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Some New Applications of the q -Analogues of Differential and Integral Operators for New Subclasses of q -Starlike and q -Convex Functions

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Abstract: In the geometric function theory of complex analysis, the investigation of the geometric properties of analytic functions using q -analogues of differential and integral operators is an important area of study, offering powerful tools for applications in numerical analysis and the solution of differential equations. Many topics, including complex analysis, hypergeometric series, and particle physics, have been generalized in q -calculus. In this study, first of all, we define the q -analogues of a differential operator $(DR_{\lambda,q}^{m,n})$ by using the basic idea of q -calculus and the definition of convolution. Additionally, using the newly constructed operator $(DR_{\lambda,q}^{m,n})$, we establish the q -analogues of two new integral operators $(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}$ and $G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q})$, and by employing these operators, new subclasses of the q -starlike and q -convex functions are defined. Sufficient conditions for the functions (f) that belong to the newly defined classes are investigated. Additionally, certain subordination findings for the differential operator $(DR_{\lambda,q}^{m,n})$ and novel geometric characteristics of the q -analogues of the integral operators in these classes are also obtained. Our results are generalizations of results that were previously proven in the literature.

Keywords: analytic functions; convolution; quantum (or q -) calculus; q -difference operator; q -integral operator; q -starlike and q -convex functions; differential subordination

MSC: 05A30; 30C45; 11B65; 47B38



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1. Introduction and Definitions

Since the dawn of analytic function theory, when Alexander [1] introduced the first integral operator in 1915, differential and integral operators have been the subject of scholarly research. Novel combinations of differential and integral operators are constantly being invented (see [2,3]). Sălăgean and Ruscheweyh operators have great importance in research [4–7]. Recent research on differential and integral operators from several perspectives, including quantum calculus, has produced remarkable findings that have applications in other branches of physics and mathematics. Some fascinating uses of differential and integral operators are highlighted in a recent survey-cum-expository review study [8]. Some examples of publications on the extension of Sălăgean differential operators are included in [9,10], with examples of q -extensions in [11–18].

The theory of real and complex-order integrals and derivatives has been used in the study of geometric functions, and it has also shown potential for mathematical modeling and analysis of practical concerns in the applied sciences. Analyzing the dynamics of dengue transmission [19] and creating a novel model of the human liver [20] are both examples of studies that are included within the aforementioned field of research.

In particular, the family of integral operators related to the first-kind Lommel functions was introduced in [21] and is crucial for understanding both pure and applied mathematics. It is now possible to examine differential equations from the perspectives of functional analysis and operator theory due to differential operators. Differential operator properties are employed to solve differential equations using the operator technique. For the integral operators introduced in this work, several interesting geometric and mapping features are also deduced. In this line of study, we use the concepts of quantum operator theory and introduce the q -analogues of the differential operator, then consider this operator. We also introduce two new integral operators in this paper. From the viewpoints of operator theory and functional analysis, the study of differential equations utilizes operators, and with more investigation, it might be discovered that such operators play a role in solving partial differential equations.

In the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, let \mathcal{A} stand for the collection of all analytic functions, and let every $f \in \mathcal{A}$ in this set have a series of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, z \in U. \quad (1)$$

The class T is a subclass of \mathcal{A} , and every $f \in T$ has a series of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, z \in U. \quad (2)$$

For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ stand for the set of all star-shaped functions of order α , which we define as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \right\}.$$

For $\alpha = 0$,

$$\mathcal{S}^*(0) = \mathcal{S}^*.$$

The convolution of the functions $f, g \in \mathcal{A}$ is denoted by

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), \quad z \in U,$$

where $f(z)$ is defined by Equation (1), and

$$g(z) = z + \sum_{j=2}^{\infty} b_j z^j.$$

Definition 1 ([22]). If \mathcal{K}_1 and \mathcal{K}_2 are two analytic functions in the open unit disk (U), if there is an analytic function (u_0) in U , then \mathcal{K}_1 is subordinate to \mathcal{K}_2 , ($\mathcal{K}_1 \prec \mathcal{K}_2$) with

$$u_0(0) = 0, \text{ and } |u_0(z)| < 1$$

the set of all $z \in U$ then

$$\mathcal{K}_1(z) = \mathcal{K}_2(u_0(z)).$$

If \mathcal{K}_2 is univalent, then

$$\mathcal{K}_1 \prec \mathcal{K}_2 \iff \mathcal{K}_1(0) = \mathcal{K}_2(0)$$

and

$$\mathcal{K}_1(U) \subseteq \mathcal{K}_2(U).$$

Definition 2 ([22]). Let $\psi : U \times \mathbb{C}^3 \rightarrow \mathbb{C}$ and h is univalent in U . If s is analytic in U and the following differential subordination conditions hold:

$$\psi(s(z), zs'(z), z^2s''(z); z) \prec h(z), \text{ for all } z \in U, \quad (3)$$

then s is the solution of the differential subordination. Dominant refers to the univalent function (r) if $s \prec r$ for all s satisfying (3). A dominant \tilde{r} satisfying $\tilde{r} \prec r$ for all dominants r of (3) is said to be the best dominant of (3). Up to a rotation of U , the best dominant is unique.

Geometric function theory, q -difference equations, and q -integral equations are only a few examples of the recent generalization of quantum (q -) calculus across many areas of mathematics and science. Starting with the basics of q -calculus theory, Jackson [23] introduced the q -derivative and q -integral operators; then, Ismail et al. [24] defined q -starlike functions using the same ideas. After the q -difference operator was introduced, a rush of studies examined the q -analogues of other differential operators. In order to build a new class of analytic functions in the conic domain, Kanas and Raducanu [25] created the q -analogue of the Ruscheweyh differential operator. The multivalent generalizations were later provided by Arif et al. [26]. Using the basics of q -calculus, Zang et al. [27] constructed a generalized conic domain and studied a new category of q -starlike functions in this context. Geometric function theory (GFT) and q -calculus theory both have been the subject of a great deal of research by numerous mathematicians to date (for details, see [28–34]). It has been established that time-scale calculus, a more general branch of mathematics, involves quantum calculus. Time-scale calculus enables the investigation of dynamic equations according to a cogent framework in both discrete and continuous domains.

The main contribution of this study is the quantum calculus operator theory. We develop several new forms of q -analogues of the differential and integral operators using the fundamental principles of quantum calculus operator theory and the q -difference operator. Using these operators, we build many new classes of q -starlike and q -convex functions and investigate several interesting features of the analytic function (f) that belongs to these classes.

