+U_s(t,t)\right)P(t)\\
-\int_t^{\frac{\pi}{2}}U_c(x,t)Q(x)dx-\int_t^{\frac{\pi}{2}}\left(K'(t)U_c(x,t)-\frac{\partial H_s(x,t)}{\partial t}\right)P(x)dx.
\end{cases}\tag{2.13}$$

If we mark this

$$S(t)=\left(Q(t),P(t),P'(t)\right)^T$$

and

$$K(x,t)=\begin{pmatrix}
U_c(x,t) & K'(t)U_c(x,t)-\frac{\partial U_s(x,t)}{\partial t} & -\left(K'(t)+U_s(t,t)\right)\\
0 & 0 & 1\\
U_s(x,t) & K'(t)U_s(x,t)+\frac{\partial U_s(x,t)}{\partial t} & U_c(x,t)
\end{pmatrix}$$

Equations (2.13) can be reduced to a vector from

$$S(t)+\int_t^{\frac{\pi}{2}}K(x,t)S(x)dx=0\tag{2.14}$$

for $0 < t < \frac{\pi}{2}$.

Since equation (2.14) is a homogenous Volterra integral equation, it only has the trivial solution. Thus, we obtain $S(t) = 0$ for $0 < t < \frac{\pi}{2}$. This gives us $Q(t) = P(t) = 0$ for $0 < t < \frac{\pi}{2}$.

Thus, we obtain $q(x) = \tilde{q}(x)$ and $p(x) = \tilde{p}(x)$ on $(0, \pi)$. The proof is completed. $\square$

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