

# An efficient ADE-WLP-FDTD method with new WLPs and factorized splitting scheme for dispersive simulation

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**Abstract** Based on an auxiliary differential equation (ADE) and new weighted Laguerre polynomials (WLPs), an efficient 3-D finite-difference time-domain method (FDTD) with factorized-splitting (FS) scheme is proposed to calculate wave propagation in general dispersive materials. The ADE technique is introduced to model general dispersive materials. Using a new temporal basis, the new WLPs can improve computational efficiency and save computing resources. The FS scheme is used to efficiently solve the huge sparse matrix equation. A numerical example is given to verify the accuracy and the efficiency of the proposed method. The results show its superiority compared with existing methods.

**Keywords:** auxiliary differential equation (ADE), dispersive materials, finite-difference time-domain (FDTD), weighted Laguerre polynomials (WLP)

**Classification:** Electromagnetic theory

## 1. Introduction

The unconditionally stable FDTD method based on WLPs is an efficient method to analyze electromagnetic problems with multi-scale structures [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. However, this method can only be used to model lossless and non-dispersive material. M. Ha and M. Swaminathan proposed an efficient WLP-FDTD method that can simulate frequency-dependent dispersive materials [13], which requires complicated Laguerre transformation of the convolution operator. In order to simulate general dispersive materials and avoid complicated Laguerre transform of convolution operator, ADE-WLP-FDTD method was proposed [14], but there is an accumulation term of the auxiliary differential variable during the operation in this method, which consumes considerable computer resources. Zhang *et al.* introduced a new WLP temporal basis which is a linear combination of three adjacent order weighted Laguerre polynomials [15]. The methods with this new WLP temporal basis are more efficient than the conventional WLP-FDTD method and are still unconditionally stable. Based on the new WLP temporal basis, we proposed a general ADE-WLP-FDTD method in [16]. This method can save computing resources and accelerate computational process because it eliminates the accumulation term of the conventional ADE-WLP-FDTD method. All the above methods lead to a huge sparse matrix equation in the calculation process

[17, 18], which is very challenging to solve. In order to solve this problem, the factorization-splitting (FS) and the domain decomposition techniques are regularly introduced. Duan *et al.* proposed an efficient WLP-FDTD method [19, 20], which introduced the FS scheme to solve the huge matrix equations into sub-steps procedure. He *et al.* proposed a domain decomposition WLP-FDTD method [21, 22], which decomposed the huge sparse matrix equation into several small subsystems for solving. These two methods of solving matrix equations are more efficient than direct calculation.

In this letter, a general 3-D ADE-WLP-FDTD with new WLPs and FS scheme is proposed for dispersive media simulation. By introducing the new WLPs, ADE and FS scheme simultaneously to the conventional WLP-FDTD method, our proposed method can accelerate the computational process with less computing resources while maintaining a comparable accuracy. A numerical example is given to verify the superiority of our proposed method to the existed ones.

## 2. Mathematical formulation

With reference to [16], by introducing the ADE and new WLPs, for lossless and dispersive media, the Maxwell's equations in Laguerre domain can be written as

$$\left(1 + \sum_{n=1}^{N_d} A_{1n}\right) \mathbf{E}^p(\mathbf{r}) = \frac{2}{s\epsilon_0\epsilon_\infty} \nabla \times \left(\mathbf{H}^p(\mathbf{r}) - 2\mathbf{H}^{p-1}(\mathbf{r}) + \mathbf{H}^{p-2}(\mathbf{r})\right) + \sum_{n=1}^{N_d} 2A_{1n}\mathbf{E}^{p-1}(\mathbf{r}) - \left(\sum_{n=1}^{N_d} A_{1n} - 1\right) \mathbf{E}^{p-2}(\mathbf{r}) \quad (1)$$