Definition 3. Jackson [23] provided the following definition of the q -difference (or derivative) operator (∂_q) for analytic functions (f), where $q \in (0, 1)$.

$$\begin{aligned} \partial_q f(z) &= \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0, \\ &= 1 + \sum_{j=2}^{\infty} [j]_q a_j z^{j-1}, \end{aligned} \quad (4)$$

where $[j]_q$ is the q -number and defined as:

$$\begin{aligned} [j]_q &= \frac{1 - q^j}{1 - q} \\ &= 1 + q + q^2 + \dots + q^{j-1}, \quad j \in \mathbb{N} \end{aligned}$$

and

$$[0]_q = 0.$$

The factorial of q , $[j]_q!$ is identified as follows:

$$[j]_q! = [j]_q [j-1]_q [j-2]_q \dots [2]_q [1]_q$$

and

$$[0]_q! = 1.$$

Definition 4. Jackson [35] defined the q -integral for the function $f \in \mathcal{A}$ as follows:

$$\int f(z) d_q(z) = (1-q)z \sum_{j=0}^{\infty} f(q^j z) q^j.$$

By using the same technique of the Al-Oboudi differential operator [36], now we define the q -analogues of the Al-Oboudi differential operator $(D_{\lambda,q}^m)$ for analytic functions as follows:

Definition 5. For $\lambda \geq 0$, $q \in (0,1)$, $m, n \in \mathbb{N}$, and $f \in \mathcal{A}$, the operator $D_{\lambda,q}^m : \mathcal{A} \rightarrow \mathcal{A}$, is defined by

$$\begin{aligned} D_{\lambda,q}^0 f(z) &= f(z), \\ D_{\lambda,q}^1 f(z) &= (1-\lambda)f(z) + \lambda z \partial_q f(z) = D_{\lambda,q} f(z) \\ &\dots \\ D_{\lambda,q}^m f(z) &= (1-\lambda)D_{\lambda,q}^{m-1} f(z) + \lambda z \partial_q (D_{\lambda,q}^{m-1} f(z)) = D_{\lambda,q} (D_{\lambda,q}^{m-1} f(z)). \end{aligned}$$

After some simple calculation, we have

$$D_{\lambda,q}^m f(z) = z + \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m a_j z^j. \quad (5)$$

Remark 1. For the function (f) of the form (2), the series expansion of $D_{\lambda,q}^m$ is given by:

$$D_{\lambda,q}^m f(z) = z - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m a_j z^j.$$

Remark 2. Specifically, when $\lambda = 1$, the operator $D_{\lambda,q}^m$ simplifies to the Sălăgean q -differential operator given by [37].

Remark 3. If $q \rightarrow 1-$, then we obtain the Al-Oboudi differential operator studied in [36].

Remark 4. If $\lambda = 1$, and $q \rightarrow 1-$, then we obtain the Sălăgean differential operator defined in [38].

The Ruscheweyh q -differential operator (R_q^n) was developed by Kanas and Raducanu utilizing fundamental concepts from operator theory in quantum mechanics. Very intriguing aspects of this operator in the conic domain were explored; they also created a new subclass of q -starlike functions connected to the conic domain.

Definition 6 ([25]). To define the operator $R_q^n : \mathcal{A} \rightarrow \mathcal{A}$ for $n \in \mathbb{N}$ and $f \in \mathcal{A}$, we write

$$\begin{aligned} R_q^0 f(z) &= f(z), \\ R_q^1 f(z) &= z \partial_q f(z) \\ &\dots \\ R_q^n f(z) &= \frac{z \partial_q^n (z^{n-1} f(z))}{[n]!}, \quad z \in U \end{aligned}$$

or

$$\begin{aligned} R_q^n f(z) &= z + \sum_{j=2}^{\infty} \frac{\Gamma_q(j+n)}{[j-1]!\Gamma_q(1+n)} a_j z^j \\ &= z + \sum_{j=2}^{\infty} \frac{[n+1]_{j-1}}{[j-1]!} a_j z^j. \end{aligned} \quad (6)$$

The standard quantum calculus has been extensively studied by numerous mathematicians, physicists, and engineers. Applications in areas including engineering, economics, mathematics, and other disciplines have helped q -calculus improve in a number of ways. If we consider the above facts about q -calculus in many areas, it is safe to assume that q -calculus has functioned as the interface between mathematics and physics throughout the last three decades. In addition, the q -calculus operator, the q -integral operator, and the q -derivative operator are used to build several classes of regular functions and play an intriguing role, since they are used and applied in many different branches of mathematics, including the theory of relativity, the calculus of variations, orthogonal polynomials, and basic hypergeometric functions. In [39], Akça et al. used the q -derivative and generated solutions to some differential equations. Therefore, we have also made use of q -calculus and provide certain important new types of q -analogues of differential and integral operators, as mentioned in this paper. Non-commutative q -calculus is a generalization of classical calculus as developed by Newton and Leibnitz. This q -derivative may be used with any function whose domain of definition does not include 0. When q equals 1, the result simplifies to the standard derivative; that is, the results obtained by the q -differential and integral operators are quite effective and efficient.

Here, we define the q -analogues of differential operator $DR_{\lambda,q}^{m,n}$ by using the definition of convolution on the newly defined differential operator $D_{\lambda,q}^m$ and the Ruscheweyh q -differential operator R_q^n . This newly defined operator will help us to define two new integral operators introduced in this study.

Definition 7. For $f \in \mathcal{A}$, $n, m \in \mathbb{N} = \{1, 2, 3, \dots\}$ and $\lambda \geq 0$, the q -analogues of differential operator $DR_{\lambda,q}^{m,n}$ is defined by

$$DR_{\lambda,q}^{m,n} f(z) = D_{\lambda,q}^m f(z) * R_q^n f(z), \quad z \in U. \quad (7)$$

Using (5) and (6) in (7) and applying the definition of convolution, we obtain the following series expansion of $DR_{\lambda,q}^{m,n}$:

$$DR_{\lambda,q}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]!\Gamma_q(1+n)} a_j^2 z^j, \quad z \in U.$$

Remark 5. The series expansion of $DR_{\lambda,q}^{m,n}$ for the function (f) of type (2) is as follows:

$$DR_{\lambda,q}^{m,n} f(z) = z - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]!\Gamma_q(1+n)} a_j^2 z^j, \quad z \in U,$$

where $\lambda \geq 0$, $m, n \in \mathbb{N}$. The following identity holds for the function $f \in T$:

$$DR_{\lambda,q}^{m+1,n} f(z) = \left(1 - \frac{[\lambda]_q}{q^\lambda} \right) DR_{\lambda,q}^{m,n} f(z) + \left(\frac{[\lambda]_q}{q^\lambda} \right) z \partial_q \left(DR_{\lambda,q}^{m,n} f(z) \right). \quad (8)$$

The following formulation introduces two new integral operators, $F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}$ and $G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}$, while considering the convolution operator $DR_{\lambda,q}^{m,n} f(z)$:

Definition 8. For functions $f_i \in T$, and $\gamma_i \in \mathbb{R}$, $i \in \{1, 2, 3, \dots, l\}$, the integral operators $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}$ and $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}$ are defined as follows:

$$F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q} = \int_0^z \left(\frac{DR_{\lambda, q}^{m, n} f_1(t)}{t} \right)^{\gamma_1} \dots \left(\frac{DR_{\lambda, q}^{m, n} f_l(t)}{t} \right)^{\gamma_l} d_q t \quad (9)$$

and

$$G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q} = \int_0^z \left(\partial_q \left(\frac{DR_{\lambda, q}^{m, n} f_1(t)}{t} \right) \right)^{\gamma_1} \dots \left(\partial_q \left(\frac{DR_{\lambda, q}^{m, n} f_l(t)}{t} \right) \right)^{\gamma_l} d_q t, \quad (10)$$

where

$$\lambda \geq 0, q \in (0, 1), m, n \in \mathbb{N}, \text{ and } z \in U.$$

Remark 6. For $\lambda = 0$, $m = 0$, and $q \rightarrow 1-$, we obtain the integral operators introduced by Breaz and Breaz in [40,41].

