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Existence and stability results for nonlinear fractional integrodifferential coupled systems

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Abstract

In this paper, a class of nonlinear ψ-Hilfer fractional integrodifferential coupled systems on a bounded domain is investigated. The existence and uniqueness results for the coupled systems are proved based on the contraction mapping principle. Moreover, the Ulam–Hyers–Rassias, Ulam–Hyers, and semi-Ulam–Hyers–Rassias stabilities to the initial value problem are obtained.

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1 Introduction

The objective of the present paper is to investigate the existence, uniqueness, and stability of solutions for a class of nonlinear ψ-Hilfer fractional integrodifferential coupled systems on a bounded domain. The system is described as follows

$$\begin{cases} {}^{H}D_{a^+}^{\alpha,\beta;\psi}x(t) = f(t,y(t),{}^{H}D_{a^+}^{u,v;\psi}y(t)) + \int_a^t F(t,\tau,y(\tau),y(\delta(\tau)))\,d\tau, & t \in J = [a,b], \\ {}^{H}D_{a^+}^{\alpha',\beta';\psi}y(t) = g(t,x(t),{}^{H}D_{a^+}^{u,v;\psi}x(t)) + \int_a^t G(t,\tau,x(\tau),x(\delta(\tau)))\,d\tau, & t \in J, \\ I_{a^+}^{1-\gamma;\psi}x(a) = 0, \qquad I_{a^+}^{1-\gamma';\psi}y(a) = 0, \end{cases} \tag{1.1}$$

where ψ-Hilfer fractional derivatives ${}^{H}D_{a^+}^{\alpha,\beta;\psi}(\cdot)$, ${}^{H}D_{a^+}^{\alpha',\beta';\psi}(\cdot)$, ${}^{H}D_{a^+}^{u,v;\psi}(\cdot)$ [1] of order $0 < u < \alpha,\alpha' < 1$ with type $0 \leq \beta,\beta',v \leq 1$ and ψ-Riemann–Liouville fractional integral $I_{a^+}^{1-\gamma;\psi}(\cdot)$, $I_{a^+}^{1-\gamma';\psi}(\cdot)$ [1] of order $1-\gamma$, where $\gamma = \alpha + \beta(1-\alpha)$, $\gamma' = \alpha + \beta(1-\alpha)$, and $w = u + v(1-u)$. $f: J \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ and $g: J \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ are continuously differentiable functions and $\mathbb{Y}$ is a real Banach space. $F: J \times J \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ and $G: J \times J \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ are continuous functions. $\delta: J \to J$ is a continuous delay function with $\delta(t) \leq t$.

Fractional differential equations help practical problems to be described more accurately compared with integer differential equations. In recent years, numerous fractional derivatives, such as Riemann–Liouville, Caputo, and Hilfer are widely used in the fields of finance, physics, biology, image processing, etc., which can be found in [2–10] and the references cited therein.

As is known, a variety of new concepts about fractional derivatives have been defined, for instance, Kilbas introduced a new fractional derivative operator of a function with respect to another function ψ in [11], named the ψ-Riemann–Liouville fractional derivative. Similarly, Almeida proposed the ψ-Caputo fractional derivative on the basis of the classical Caputo fractional derivative [12]. It is well known that the ψ-Hilfer fractional derivative is given by Sousa and Oliveira [1] in the same way. It is not difficult to discover that the ψ-Hilfer fractional derivative contains the Riemann–Liouville, Caputo, and Hilfer fractional derivatives, i.e., the ψ-Hilfer fractional derivative is compatible with classical fractional derivatives based on Definition 2.1. Therefore, it is a better method to choose fractional differential systems involving the ψ-Hilfer fractional derivative that includes many fractional differential equations as special cases for resolving problems of finance, physics, biology, image processing, etc., mentioned above in the real world.