$$- \sum_{n=1}^{N_d} A_{2n}\mathbf{S}_n^{p-1}(\mathbf{r}) + \sum_{n=1}^{N_d} (A_{3n} + 1)\mathbf{S}_n^{p-2}(\mathbf{r}) - \frac{2}{s\epsilon_0\epsilon_\infty} \left(\mathbf{J}^p(\mathbf{r}) - 2\mathbf{J}^{p-1}(\mathbf{r}) + \mathbf{J}^{p-2}(\mathbf{r})\right) \mathbf{S}_n^p(\mathbf{r}) = A_{1n} \left(\mathbf{E}^p(\mathbf{r}) - 2\mathbf{E}^{p-1}(\mathbf{r}) + \mathbf{E}^{p-2}(\mathbf{r})\right) + A_{2n}\mathbf{S}_n^{p-1}(\mathbf{r}) - A_{3n}\mathbf{S}_n^{p-2}(\mathbf{r}) \quad (2)$$

$$\mathbf{H}^p(\mathbf{r}) = -\frac{2}{su_0} \nabla \times \left(\mathbf{E}^p(\mathbf{r}) - 2\mathbf{E}^{p-1}(\mathbf{r}) + \mathbf{E}^{p-2}(\mathbf{r})\right) + \mathbf{H}^{p-2}(\mathbf{r}) \quad (3)$$

where  $S$  is the auxiliary differential variable,  $\epsilon_0$  is the electric permittivity in free space,  $\epsilon_\infty$  is the infinite dielectric constant,  $\mu_0$  is the magnetic permeability of free space,  $s$  is

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time-scale factor,  $p$  is the order of Laguerre functions,  $a_n$ ,  $b_n$ ,  $c_n$ , and  $d_n$  are known constants determined by the properties of the electric fields [23, 24],  $A_{1n} = a_n/(b_n + 0.5c_ns + 0.25d_ns^2)$ ,  $A_{2n} = (2b_n - 0.5d_ns^2)/(b_n + 0.5c_ns + 0.25d_ns^2)$ ,  $A_{3n} = (b_n - 0.5c_ns + 0.25d_ns^2)/(b_n + 0.5c_ns + 0.25d_ns^2)$ . The WLP new temporal basis function is linear combination of three successive order weighted Laguerre polynomials. The resource  $\mathbf{J}^p$  can be derived by

$$\mathbf{J}^p(\mathbf{r}) = \int_0^\infty \mathbf{J}^p(\mathbf{r}, t) \varphi_p(st) d(st) + 2\mathbf{J}^{p-1}(\mathbf{r}) - \mathbf{J}^{p-2}(\mathbf{r}) \quad (4)$$

Eqs. (1) and (3) can be written as a matrix equation with three electric field components and three magnetic field components, which is challenging to solve. In order to efficiently solve this matrix equation, we introduce the FS scheme in the following derivation.

For simplicity, we specify the single pole dispersive media ( $N_d = 1$ ) in the following sections to describe the procedures for deriving the method presented in this letter. Expand (1) and (3) as

$$\begin{aligned} E_x^p = & \alpha (D_y H_z^p - D_z H_y^p - 2D_y H_z^{p-1} + 2D_z H_y^{p-1} \\ & + D_y H_z^{p-2} - D_z H_y^{p-2}) + M_1 E_x^{p-1} - M_2 E_x^{p-2} \\ & - M_3 S_x^{p-1} + M_4 S_x^{p-2} - \alpha (J_x^p - 2J_x^{p-1} + J_x^{p-2}) \end{aligned} \quad (5)$$

$$\begin{aligned} E_y^p = & \alpha (D_z H_x^p - D_x H_z^p - 2D_z H_x^{p-1} + 2D_x H_z^{p-1} \\ & + D_z H_x^{p-2} - D_x H_z^{p-2}) + M_1 E_y^{p-1} - M_2 E_y^{p-2} \\ & - M_3 S_y^{p-1} + M_4 S_y^{p-2} - \alpha (J_y^p - 2J_y^{p-1} + J_y^{p-2}) \end{aligned} \quad (6)$$