We establish several new types of q -starlike and q -convex functions by utilizing the q -difference operator and the q -analogues of the differential operator $DR_{\lambda, q}^{m, n}$ provided in Definition 7.

Definition 9. Let an analytic function (f) of the form (2) be a member of class $R(\delta, q)$, if it satisfies the following inequality

$$\operatorname{Re} \left(\frac{z \partial_q (DR_{\lambda, q}^{m, n} f(z))}{DR_{\lambda, q}^{m, n} f(z)} \right) < \delta, \text{ for all } z \in U, \text{ and } \delta > 1.$$

Definition 10. Let an analytic function (f) of the form (2) be a member of class $C(\delta, q)$ if it satisfies the following inequality

$$\operatorname{Re} \left(1 + \frac{z \partial_q^2 (DR_{\lambda, q}^{m, n} f(z))}{\partial_q (DR_{\lambda, q}^{m, n} f(z))} \right) < \delta, \text{ for all } z \in U, \text{ and } \delta > 1.$$

Definition 11. Let an analytic function (f) of the form (2) be a member of class $RA(\beta, \mu, q)$, if

$$\left| \frac{z \partial_q (DR_{\lambda, q}^{m, n} f(z))}{DR_{\lambda, q}^{m, n} f(z)} - 1 \right| < \mu \left| \beta \left(\frac{z \partial_q (DR_{\lambda, q}^{m, n} f(z))}{DR_{\lambda, q}^{m, n} f(z)} \right) - 1 \right|, z \in U,$$

where

$$0 \leq \beta < 1, \text{ and } 0 < \mu \leq 1.$$

Definition 12. Let an analytic function (f) of the form (2) be a member of class $CA(\beta, \mu, q)$, if

$$\left| \frac{z \partial_q^2 (DR_{\lambda, q}^{m, n} f(z))}{\partial_q (DR_{\lambda, q}^{m, n} f(z))} \right| < \mu \left| \beta \left(1 + \frac{z \partial_q^2 (DR_{\lambda, q}^{m, n} f(z))}{\partial_q (DR_{\lambda, q}^{m, n} f(z))} \right) + 1 \right|, z \in U,$$

where

$$0 \leq \beta < 1, \text{ and } 0 < \mu \leq 1.$$

In the following definitions, we consider integral operators $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}$ and $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}$ given in Definition 8, and we define two new subclasses of q -convex functions:

Definition 13. Let an analytic function $(f_i, i \in \{1, 2, \dots, l\})$ of the form (2) be a member of class $LA F(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$ if

$$\operatorname{Re} \left(1 + \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right) \geq \beta \left| \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right| + \mu, \quad z \in U,$$

where $\lambda \geq 0, \beta \geq 0, -1 \leq \mu \leq 1$, and $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z)$ is defined by (9).

Definition 14. Let an analytic function $(f_i, i \in \{1, 2, \dots, l\})$ of the form (2) be a member of class $LAG(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$ if

$$\operatorname{Re} \left(1 + \frac{z \partial_q^2 (G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right) \geq \beta \left| \frac{z \partial_q^2 (G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right| + \mu, \quad z \in U,$$

where $\lambda \geq 0, \beta \geq 0, -1 \leq \mu \leq 1$, and $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z)$ is defined in (10).

This article is composed of four sections. We briefly reviewed some fundamental geometric function theory ideas, investigated some new q -analogues of differential and integral operators, and considered these operators to define a number of new subclasses of q -starlike and q -convex functions in Section 1 because they were important to our main finding. In Section 2, we provide some known lemmas and investigate some new lemmas that are used to prove our main results. In Section 3, we present our key findings, and in Section 4, we provide concluding remarks.

2. Set of Lemmas

Here, we provide some previously established lemmas and prove four new ones that are used in the proof of our key findings.

Lemma 1 ([42]). For convex univalent function p and

$$\operatorname{Re} \left[\frac{1 - \vartheta}{\vartheta} + 2p(z) + \left(1 + \frac{zp''(z)}{p'(z)} \right) \right] > 0.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{f(z)} + \vartheta z^2 \frac{f''(z)}{f'(z)} \prec (1 - \vartheta)p(z) + \vartheta p^2(z) + \gamma zp'(z),$$

then,

$$\frac{zf'(z)}{f(z)} \prec p(z),$$

where $0 < \vartheta \leq 1$, and $p(z)$ is the best dominant.

Lemma 2 ([42]). Let an analytic function (p) be in the open unit disk (U) and

$$p(0) = 1, \text{ and } h(z) = \frac{zp'(z)}{p(z)}$$

is starlike and univalent in U . If $f \in \mathcal{A}$ satisfies

$$\frac{(zf(z))''}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec h(z)$$

then,

$$\frac{z^2 f'(z)}{f^2(z)} \prec p(z),$$

where $p(z)$ is the best dominant.

Lemma 3 ([22]). Consider the case when p is univalent and ϕ is analytic in the set of all $p(U)$. If

$$\frac{zp'(z)}{\phi(p(z))}$$

is starlike and

$$\phi(\psi(z))z\psi'(z) \prec \phi(p(z))zp'(z), \quad z \in U,$$

then,

$$\psi(z) \prec p(z),$$

where $p(z)$ is the best dominant.

Lemma 4 ([43]). For complex numbers α, β and γ and $\gamma \neq 0$. Let analytic functions s and p be in U , and p be a convex univalent; suppose that

$$\operatorname{Re} \left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} p(z) + \left(1 + \frac{zp''(z)}{p'(z)} \right) \right] > 0.$$

If $s(z) = 1 + c_1 z + \dots$ is analytic in U and

$$\alpha s(z) + \beta s^2(z) + \gamma zs'(z) \prec \alpha p(z) + \beta p^2(z) + \gamma zp'(z),$$

then, $s(z) \prec p(z)$, and the function $p(z)$ is the best dominant.

Now, we generalize the lemmas introduced in [22,43] by using the fundamentals of q -calculus operator theory.

Lemma 5. Consider the case when p is univalent and ϕ is analytic in the set of all $p(U)$. If

$$\frac{z\partial_q p(z)}{\phi(p(z))} \tag{11}$$

is starlike and

$$\phi(\psi(z))z\partial_q \psi(z) \prec \phi(p(z))z\partial_q p(z), \quad z \in U, \tag{12}$$

then, $\psi(z) \prec p(z)$, and $p(z)$ is the best dominant.