Very recently, numerous monographs have appeared concerning the results of the existence and stability of Ulam–Hyers–Rassias and Ulam–Hyers of nonlinear fractional differential equations focused on Riemann–Liouville, Caputo, Hilfer, etc. The readers can refer to the papers of Wang and Xu [13], Rajan et al. [14] and Haider et al. [15] so on [16–23]. However, there are few research results on the existence and stability of solutions for the ψ-Hilfer fractional derivative system except for [24, 25]. Sousa and Oliveira have studied the existence and uniqueness of solutions for the initial value problems of fractional derivative systems on a finite interval by using the fixed- point method. Furthermore, Sousa and Oliveira [26] discussed Ulam–Hyers–Rassias, Ulam–Hyers, and semi-Ulam–Hyers–Rassias stability on a finite interval $[a,b]$. As a generalization, they also discussed the stability of Ulam–Hyers–Rassias on the semiinfinite interval $[a,\infty)$ in a weighted space for the following nonlinear ψ-Hilfer fractional integrodifferential equation:

$$\begin{cases} {}^{H}D_{a^+}^{\alpha,\beta;\psi} y(x) = f(x, y(x), \int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))\,d\tau), \quad t\in[a,b], \\ I_{a^+}^{1-\gamma;\psi} y(a) = c, \end{cases} \tag{1.2}$$

where ${}^{H}D_{a^+}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative of order $\alpha\in(0,1)$ and type $\beta\in[0,1]$, $I_{a^+}^{1-\gamma;\psi}(\cdot)$ is the ψ-Riemann–Liouville fractional integral, with $1-\gamma$ and $\gamma=\alpha+\beta(1-\alpha)$. $a,b\in\mathbb{R}$, such that $y\in C^1[a,b]$ for all $x\in[a,b]$. $\delta:[a,b]\to[a,b]$ is a continuous delay function with $\delta(t)\le t$ for all $t\in[a,b]$. Moreover, continuous functions $f:[a,b]\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ and $K:[a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ satisfy the following Lipschitz conditions, respectively:

$$\begin{aligned} \left|f(x,u,g)-f(x,v,h)\right| &\le M\big(|u-v|+|g-h|\big), \\ \left|K(x,u,g)-K(x,v,h)\right| &\le L|g-h|, \end{aligned}$$

with $M>0$, $L>0$.

Motivated by [27–30] and the references therein, we will deal with the existence and stability of a solution to the initial value problem for the nonlinear integrodifferential equation (1.1) by virtue of an appropriate Banach space given by

$$\mathbb{X}=\left\{x \mid x(t)\in C^1(J,\mathbb{Y}), {}^{H}D_{a^+}^{u,v;\psi}x(t)\in C^1(J,\mathbb{Y})\right\},$$

where $\mathbb{Y}$ is a real Banach space, endowed with the associated norm:

$$\|x\|_{\mathbb{X}} = \max\Big\{\sup_{t\in J}\big\|x(t)\big\|, \sup_{t\in J}\big\|{}^{H}D_{a^+}^{u,v;\psi}x(t)\big\|\Big\},$$

where $\|x(t)\| = \max_{t\in J}|x(t)|$, $\|{}^{H}D_{a^+}^{u,v;\psi}x(t)\| = \max_{t\in J}|{}^{H}D_{a^+}^{u,v;\psi}x(t)|$.

We establish a special Banach space

$$\mathbb{X}\times\mathbb{X} = \big\{(x,y) \mid x\in\mathbb{X}, y\in\mathbb{X}\big\},$$

endowed with the associated norm:

$$\big\|(x,y)\big\|_{\mathbb{X}\times\mathbb{X}} = \max\big\{\|x\|_{\mathbb{X}}, \|y\|_{\mathbb{X}}\big\},$$

by the method given in [31–33], we can easily obtain that $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{X}\times\mathbb{X}, \|\cdot\|_{\mathbb{X}\times\mathbb{X}})$ are Banach spaces.

The rest of this paper is organized as follows. In Sect. 2, we present some necessary material related to our study. Section 3 contains the main results about the existence and stability of solution for the nonlinear ψ-Hilfer fractional integrodifferential equation (1.1) that rely on the Banach contraction mapping principle.

2 Preliminaries

Let us begin this section with some basic concepts and conclusions used in our study.

Definition 2.1 (ψ-Hilfer fractional derivative, [1]) Let $n-1<\alpha<n$ with $n\in\mathbb{N}$. $I=[a,b]$ be the interval such that $-\infty\le a<b\le\infty$ and $f,\psi\in C^n([a,b],\mathbb{R})$ are two functions such that ψ is increasing and $\psi'(x)\neq 0$, for all $x\in I$. The left-sided ψ-Hilfer fractional derivative ${}^{H}D_{a^+}^{\alpha,\beta;\psi}(\cdot)$ of function of order α and type $0\le\beta\le 1$, is defined by

$${}^{H}D_{a^+}^{\alpha,\beta;\psi}f(x) = I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}f(x),$$

the right-sided ψ-Hilfer fractional derivative ${}^{H}D_{b^-}^{\alpha,\beta;\psi}(\cdot)$ is defined as follows:

$${}^{H}D_{b^-}^{\alpha,\beta;\psi}f(x) = I_{b^-}^{\beta(n-\alpha);\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b^-}^{(1-\beta)(n-\alpha);\psi}f(x).$$