$$\begin{aligned} E_z^p = & \alpha (D_x H_y^p - D_y H_x^p - 2D_x H_y^{p-1} + 2D_y H_x^{p-1} \\ & + D_x H_y^{p-2} - D_y H_x^{p-2}) + M_1 E_z^{p-1} - M_2 E_z^{p-2} \\ & - M_3 S_z^{p-1} + M_4 S_z^{p-2} - \alpha (J_z^p - 2J_z^{p-1} + J_z^{p-2}) \end{aligned} \quad (7)$$

$$\begin{aligned} H_x^p = & \beta (D_z E_y^p - D_y E_z^p - 2D_z E_y^{p-1} + 2D_y E_z^{p-1} \\ & + D_z E_y^{p-2} - D_y E_z^{p-2}) + H_x^{p-2} \end{aligned} \quad (8)$$

$$\begin{aligned} H_y^p = & \beta (D_x E_z^p - D_z E_x^p - 2D_x E_z^{p-1} + 2D_z E_x^{p-1} \\ & + D_x E_z^{p-2} - D_z E_x^{p-2}) + H_y^{p-2} \end{aligned} \quad (9)$$

$$\begin{aligned} H_z^p = & \beta (D_y E_x^p - D_x E_y^p - 2D_y E_x^{p-1} + 2D_x E_y^{p-1} \\ & + D_y E_x^{p-2} - D_x E_y^{p-2}) + H_z^{p-2} \end{aligned} \quad (10)$$

where  $\alpha = 2/s\epsilon_0\epsilon_\infty(1 + A_1)$ ,  $\beta = 2/su_0$ ,  $J_x$ ,  $J_y$ , and  $J_z$  are the excitation sources along  $x$ ,  $y$ , and  $z$  axes.  $D_x$ ,  $D_y$  and  $D_z$  are the first-order central difference operators along  $x$ ,  $y$ , and  $z$  axes, respectively,  $M_1 = 2A_1/(1 + A_1)$ ,  $M_2 = (A_1 - 1)/(1 + A_1)$ ,  $M_3 = A_2/(1 + A_1)$ ,  $M_4 = (1 + A_3)/(1 + A_1)$  and

$$S_\xi^p = A_1 (E_\xi^p - 2E_\xi^{p-1} + E_\xi^{p-2}) + A_2 S_\xi^{p-1} - A_3 S_\xi^{p-2} \quad (11)$$

$\xi = x, y, z$ . According to [25], with some manipulations, we can write (5)–(10) in the following matrix forms:

$$\begin{aligned} \mathbf{W}_E^p = & \alpha \mathbf{D}_H \mathbf{W}_H^p - 2\alpha \mathbf{D}_H \mathbf{W}_H^{p-1} + \alpha \mathbf{D}_H \mathbf{W}_H^{p-2} \\ & + \mathbf{W}_E^{p-2} + M_1 \mathbf{W}_E^{p-1} - M_1 \mathbf{W}_E^{p-2} - M_3 \mathbf{S}^{p-1} \\ & + M_4 \mathbf{S}^{p-2} - \alpha (\mathbf{J}_E^p - 2\mathbf{J}_E^{p-1} + \mathbf{J}_E^{p-2}) \end{aligned} \quad (12)$$

$$\mathbf{W}_H^p = \beta \mathbf{D}_E \mathbf{W}_E^p - 2\beta \mathbf{D}_E \mathbf{W}_E^{p-1} + \beta \mathbf{D}_E \mathbf{W}_E^{p-2} + \mathbf{W}_H^{p-2} \quad (13)$$

Define a set of auxiliary matrices as

$$\mathbf{W}_E^q = (E_x^q \ E_y^q \ E_z^q)^T \quad (14)$$

$$\mathbf{S}^q = (S_x^q \ S_y^q \ S_z^q)^T \quad (15)$$

$$\mathbf{W}_H^q = (H_x^q \ H_y^q \ H_z^q)^T \quad (16)$$

$$\mathbf{J}_E^p = (J_x^p \ J_y^p \ J_z^p)^T \quad (17)$$

$$\mathbf{D}_H = (\mathbf{D}_E)^T = \begin{pmatrix} 0 & -D_z & D_y \\ D_z & 0 & -D_x \\ -D_y & D_x & 0 \end{pmatrix} \quad (18)$$