Proof. Suppose that ϕ is analytic in a domain containing $p(U)$ and p is analytic in U . Letting $q \rightarrow 1-$ in (11) and (12) yields

$$\frac{zp'(z)}{\phi(p(z))},$$

which is starlike; then,

$$z\psi'(z)\phi(\psi(z)) \prec zp'(z)\phi(p(z)), \quad z \in U.$$

Then, from the lemma in [22], we obtain $\psi(z) \prec p(z)$, and $p(z)$ is the best dominant. \square

Lemma 6. We assume that p and h are analytic in U and that h is convex and univalent in U , where $\alpha, \vartheta, \gamma \in \mathbb{C}$. Furthermore, we assume

$$\operatorname{Re} \left[\frac{\alpha}{\gamma} + \frac{2\vartheta}{\gamma} h(z) + \left(1 + \frac{z \partial_q^2 h(z)}{h(z)} \right) \right] > 0. \quad (13)$$

If $p(z)$ is analytic in U and

$$\alpha p(z) + \vartheta p^2(z) + \gamma z \partial_q p(z) \prec \alpha h(z) + \vartheta h^2(z) + \gamma z \partial_q h(z), \quad (14)$$

then, $p(z) \prec h(z)$, and $h(z)$ is the best dominant.

Proof. Suppose that p and h are analytic in U . Letting $q \rightarrow 1-$ in (13) and (14), we have

$$\operatorname{Re} \left[\frac{\alpha}{\gamma} + \frac{2\vartheta}{\gamma} p(z) + \left(1 + \frac{z p''(z)}{p(z)} \right) \right] > 0.$$

If $p(z)$ is analytic in U and

$$\alpha p(z) + \vartheta p^2(z) + \gamma z p'(z) \prec \alpha h(z) + \vartheta h^2(z) + \gamma z h''(z), \quad z \in U,$$

then, from the lemma in [43], we obtain $p(z) \prec h(z)$, and $h(z)$ is the best dominant. \square

Lemma 7. For $f_i(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j \in T$, $i \in \{1, 2, \dots, l\}$, we get

$$\frac{z \partial_q^2 \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)}{\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)} = \sum_{i=1}^l \gamma_i \left(\frac{- \sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j} \right),$$

where $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z)$ is defined in (9).

Proof. For $f_i(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j$, $i \in \{1, 2, \dots, l\}$, then

$$\partial_q \left(DR_{\lambda, q}^{m, n} f_i(z) \right) = 1 - \sum_{j=2}^{\infty} [j]_q \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}.$$

We obtain

$$\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right) = \left(\frac{DR_{\lambda, q}^{m, n} f_1(z)}{z} \right)^{\gamma_1} \cdots \left(\frac{DR_{\lambda, q}^{m, n} f_l(z)}{z} \right)^{\gamma_l},$$

so

$$\begin{aligned} \partial_q^2 \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right) &= E_1 \left(\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right) \right) \frac{z}{DR_{\lambda, q}^{m, n} f_1(z)} \\ &\quad + \dots + E_l \left(\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right) \right) \frac{z}{DR_{\lambda, q}^{m, n} f_l(z)}, \end{aligned}$$

where

$$E_i = \gamma_i \left(\frac{z \partial_q \left(DR_{\lambda,q}^{m,n} f_i(z) \right) - DR_{\lambda,q}^{m,n} f_i(z)}{z^2} \right).$$

We calculate the expression

$$\frac{z \partial_q^2 \left(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right)}{\partial_q \left(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right)} = \sum_{i=1}^l \gamma_i \left[\frac{z \partial_q \left(DR_{\lambda,q}^{m,n} f_i(z) \right)}{DR_{\lambda,q}^{m,n} f_i(z)} - 1 \right].$$

We find

$$\begin{aligned} & \frac{z \partial_q^2 \left(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right)}{\partial_q \left(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right)} \\ &= \sum_{i=1}^l \gamma_i \left(\frac{z - \sum_{j=2}^{\infty} [j]_q \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j}{z - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j} - 1 \right) \\ &= \sum_{i=1}^l \gamma_i \left(\frac{- \sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j}{z - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j} \right) \\ &= \sum_{i=1}^l \gamma_i \left(\frac{- \sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j} \right). \end{aligned}$$

□

Lemma 8. For $f_i(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j$, $i \in \{1, 2, \dots, l\}$, we get

$$\frac{z \partial_q^2 \left(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right)}{\partial_q \left(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right)} = - \sum_{i=1}^l \gamma_i \left(\frac{\sum_{j=2}^{\infty} [j]_q ([j-1]_q) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j}{1 - \sum_{j=2}^{\infty} [j]_q \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j} \right),$$

where $G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z)$ is defined in (10).

Proof. For $f_i(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j$, $i \in \{1, 2, \dots, l\}$, we obtain

$$\partial_q \left(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right) = \left(\partial_q \left(DR_{\lambda,q}^{m,n} f_1(z) \right) \right)^{\gamma_1} \dots \left(\partial_q \left(DR_{\lambda,q}^{m,n} f_l(z) \right) \right)^{\gamma_l},$$

so

$$\partial_q^2 \left(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right) = \sum_{i=1}^l \gamma_i \left(\partial_q \left(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z) \right) \right) \frac{\partial_q^2 \left(DR_{\lambda,q}^{m,n} f_i(z) \right)}{\partial_q \left(DR_{\lambda,q}^{m,n} f_i(z) \right)}.$$

We calculate the expression $\frac{z\partial_q^2(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))}{\partial_q(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))}$

$$\frac{z\partial_q^2(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))}{\partial_q(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))} = \sum_{i=1}^l \gamma_i \left[\frac{z\partial_q^2(DR_{\lambda,q}^{m,n}f_i(z))}{\partial_q(DR_{\lambda,q}^{m,n}f_i(z))} \right].$$

We find

$$\begin{aligned} & \frac{z\partial_q^2(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))}{\partial_q(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))} \\ &= \sum_{i=1}^l \gamma_i \left(\frac{-\sum_{j=2}^{\infty} [j]_q ([j-1]_q) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} [j]_q \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right). \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{z\partial_q^2(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))}{\partial_q(G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))} \\ &= -\sum_{i=1}^l \gamma_i \left(\frac{\sum_{j=2}^{\infty} [j]_q ([j-1]_q) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} [j]_q \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right). \end{aligned}$$

□

3. Main Results

We then provide necessary and sufficient criteria for the classes $LAF(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$ and $LAG(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$, where

$$\lambda \geq 0, \beta \geq 0, \text{ and } -1 \leq \mu \leq 1.$$

Theorem 1. For $i \in \{1, 2, 3, \dots, l\}$, let $f_i \in T$. Then, $f_i \in LAF(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$ if and only if

$$\sum_{i=1}^l \gamma_i (\beta + 1) \left(\frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^j} \right) \leq 1 - \mu, \quad (15)$$

where $\beta \geq 0, -1 \leq \mu \leq 1$.