Here we combine (12) and (13) into a matrix form as

$$\begin{aligned} \begin{bmatrix} \mathbf{W}_E^p \\ \mathbf{W}_H^p \end{bmatrix} = & \begin{bmatrix} 0 & \alpha \mathbf{D}_H \\ \beta \mathbf{D}_E & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_E^p \\ \mathbf{W}_H^p \end{bmatrix} - 2 \begin{bmatrix} 0 & \alpha \mathbf{D}_H \\ \beta \mathbf{D}_E & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_E^{p-1} \\ \mathbf{W}_H^{p-1} \end{bmatrix} \\ & + \begin{bmatrix} 0 & \alpha \mathbf{D}_H \\ \beta \mathbf{D}_E & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_E^{p-2} \\ \mathbf{W}_H^{p-2} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_E^{p-2} \\ \mathbf{W}_H^{p-2} \end{bmatrix} + M_1 \begin{bmatrix} \mathbf{W}_E^{p-1} \\ 0 \end{bmatrix} \\ & - M_1 \begin{bmatrix} \mathbf{W}_E^{p-2} \\ 0 \end{bmatrix} - M_3 \begin{bmatrix} \mathbf{S}^{p-1} \\ 0 \end{bmatrix} + M_4 \begin{bmatrix} \mathbf{S}^{p-2} \\ 0 \end{bmatrix} \\ & - \alpha \begin{bmatrix} \mathbf{J}_E^p - 2\mathbf{J}_E^{p-1} + \mathbf{J}_E^{p-2} \\ 0 \end{bmatrix} \end{aligned} \quad (19)$$

Let  $\mathbf{W}^p = [\mathbf{W}_E^p \ \mathbf{W}_H^p]^T$ ,  $\mathbf{W}^{p-1} = [\mathbf{W}_E^{p-1} \ \mathbf{W}_H^{p-1}]^T$ ,  $\mathbf{W}^{p-2} = [\mathbf{W}_E^{p-2} \ \mathbf{W}_H^{p-2}]^T$ ,  $\mathbf{W}_{E0}^{p-1} = [\mathbf{W}_E^{p-1} \ 0]^T$ ,  $\mathbf{W}_{E0}^{p-2} = [\mathbf{W}_E^{p-2} \ 0]^T$ ,  $\mathbf{J}_{E0}^p = [\mathbf{J}_E^p \ 0]$ ,  $\mathbf{S}_0^{p-1} = [\mathbf{S}^{p-1} \ 0]^T$ ,  $\mathbf{S}_0^{p-2} = [\mathbf{S}^{p-2} \ 0]^T$ ,  $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 0 & \alpha \mathbf{D}_H \\ \beta \mathbf{D}_E & 0 \end{bmatrix}$ , then we can write (19) as

$$\begin{aligned} (\mathbf{I} - \mathbf{A} - \mathbf{B}) \mathbf{W}^p = & -2(\mathbf{A} + \mathbf{B}) \mathbf{W}^{p-1} + (\mathbf{A} + \mathbf{B}) \mathbf{W}^{p-2} + \mathbf{W}^{p-2} \\ & + M_1 \mathbf{W}_{E0}^{p-1} - M_1 \mathbf{W}_{E0}^{p-2} - M_3 \mathbf{S}_0^{p-1} \\ & + M_4 \mathbf{S}_0^{p-2} - \alpha (\mathbf{J}_{E0}^p - 2\mathbf{J}_{E0}^{p-1} + \mathbf{J}_{E0}^{p-2}) \end{aligned} \quad (20)$$

where  $\mathbf{I}$  is the identity matrix. Adding a perturbation term  $\mathbf{AB}(\mathbf{W}^p - \mathbf{W}^{p-2})$  to (20), using the splitting scheme, we can obtain the factorized form of (20)