Proof. In order to demonstrate that (15) is true, we must prove that

$$\beta \left| \frac{z\partial_q^2(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))}{\partial_q(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))} \right| - \operatorname{Re} \left(\frac{z\partial_q^2(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))}{\partial_q(F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z))} \right) \leq 1 - \mu.$$

We have

$$\beta \left| \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right| - \operatorname{Re} \left(\frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right) \leq (\beta + 1) \left| \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right|.$$

Applying Lemma 7, we obtain

$$\begin{aligned} & (\beta + 1) \left| \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right| \\ &= (\beta + 1) \left| \sum_{i=1}^l \gamma_i \left(\frac{- \sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right) \right| \\ &\leq (\beta + 1) \sum_{i=1}^l \gamma_i \left\{ \frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 |z^{j-1}|}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 |z^{j-1}|} \right\} \\ &\leq (\beta + 1) \sum_{i=1}^l \gamma_i \left\{ \frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2} \right\} \leq 1 - \mu. \end{aligned}$$

Therefore, we deduce

$$\beta \left| \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right| - \operatorname{Re} \left(\frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right) \leq 1 - \mu,$$

or, equivalently,

$$\operatorname{Re} \left(1 + \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right) \geq \beta \left| \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z))} \right| + \mu.$$

Thus, $f_i \in LAF(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$.

Contrarily, assume that $f_i \in LAF(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$. Lemma 7 and (15) allow us to derive

$$\begin{aligned} & 1 - \sum_{i=1}^l \gamma_i \left[\frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 |z^{j-1}|}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 |z^{j-1}|} \right] \\ &\geq \beta \sum_{i=1}^l \gamma_i \left| \frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right| + \mu \\ &\geq \beta \sum_{i=1}^l \gamma_i \left(\frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right) + \mu, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \sum_{i=1}^l \beta \gamma_i \left[\frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right] \\ & + \sum_{i=1}^l \gamma_i \left[\frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right] \\ & \leq 1 - \mu, \end{aligned}$$

which reduces to

$$\sum_{i=1}^l (\beta + 1) \gamma_i \left[\frac{\sum_{j=2}^{\infty} ([j]_q - 1) \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2 z^{j-1}} \right] \leq 1 - \mu,$$

Inequality (15) is found when $z \rightarrow 1-$ is on the real axis. \square

For $q \rightarrow 1-$, we obtain known result that were proven in [44].

Corollary 1 ([44]). For $i \in \{1, 2, 3, \dots, l\}$, let $f_i \in T$. Then, $f_i \in LAF(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l)$ if and only if

$$\sum_{i=1}^l \gamma_i (\beta + 1) \left(\frac{\sum_{j=2}^{\infty} \left\{ \lambda (j-1) + 1 \right\}^m \frac{(n+j-1)!}{n!(j-2)!} a_{i,j}^2 z^j}{1 - \sum_{j=2}^{\infty} \left\{ \lambda (j-1) + 1 \right\}^m \frac{(n+j-1)!}{n!(j-1)!} a_{i,j}^2 z^j} \right) \leq 1 - \mu.$$

Theorem 2. For $i \in \{1, 2, 3, \dots, l\}$, let $f_i \in T$. Then, $f_i \in LAG(\lambda, \beta, \mu, \gamma_1, \gamma_2, \dots, \gamma_l, q)$ if and only if

$$\sum_{i=1}^l \gamma_i (\beta + 1) \left(\frac{\sum_{j=2}^{\infty} [j]_q [j-1]_q \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2}{1 - \sum_{j=2}^{\infty} [j]_q \left\{ \lambda ([j]_q - 1) + 1 \right\}^m \frac{\Gamma_q(j+n)}{[j-1]! \Gamma_q(1+n)} a_{i,j}^2} \right) \leq 1 - \mu,$$

where $\beta \geq 0, -1 \leq \mu \leq 1$.

Proof. Using Lemma 8 and the method used to prove Theorem 1, we arrive at Theorem 2. \square

We now demonstrate some characteristics of the integral operators $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z)$ and $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z)$ for the families $R(\delta, q)$, $C(\delta, q)$, $RA(\beta, \mu, q)$, and $CA(\beta, \mu, q)$.

Theorem 3. Let $f_i \in T$ and $\left| \frac{\partial_q (DR_{\lambda, q}^{m, n} f(z))}{DR_{\lambda, q}^{m, n} f(z)} \right| < M_i$. If $f_i \in RA(\beta_i, \mu_i, q)$, then $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i \mu_i (\beta_i M_i + 1), \quad z \in U,$$

where

$$\gamma_i \in R, \quad \gamma_i > 0, \quad i \in \{1, 2, 3, \dots, l\}.$$

Proof. As shown in (9), $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q} \in T$. Upon differentiating $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z)$ as shown in (9), we obtain

$$\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right) = \prod_{i=1}^l \left(\frac{DR_{\lambda, q}^{m, n} f(z)}{z} \right)^{\gamma_i}. \quad (16)$$

Taking the logarithmic differentiation of (16) and multiplying by z , we obtain

$$\frac{z \partial_q^2 \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)}{\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)} = \sum_{i=1}^l \gamma_i \left(\frac{z \partial_q \left(DR_{\lambda, q}^{m, n} f(z) \right)}{DR_{\lambda, q}^{m, n} f(z)} - 1 \right),$$

or, equivalently,

$$1 + \frac{z \partial_q^2 \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)}{\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)} = 1 + \sum_{i=1}^l \gamma_i \left(\frac{z \partial_q \left(DR_{\lambda, q}^{m, n} f(z) \right)}{DR_{\lambda, q}^{m, n} f(z)} - 1 \right). \quad (17)$$

By taking a real part from either side of (17), we obtain

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{z \partial_q^2 \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)}{\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)} \right) &= 1 + \sum_{i=1}^l \gamma_i \left(\operatorname{Re} \left(\frac{z \partial_q \left(DR_{\lambda, q}^{m, n} f(z) \right)}{DR_{\lambda, q}^{m, n} f(z)} \right) - 1 \right) \\ &\leq 1 + \sum_{i=1}^l \gamma_i \left| \frac{z \partial_q \left(DR_{\lambda, q}^{m, n} f(z) \right)}{DR_{\lambda, q}^{m, n} f(z)} - 1 \right|. \end{aligned}$$

Since $f_i \in RA(\beta_i, \mu_i, q)$, we deduce that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{z \partial_q^2 \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)}{\partial_q \left(F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)} \right) &< 1 + \sum_{i=1}^l \gamma_i \mu_i \left| \beta_i \frac{z \partial_q \left(DR_{\lambda, q}^{m, n} f(z) \right)}{DR_{\lambda, q}^{m, n} f(z)} + 1 \right| \\ &< 1 + \sum_{i=1}^l \gamma_i \mu_i \beta_i \left| \frac{z \partial_q \left(DR_{\lambda, q}^{m, n} f(z) \right)}{DR_{\lambda, q}^{m, n} f(z)} \right| + \sum_{i=1}^l \gamma_i \mu_i \beta_i \\ &< 1 + \sum_{i=1}^l \gamma_i \mu_i (\beta_i M_i + 1). \end{aligned}$$

Furthermore,

$$\sum_{i=1}^l \gamma_i \mu_i (\beta_i M_i + 1) > 0, \text{ and } F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \in \mathcal{D}(\delta'),$$

where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i \mu_i (\beta_i M_i + 1), \quad z \in U.$$

□

For $q \rightarrow 1-$, we obtain the result proven in [44].

Corollary 2. Let $\gamma_i \in \mathbb{R}$, $\gamma_i > 0$, $i \in \{1, 2, 3 \dots l\}$, $f_i \in T$ and $\left| \frac{(DR_{\lambda}^{m,n} f(z))'}{DR_{\lambda}^{m,n} f(z)} \right| < M_i$. If $f_i \in RA(\beta_i, \mu_i)$, then $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m,n}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i \mu_i (\beta_i M_i + 1), \quad z \in U.$$

The following is a corollary of Theorem 3 under the assumptions that $l = 1$, $\gamma_1 = \gamma$, $\delta_1 = \delta$, and $f_1 = f$.

Corollary 3. Let $f \in T$ and $\left| \frac{\partial_q f(z)}{f(z)} \right| < M$. If $f \in RA(\beta, \mu, q)$, then $\int_0^z \left(\frac{f(t)}{t} \right)^\gamma d_q(t) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \gamma \mu (\beta M + 1),$$

and $\gamma \in \mathbb{R}$, $\gamma > 0$, $z \in U$.

Theorem 4. Let $f_i \in T$. Then, $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m,n,q}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i (\delta_i - 1), \quad z \in U$$

and

$$\gamma_i \in \mathbb{R}, \quad \delta_i > 1, \quad \gamma_i > 0, \quad i \in \{1, 2, 3 \dots l\}.$$

Proof. From (17), we have

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{z \partial_q^2 (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m,n,q}(z))}{\partial_q (F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m,n,q}(z))} \right) &= 1 + \sum_{i=1}^l \gamma_i \operatorname{Re} \left(\frac{z \partial_q (DR_{\lambda, q}^{m,n} f_i(z))}{DR_{\lambda, q}^{m,n} f_i(z)} \right) - \sum_{i=1}^l \gamma_i \\ &< 1 + \sum_{i=1}^l \gamma_i \delta_i - \sum_{i=1}^l \gamma_i = 1 + \sum_{i=1}^l \gamma_i (\delta_i - 1). \end{aligned}$$

Since $\delta_i > 1$, evidently, $\sum_{i=1}^l \gamma_i (\delta_i - 1) > 0$; hence, $F_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m,n,q}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i (\delta_i - 1), \quad z \in U.$$

□

The following is a corollary of Theorem 4 under the assumptions that $l = 1$, $\gamma_1 = \gamma$, $\delta_1 = \delta$, and $f_1 = f$.

Corollary 4. Let $f \in R(\delta)$. Then, $\int_0^z \left(\frac{f(t)}{t} \right)^\gamma d_q(t) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \gamma (\delta - 1)$$

and

$$\delta > 1, \quad \gamma > 0, \quad z \in U.$$

Theorem 5. Let $f_i \in \mathcal{D}(\delta_i)$. Then, $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i(\delta_i - 1), \quad z \in U$$

and

$$\gamma_i > 0, \quad i \in \{1, 2, 3, \dots, l\}, \quad \delta_i > 1.$$

Proof. From the definition of $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z)$ given by (10), we have

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{z \partial_q^2 \left(G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)}{\partial_q \left(G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)} \right) &= 1 + \sum_{i=1}^l \gamma_i \operatorname{Re} \left(\frac{z \partial_q \left(DR_{\lambda, q}^{m, n} f_i(z) \right)}{DR_{\lambda, q}^{m, n} f_i(z)} \right) - \sum_{i=1}^l \gamma_i \\ &< 1 + \sum_{i=1}^l \gamma_i \delta_i - \sum_{i=1}^l \gamma_i = 1 + \sum_{i=1}^l \gamma_i (\delta_i - 1). \end{aligned}$$

Since $\delta_i > 1$, it seems to reason that $\sum_{i=1}^l \gamma_i (\delta_i - 1) > 0$ and that $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i (\delta_i - 1), \quad z \in U.$$

□

The following is a corollary of Theorem 5 under the assumptions $l = 1$, $\gamma_1 = \gamma$, $\delta_1 = \delta$, and $f_1 = f$.

Corollary 5. Let $f \in \mathcal{D}(\delta)$. Then, $\int_0^z \left(f'(t) \right)^\gamma d_q(t) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \gamma(\delta - 1)$$

and

$$\gamma > 0, \quad \delta > 1.$$

Theorem 6. Let $f_i \in DA(\beta_i, \mu_i, q)$ and $\left| \frac{\partial_q^2 \left(DR_{\lambda, q}^{m, n} f_i(z) \right)}{\partial_q \left(DR_{\lambda, q}^{m, n} f_i(z) \right)} \right| < M_i$. Then, $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i \mu_i (\beta_i M_i + 1)$$

and

$$\gamma_i \in \mathbb{R}, \quad \gamma_i > 0, \quad z \in U, \quad i \in \{1, 2, 3, \dots, l\}.$$

Proof. The following is derived from the definition of $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}$ in (10):

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{z \partial_q^2 \left(G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)}{\partial_q \left(G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m, n, q}(z) \right)} \right) &\leq \sum_{i=1}^l \gamma_i \left| \frac{z \partial_q^2 \left(DR_{\lambda, q}^{m, n} f_i(z) \right)}{\partial_q \left(DR_{\lambda, q}^{m, n} f_i(z) \right)} \right| \\ &< \sum_{i=1}^l \gamma_i \mu_i \left| \beta_i \left(1 + \frac{z \partial_q^2 \left(DR_{\lambda, q}^{m, n} f_i(z) \right)}{\partial_q \left(DR_{\lambda, q}^{m, n} f_i(z) \right)} + 1 \right) \right| + 1 \end{aligned}$$

$$\begin{aligned}
&< 1 + \sum_{i=1}^l \gamma_i \mu_i \beta_i \left(1 + \left| \frac{z \partial_q^2 (DR_{\lambda,q}^{m,n} f(z))}{\partial_q (DR_{\lambda,q}^{m,n} f(z))} \right| \right) + \sum_{i=1}^l \gamma_i \mu_i + 1 \\
&< \sum_{i=1}^l [\beta_i (1 + M_i) + 1] \gamma_i \mu_i + 1.
\end{aligned}$$

Because

$$\sum_{i=1}^l [\beta_i (1 + M_i) + 1] \gamma_i \mu_i > 0,$$

we draw the following conclusion

$$G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m,n,q}(z) \in \mathcal{D}(\delta'),$$

where

$$\delta' = 1 + \sum_{i=1}^l [\beta_i (1 + M_i) + 1] \gamma_i \mu_i, \quad z \in U.$$

□

For $q \rightarrow 1-$, we obtain the result proven in [44].