$$\begin{aligned} (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{B}) \mathbf{W}^p = & \mathbf{ABW}^{p-2} + (\mathbf{A} + \mathbf{B} + \mathbf{I}) \mathbf{W}^{p-2} \\ & - 2(\mathbf{A} + \mathbf{B}) \mathbf{W}^{p-1} + M_1 \mathbf{W}_{E0}^{p-1} - M_1 \mathbf{W}_{E0}^{p-2} - M_3 \mathbf{S}_0^{p-1} \\ & + M_4 \mathbf{S}_0^{p-2} - \alpha (\mathbf{J}_{E0}^p - 2\mathbf{J}_{E0}^{p-1} + \mathbf{J}_{E0}^{p-2}) \end{aligned} \quad (21)$$

Equation (21) can be solved into two sub-steps with a factorization-splitting scheme as follows:

$$\begin{aligned} (\mathbf{I} - \mathbf{A}) \mathbf{W}^{*p} = & \mathbf{BW}^{p-2} - 2(\mathbf{A} + \mathbf{B}) \mathbf{W}^{p-1} + (\mathbf{A} + \mathbf{B} + \mathbf{I}) \mathbf{W}^{p-2} \\ & + M_1 \mathbf{W}_{E0}^{p-1} - M_1 \mathbf{W}_{E0}^{p-2} - M_3 \mathbf{S}_0^{p-1} + M_4 \mathbf{S}_0^{p-2} \\ & - \alpha (\mathbf{J}_{E0}^p - 2\mathbf{J}_{E0}^{p-1} + \mathbf{J}_{E0}^{p-2}) \end{aligned} \quad (22)$$

$$(\mathbf{I} - \mathbf{B}) \mathbf{W}^p = \mathbf{W}^{*p} - \mathbf{BW}^{p-2} \quad (23)$$

$\mathbf{W}^{*p} = [\mathbf{W}_E^{*p} \ \mathbf{W}_H^{*p}]^T = [E_x^{*p} \ E_y^{*p} \ E_z^{*p} \ H_x^{*p} \ H_y^{*p} \ H_z^{*p}]^T$  is a nonphysical intermediate value. To solve (22) and (23), we choose

the matrices  $\mathbf{A}$  and  $\mathbf{B}$  as

$$\mathbf{A} = \begin{bmatrix} 0 & -\alpha \mathbf{D} \\ -\beta \mathbf{D}^T & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & \alpha \mathbf{D}^T \\ \beta \mathbf{D} & 0 \end{bmatrix} \quad (24)$$

where

$$\mathbf{D} = \begin{pmatrix} 0 & D_z & 0 \\ 0 & 0 & D_x \\ D_y & 0 & 0 \end{pmatrix}, \quad \mathbf{D}^T = \begin{pmatrix} 0 & 0 & D_y \\ D_z & 0 & 0 \\ 0 & D_x & 0 \end{pmatrix} \quad (25)$$

Inserting (24) into (22) and (23), and with some manipulations to eliminate  $\mathbf{W}_H^{*q}$ , leads to

$$\begin{aligned} (\mathbf{I} - \alpha \beta \mathbf{D} \mathbf{D}^T) \mathbf{W}_E^{*p} = & \\ -\alpha \beta \mathbf{D} \mathbf{D} \mathbf{W}_E^{p-2} + \alpha \mathbf{D}^T \mathbf{W}_H^{p-2} - 2\alpha \mathbf{D}_H \mathbf{W}_H^{p-1} & \\ + (\mathbf{I} - \alpha \beta \mathbf{D} \mathbf{D}_E) \mathbf{W}_E^{p-2} + \alpha (\mathbf{D}_H - \mathbf{D}) \mathbf{W}_H^{p-2} & \\ + 2\alpha \beta \mathbf{D} \mathbf{D}_E \mathbf{W}_E^{p-1} + \mathbf{M}_1 \mathbf{W}_E^{p-1} - \mathbf{M}_1 \mathbf{W}_E^{p-2} - \mathbf{M}_3 \mathbf{S}^{p-1} & \\ + \mathbf{M}_4 \mathbf{S}^{p-2} - \alpha (\mathbf{J}^p - 2\mathbf{J}^{p-1} + \mathbf{J}^{p-2}) & \end{aligned} \quad (26)$$