Corollary 6 ([44]). Let $\gamma_i \in \mathbb{R}, \gamma_i > 0, i \in \{1, 2, 3, \dots, l\}, f_i \in DA(\beta_i, \mu_i)$, and $\left| \frac{(DR_{\lambda,q}^{m,n} f(z))''}{(DR_{\lambda,q}^{m,n} f(z))'} \right| < M_i$. Then, $G_{\lambda, \gamma_1, \gamma_2, \dots, \gamma_l}^{m,n}(z) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \sum_{i=1}^l \gamma_i \mu_i (\beta_i M_i + 1), \quad z \in U.$$

The following is a corollary of Theorem 6 under the assumptions $l = 1, \gamma_1 = \gamma, M_1 = 1$, and $f_1 = f$.

Corollary 7. Let $f \in DA(\beta, \mu, q)$ and $\left| \frac{\partial_q f(z)}{f(z)} \right| < M$, where M is fixed. Then, $\int_0^z (f'(t))^\gamma d_q(t) \in \mathcal{D}(\delta')$, where

$$\delta' = 1 + \gamma \mu \beta [1 + M] + 1$$

and

$$\gamma \in \mathbb{R}, \gamma > 0, z \in U.$$

Subordination Results:

In this paper, we generalize Lemmas 1 and 2 to the operator $DR_{\lambda,q}^{m,n} f(z)$.

Theorem 7. Assuming h is both convex and univalent, $\varsigma \neq 0$, and

$$\operatorname{Re} \left\{ \frac{(1 - \varsigma) q^\lambda}{[\lambda]_q \varsigma} + \frac{2q^\lambda}{[\lambda]_q} h(z) + \left(1 + \frac{z \partial_q^2 h(z)}{\partial_q h(z)} \right) \right\} > 0.$$

If the differential subordination condition for $f \in T$ holds, then

$$\begin{aligned} & \left(\gamma \frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} + 1 - \varsigma \right) \frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \\ & \prec (1 - \varsigma)h(z) + \varsigma h^2(z) + \frac{\varsigma[\lambda]_q}{q^\lambda} z \partial_q h(z) \end{aligned} \quad (18)$$

then,

$$\frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \prec h(z), \quad z \in U.$$

Proof. Consider

$$p(z) = \frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)}, \quad z \in U. \quad (19)$$

We achieved

$$\begin{aligned} \frac{\partial_q p(z)}{p(z)} &= \frac{DR_{\lambda,q}^{m,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} \left\{ \frac{(DR_{\lambda,q}^{m,n} f(z)) \partial_q (DR_{\lambda,q}^{m+1,n} f(z)) - DR_{\lambda,q}^{m+1,n} f(z) \partial_q (DR_{\lambda,q}^{m,n} f(z))}{(DR_{\lambda,q}^{m,n} f(z))^2} \right\} \\ &= \frac{\partial_q (DR_{\lambda,q}^{m+2,n} f(z))}{DR_{\lambda,q}^{m+1,n} f(z)} - \frac{\partial_q (DR_{\lambda,q}^{m+1,n} f(z))}{DR_{\lambda,q}^{m,n} f(z)}. \end{aligned}$$

Thus,

$$\frac{z \partial_q p(z)}{p(z)} = \frac{z \partial_q (DR_{\lambda,q}^{m+2,n} f(z))}{DR_{\lambda,q}^{m+1,n} f(z)} - \frac{z \partial_q (DR_{\lambda,q}^{m+1,n} f(z))}{DR_{\lambda,q}^{m,n} f(z)}. \quad (20)$$

By using (8) in (20), we obtain

$$\begin{aligned} \frac{z \partial_q p(z)}{p(z)} &= \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} \right) - \frac{q^\lambda}{[\lambda]_q} \left(1 - \frac{[\lambda]_q}{q^\lambda} \right) - \\ & \quad \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \right) + \frac{q^\lambda}{[\lambda]_q} \left(1 - \frac{[\lambda]_q}{q^\lambda} \right). \end{aligned}$$

$$\frac{[\lambda]_q}{q^\lambda} \left(\frac{z \partial_q p(z)}{p(z)} \right) = \left(\frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} \right) - p(z)$$

$$\frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} = \frac{[\lambda]_q}{q^\lambda} \left(\frac{z \partial_q p(z)}{p(z)} + \frac{q^\lambda}{[\lambda]_q} p(z) \right).$$

We deduce from (8) that

$$\begin{aligned} & \frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \left\{ \varsigma \frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} + 1 - \varsigma \right\} \\ &= p(z) \left\{ \frac{\varsigma[\lambda]_q}{q^\lambda} \left(\frac{z \partial_q p(z)}{p(z)} + \frac{q^\lambda}{[\lambda]_q} p(z) \right) + 1 - \varsigma \right\} \\ &= (1 - \varsigma)p(z) + \varsigma p^2(z) + \frac{\varsigma[\lambda]_q}{q^\lambda} z \partial_q p(z). \end{aligned}$$

Therefore, the differential subordination in (18) becomes

$$\begin{aligned} & (1 - \varsigma)p(z) + \varsigma p^2(z) + \frac{\varsigma[\lambda]_q}{q^\lambda} z \partial_q p(z) \\ & \prec (1 - \varsigma)h(z) + \varsigma h^2(z) + \frac{\varsigma[\lambda]_q}{q^\lambda} z \partial_q p(z). \end{aligned}$$

Using Lemma 6, we obtain

$$\frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \prec h(z),$$

where h is the best dominant. \square

For $q \rightarrow 1-$, we obtain the result proven in [44].

Corollary 8 ([44]). Let h be both convex and univalent, $\varsigma \neq 0$, and

$$\operatorname{Re} \left\{ \frac{(1 - \varsigma)}{\lambda \varsigma} + \frac{2}{\lambda} h(z) + \left(1 + \frac{zh''(z)}{h'(z)} \right) \right\} > 0.$$

If $f \in T$ satisfies the differential subordination

$$\begin{aligned} & \frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} \left(\varsigma \frac{DR_{\lambda}^{m+2,n} f(z)}{DR_{\lambda}^{m+1,n} f(z)} + 1 - \varsigma \right) \\ & \prec (1 - \varsigma)h(z) + \varsigma h^2(z) + \varsigma \lambda z h'(z), \end{aligned}$$

then

$$\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)} \prec h(z), \quad z \in U.$$

Theorem 8. For $h(0) \neq 0$, $\varsigma \neq 0$. Let h be univalent in U and $\frac{z \partial_q h(z)}{h(z)}$ be both univalent and starlike in U . If the differential subordination condition for $f \in T$ holds, then

$$\begin{aligned} & \frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} - \varsigma \frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \\ & \prec \frac{[\lambda]_q}{q^\lambda} \frac{z \partial_q h(z)}{h(z)} + 1 - \varsigma, \end{aligned} \quad (21)$$

then

$$\frac{z^{\varsigma-1} DR_{\lambda,q}^{m+1,n} f(z)}{\left(DR_{\lambda,q}^{m,n} f(z) \right)^{\varsigma}} \prec h(z), \quad z \in U, \quad (22)$$

where the best dominant function is h .