$$\begin{aligned} (\mathbf{I} - \alpha \beta \mathbf{D}^T \mathbf{D}) \mathbf{W}_E^p = \mathbf{W}_E^{*p} - \alpha \beta \mathbf{D}^T \mathbf{D}^T \mathbf{W}_E^{*p} & \\ + \alpha \beta \mathbf{D}^T \mathbf{D} \mathbf{W}_E^{p-2} + \alpha \beta \mathbf{D}^T \mathbf{D}_E \mathbf{W}_E^{p-2} & \\ - 2\alpha \beta \mathbf{D}^T \mathbf{D}_E \mathbf{W}_E^{p-1} - \alpha \beta \mathbf{D}^T \mathbf{D} \mathbf{W}_E^{p-2} & \end{aligned} \quad (27)$$

$$\begin{aligned} \mathbf{W}_H^p = \beta \mathbf{D} \mathbf{W}_E^p - \beta \mathbf{D}^T \mathbf{W}_E^{*p} + \beta \mathbf{D} \mathbf{W}_E^{p-2} & \\ + \beta \mathbf{D}_E \mathbf{W}_E^{p-2} + \mathbf{W}_H^{p-2} - 2\beta \mathbf{D}_E \mathbf{W}_E^{p-1} - \beta \mathbf{D} \mathbf{W}_E^{p-2} & \end{aligned} \quad (28)$$

Using (14)–(18) and (25) to expand (26)–(28), leads to

$$\begin{aligned} (1 - \alpha \beta D_z D_z) E_x^{*p} = & \\ 2\alpha D_z H_y^{p-1} - 2\alpha D_z H_y^{p-2} - 2\alpha D_y H_z^{p-1} + 2\alpha D_y H_z^{p-2} & \\ - 2\alpha \beta D_z D_z E_x^{p-1} + (1 + \alpha \beta D_z D_z) E_x^{p-2} + 2\alpha \beta D_z D_x E_z^{p-1} & \\ - 2\alpha \beta D_z D_x E_z^{p-2} + \mathbf{M}_1 E_x^{p-1} - \mathbf{M}_1 E_x^{p-2} - \mathbf{M}_3 S_x^{p-1} & \\ + \mathbf{M}_4 S_x^{p-2} - \alpha (J_x^p - 2J_x^{p-1} + J_x^{p-2}) & \end{aligned} \quad (29)$$

$$\begin{aligned} (1 - \alpha \beta D_x D_x) E_y^{*p} = & \\ -2\alpha D_z H_x^{p-1} + 2\alpha D_z H_x^{p-2} + 2\alpha D_x H_z^{p-1} - 2\alpha D_x H_z^{p-2} & \\ + 2\alpha \beta D_x D_y E_x^{p-1} - 2\alpha \beta D_x D_y E_x^{p-2} - 2\alpha \beta D_x D_x E_y^{p-1} & \\ + (1 + \alpha \beta D_x D_x) E_y^{p-2} + \mathbf{M}_1 E_y^{p-1} - \mathbf{M}_1 E_y^{p-2} - \mathbf{M}_3 S_y^{p-1} & \\ + \mathbf{M}_4 S_y^{p-2} - \alpha (J_y^p - 2J_y^{p-1} + J_y^{p-2}) & \end{aligned} \quad (30)$$

$$\begin{aligned} (1 - \alpha \beta D_y D_y) E_z^{*p} = & \\ 2\alpha D_y H_x^{p-1} - 2\alpha D_y H_x^{p-2} - 2\alpha D_x H_y^{p-1} + 2\alpha D_x H_y^{p-2} & \\ + 2\alpha \beta D_y D_z E_y^{p-1} - 2\alpha \beta D_y D_z E_y^{p-2} - 2\alpha \beta D_y D_y E_z^{p-1} & \\ + (1 + \alpha \beta D_y D_y) E_z^{p-2} + \mathbf{M}_1 E_z^{p-1} - \mathbf{M}_1 E_z^{p-2} - \mathbf{M}_3 S_z^{p-1} & \\ + \mathbf{M}_4 S_z^{p-2} - \alpha (J_z^p - 2J_z^{p-1} + J_z^{p-2}) & \end{aligned} \quad (31)$$

$$\begin{aligned} (1 - \alpha \beta D_y D_y) E_x^p = E_x^{*p} - \alpha \beta D_y D_x E_y^{*p} & \\ - 2\alpha \beta D_y D_y E_x^{p-1} + \alpha \beta D_y D_y E_x^{p-2} & \\ + 2\alpha \beta D_y D_x E_y^{p-1} - \alpha \beta D_y D_x E_y^{p-2} & \end{aligned} \quad (32)$$

$$\begin{aligned} (1 - \alpha \beta D_z D_z) E_y^p = E_y^{*p} - \alpha \beta D_z D_y E_z^{*p} & \\ - 2\alpha \beta D_z D_z E_y^{p-1} + \alpha \beta D_z D_z E_y^{p-2} & \\ + 2\alpha \beta D_z D_y E_z^{p-1} - \alpha \beta D_z D_y E_z^{p-2} & \end{aligned} \quad (33)$$

$$\begin{aligned} (1 - \alpha \beta D_x D_x) E_z^p = E_z^{*p} - \alpha \beta D_x D_z E_x^{*p} & \\ + 2\alpha \beta D_x D_z E_x^{p-1} - \alpha \beta D_x D_z E_x^{p-2} & \\ - 2\alpha \beta D_x D_x E_z^{p-1} + \alpha \beta D_x D_x E_z^{p-2} & \end{aligned} \quad (34)$$

$$\begin{aligned} H_x^p = -\beta D_y E_y^{*p} + H_x^{p-2} + \beta D_z E_y^p - 2\beta D_z E_y^{p-1} & \\ + \beta D_z E_y^{p-2} + 2\beta D_y E_z^{p-1} - \beta D_y E_z^{p-2} & \end{aligned} \quad (35)$$

$$\begin{aligned} H_y^p = -\beta D_z E_x^{*p} + H_y^{p-2} + 2\beta D_z E_x^{p-1} - \beta D_z E_x^{p-2} & \\ + \beta D_x E_z^p - 2\beta D_x E_z^{p-1} + \beta D_x E_z^{p-2} & \end{aligned} \quad (36)$$

$$\begin{aligned} H_z^p = -\beta D_x E_y^{*p} + H_z^{p-2} + \beta D_y E_x^p - 2\beta D_y E_x^{p-1} & \\ + \beta D_y E_x^{p-2} + 2\beta D_x E_y^{p-1} - \beta D_x E_y^{p-2} & \end{aligned} \quad (37)$$

Eqs. (29)–(37) are the new update equations for the proposed method. Applying the central-difference scheme introduced by Yee to (29)–(37), we can obtain the discrete space equations. Compared with the conventional ADE-WLP-FDTD method proposed in [14], the huge sparse matrix equation is decomposed into six small matrix equations that are easier to solve due to the adoption of the FS scheme, meanwhile the accumulation term is removed due to the introduction of the new WLPs.

### 3. Numerical results

To validate the performance of the proposed method, the transition field in a 3-D dielectric-load resonant cavity is calculated. A sinusoidal modulated Gaussian pulse is taken as the incident electric current profile

$$J_y(t) = \exp[-((t - T_c)/T_d)^2] \sin[2\pi f_c(t - T_c)] \quad (38)$$

in (38),  $T_d = 1/(2f_c)$ ,  $T_c = 3T_d$ . As shown in Fig. 1, the uniform cell of size  $8 \times 4 \times 4 \text{ cm}^3$  is divided into  $160 \times 40 \times 40$  cells, the perfect electric conductor (PEC) boundary condition is set at the terminal of the computational domain. We consider the dispersive medium as a hybrid model consisting of Debye medium and Lorentz medium with the same thickness. The relative complex permittivity of the Debye model is given by