Proof. Let

$$p(z) = \frac{z^{\varsigma-1} DR_{\lambda,q}^{m+1,n} f(z)}{\left(DR_{\lambda,q}^{m,n} f(z) \right)^{\varsigma}}, \quad (23)$$

which, when differentiated, yields

$$\begin{aligned} \partial_q p(z) &= \left\{ \frac{z^{\varsigma-2}(\varsigma-1) \left(DR_{\lambda,q}^{m+1,n} f(z) \right) + z^{\varsigma-1} \partial_q \left(DR_{\lambda,q}^{m+1,n} f(z) \right)}{\left(DR_{\lambda,q}^{m,n} f(z) \right)^{\varsigma}} \right\} \\ &\quad - \frac{\varsigma z^{\varsigma-1} \left(DR_{\lambda,q}^{m+1,n} f(z) \right) \partial_q \left(DR_{\lambda,q}^{m,n} f(z) \right)}{\left(DR_{\lambda,q}^{m,n} f(z) \right) \left(DR_{\lambda,q}^{m,n} f(z) \right)^{\varsigma}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{z \partial_q p(z)}{p(z)} &= \frac{\left(DR_{\lambda,q}^{m,n} f(z) \right)^{\varsigma}}{z^{\varsigma-1} DR_{\lambda,q}^{m+1,n} f(z)} \left\{ \frac{z^{\varsigma-2}(\varsigma-1) \left(DR_{\lambda,q}^{m+1,n} f(z) \right) + z^{\varsigma-1} \partial_q \left(DR_{\lambda,q}^{m+1,n} f(z) \right)}{\left(DR_{\lambda,q}^{m,n} f(z) \right)^{\varsigma}} \right. \\ &\quad \left. - \frac{\varsigma z^{\varsigma-1} \left(DR_{\lambda,q}^{m+1,n} f(z) \right) \partial_q \left(DR_{\lambda,q}^{m,n} f(z) \right)}{\left(DR_{\lambda,q}^{m,n} f(z) \right) \left(DR_{\lambda,q}^{m,n} f(z) \right)^{\varsigma}} \right\} \\ &= (\varsigma-1) + \frac{z \partial_q \left(DR_{\lambda,q}^{m+1,n} f(z) \right)}{DR_{\lambda,q}^{m+1,n} f(z)} - \varsigma \frac{z \partial_q \left(DR_{\lambda,q}^{m,n} f(z) \right)}{DR_{\lambda,q}^{m,n} f(z)}. \end{aligned}$$

We deduce from (8) that

$$\begin{aligned} \frac{z \partial_q p(z)}{p(z)} &= (\varsigma-1) + \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} \right) - \frac{q^\lambda}{[\lambda]_q} \left(1 - \frac{[\lambda]_q}{q^\lambda} \right) \\ &\quad - \varsigma \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \right) + \varsigma \frac{q^\lambda}{[\lambda]_q} \left(1 - \frac{[\lambda]_q}{q^\lambda} \right) \\ &= \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} \right) - \varsigma \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \right) \\ &\quad + \frac{(\varsigma-1)[\lambda]_q q^\lambda + q^\lambda(\varsigma-1)(q^\lambda - [\lambda]_q)}{[\lambda]_q q^\lambda} \\ &= \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+2,n} f(z)}{DR_{\lambda,q}^{m+1,n} f(z)} \right) - \varsigma \frac{q^\lambda}{[\lambda]_q} \left(\frac{DR_{\lambda,q}^{m+1,n} f(z)}{DR_{\lambda,q}^{m,n} f(z)} \right) \\ &\quad + \frac{q^\lambda(\varsigma-1)}{[\lambda]_q}, \end{aligned}$$

which corresponds to

$$\frac{DR_{\lambda,q}^{m+2,n}f(z)}{DR_{\lambda,q}^{m+1,n}f(z)} - \varsigma \frac{DR_{\lambda,q}^{m+1,n}f(z)}{DR_{\lambda,q}^{m,n}f(z)} = \frac{[\lambda]_q}{q^\lambda} \frac{z\partial_q p(z)}{p(z)} + (1 - \varsigma).$$

According to hypothesis (21), we have

$$\frac{z\partial_q p(z)}{p(z)} \prec \frac{z\partial_q h(z)}{h(z)}.$$

By using Lemma 5, we obtain

$$\frac{z^{\varsigma-1} DR_{\lambda,q}^{m+1,n}f(z)}{\left(DR_{\lambda,q}^{m,n}f(z) \right)^\varsigma} \prec h(z),$$

where h is the best dominant. \square

For $q \rightarrow 1-$, we obtain the result proven in [44].

Corollary 9 ([44]). For $h(0) \neq 0$, $\varsigma \neq 0$. Let h be univalent in U , and $\frac{zh'(z)}{h(z)}$ be univalent and starlike in U . The differential subordination condition is satisfied if and only if $f \in T$

$$\begin{aligned} & \frac{DR_{\lambda}^{m+2,n}f(z)}{DR_{\lambda}^{m+1,n}f(z)} - \varsigma \frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)} \\ & \prec \lambda \frac{zh'(z)}{h(z)} + 1 - \varsigma, \end{aligned}$$

then

$$\frac{z^{\varsigma-1} DR_{\lambda}^{m+1,n}f(z)}{\left(DR_{\lambda}^{m,n}f(z) \right)^\varsigma} \prec h(z), \quad z \in U,$$

where the best dominant function is h .

4. Conclusions

This study presents a modification of previous work that used quantum calculus to better understand geometric function theory. In this study, first of all, in Section 1, we defined the convolution operator $DR_{\lambda,q}^{m,n}$ inspired by the q -Sălăgean operator and the Ruscheweyh q -differential operator. Then, using the operator $DR_{\lambda,q}^{m,n}$, two new integral operators, $F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z)$ and $G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z)$, were introduced. Some new subclasses of analytic functions were introduced by means of these operators. In Section 2, four innovative lemmas that are connected to the new integral operators and were used in the justifications of the first findings in Sections 3 were proven. In Section 3, we first determined the sufficient conditions in Theorems 1 and 2 for the functions from class T to belong to classes LAF and LAG . Next, in Theorems 3–6, we proved some new properties of the integral operators $F_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z)$ and $G_{\lambda,\gamma_1,\gamma_2,\dots,\gamma_l}^{m,n,q}(z)$ for newly defined classes $R(\delta, q)$, $C(\delta, q)$, $RA(\beta, \mu, q)$, and $CA(\beta, \mu, q)$. We examined Theorems 7 and 8 by presenting the best dominants for certain differential subordinations. The results of this article are the generalizations discussed earlier in in [44].

Many new subclasses of analytic, meromorphic, and p -valent functions can be defined by utilizing the differential and integral operators introduced in this article, and a number of useful properties can be investigated for these classes.

Differential operators have allowed us to study differential equations from the perspective of operator theory and functional analysis. The use of differential operators allows

for the solution of differential equations. In the future, research might be conducted to determine whether PDEs can be solved using these operators. These novel operators may be studied for potential applications in the applied sciences and other practical sciences, where similar results have been reported for numerous differential operators.

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