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega\tau} \quad (39)$$

where  $\varepsilon_s = 4.301$ ,  $\varepsilon_\infty = 4.096$ , and  $\tau = 2.294 \times 10^{-9}$ . The relative complex permittivity of the Lorentz model can be expressed as

$$\varepsilon_r(\omega) = \varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty) \frac{G_1 \omega_1^2}{\omega_1^2 + j2\delta_1 \omega - \omega^2} \quad (40)$$

where  $\varepsilon_s = 3$ ,  $\varepsilon_\infty = 1.5$ ,  $G_1 = 0.4$ ,  $\omega_1 = 2 \times 10^9 \text{ rad/s}$ , and  $\delta_1 = 0.1 \omega_1$ . We set  $s = 2.2619 \times 10^{11}$ , the time duration  $T_f = 2 \text{ ns}$ ,  $f_c = 6 \text{ GHz}$  and step number  $N_L = 120$  [26, 27, 28, 29]. The coordinates of the incident electric current  $\mathbf{J}_y$  and the observation point are  $\mathbf{S}$  ( $20\Delta x$ ,  $20\Delta y$ ,  $20\Delta z$ ) and  $\mathbf{O}$  ( $140\Delta x$ ,  $20\Delta y$ ,  $20\Delta z$ ), respectively.

Fig. 2 shows the time-domain numerical results of four different methods in dispersive medium, it can be seen that, although the introduction of the perturbation term brings

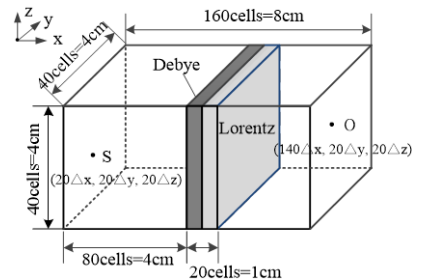
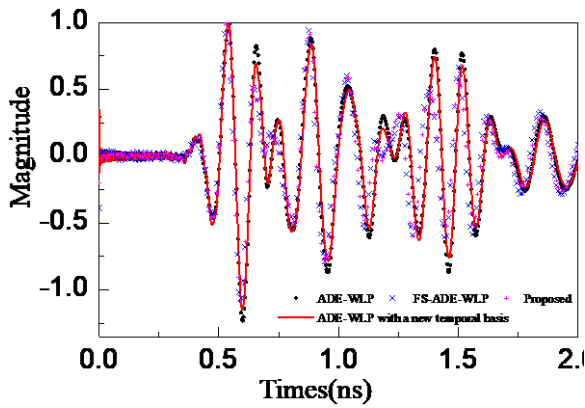


Fig. 1 Diagram of computational domain.



**Fig. 2** Normalized transient electric fields of the model in Fig. 1 calculated by conventional and proposed methods.

**Table I** Computational cost comparison for the model in Fig. 1 with different methods.

Method	CPU time(s)	Memory(MB)
ADE-WLP-FDTD [14]	2231	3391
FS-ADE-WLP-FDTD [30]	918	1672
ADE-WLP-FDTD with new WLPs[16]	1832	559
This work	858	203

splitting error, the results calculated by the two FS methods coincide with the results calculated by the other two methods with a comparable accuracy. Table I details the comparison of computing time and resources for the four different methods. We find that our proposed method has the least CPU time and memory consumption compared with the conventional methods. The simulation computer is configured with Intel (R) Core (TM) i7-11700K CPU with 32 GB RAM.

#### 4. Conclusion

In order to solve electromagnetic problems in general dispersive materials, based on the conventional WLP-FDTD method, the new WLPs, ADE and FS scheme are introduced simultaneously, and a new method consuming less computational resources is proposed in this letter. This method can accelerate computational process with a reliable accuracy and is more efficient than the conventional methods.

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