Definition 2.2 (ψ-Riemann–Liouville fractional integral, [1]) Let (a,b) $(-\infty\le a<b\le\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also, let $\psi(x)$ be an increasing, positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The left-sided ψ-Riemann–Liouville fractional integral $I_{a^+}^{\alpha;\psi}(\cdot)$ of a function f with respect to another function ψ on $[a,b]$, is defined by

$$I_{a^+}^{\alpha;\psi}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)\big(\psi(x)-\psi(s)\big)^{\alpha-1}f(s)\,ds,$$

the right-sided ψ-Riemann–Liouville fractional integral $I_{b^-}^{\alpha;\psi}(\cdot)$ is defined in a similar form.

Lemma 2.1 ([1]) *Let* $\alpha > 0$ *and* $\beta > 0$, *then we have the following semigroup property given by*

$$I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(x) = I_{a^+}^{\alpha+\beta;\psi} f(x).$$

Lemma 2.2 ([1]) *If* $f \in \mathbb{C}^n[a,b]$, $n-1 < \alpha < n$ *and* $0 \le \beta \le 1$, *then*

$$I_{a^+}^{\alpha;\psi}\, {}^H D_{a^+}^{\alpha,\beta;\psi} f(x) = f(x) - \sum_{k=1}^{n} \frac{(\psi(x)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_\psi^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a),$$

with $\gamma = \alpha + \beta(n-\alpha)$.

In particular, if $f \in \mathbb{C}^1[a,b]$, $0 < \alpha < 1$ *and* $0 \le \beta \le 1$, *then*

$$I_{a^+}^{\alpha;\psi}\, {}^H D_{a^+}^{\alpha,\beta;\psi} f(x) = f(x) - \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{(1-\beta)(1-\alpha);\psi} f(a),$$

with $(1-\alpha)(1-\beta) = 1-\gamma$.

Lemma 2.3 ([1]) *Let* $f \in \mathbb{C}^1[a,b]$, $\alpha > 0$ *and* $0 \le \beta \le 1$, *then one has*

$${}^H D_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha;\psi} f(x) = f(x).$$

Definition 2.3 (Ulam–Hyers–Rassias stability, [26]) For each continuously differentiable function $x : J \to X$, satisfying

$$\begin{aligned}
&\left\| x(t) - I_{a^+}^{\alpha;\psi} \left[f\big(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\big) + \int_a^t K\big(t,\tau,y(\tau), y\big(\delta(\tau)\big)\big)\, d\tau \right] \right\| \\
&\quad \le I_{a^+}^{\alpha;\psi} \Phi(t), \quad \forall t \in J, \\
&\left\| {}^H D_{a^+}^{u,v;\psi} x(t) - I_{a^+}^{\alpha-u;\psi} \left[f\big(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\big) + \int_a^t K\big(t,\tau,y(\tau), y\big(\delta(\tau)\big)\big)\, d\tau \right] \right\| \\
&\quad \le I_{a^+}^{\alpha-u;\psi} \Phi(t), \quad \forall t \in J, \\
&\left\| y(t) - I_{a^+}^{\alpha';\psi} \left[g\big(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\big) + \int_a^t K\big(t,\tau,x(\tau), x\big(\delta(\tau)\big)\big)\, d\tau \right] \right\| \\
&\quad \le I_{a^+}^{\alpha';\psi} \Phi(t), \quad \forall t \in J, \\
&\left\| {}^H D_{a^+}^{u,v;\psi} y(t) - I_{a^+}^{\alpha'-u;\psi} \left[g\big(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\big) + \int_a^t K\big(t,\tau,x(\tau), x\big(\delta(\tau)\big)\big)\, d\tau \right] \right\| \\
&\quad \le I_{a^+}^{\alpha'-u;\psi} \Phi(t), \quad \forall t \in J,
\end{aligned}$$

where $\Phi(t)$ is a positive, nondecreasing continuous function defined on the finite interval $[a,b]$, there exists a solution (x_0, y_0) of system (1.1) and a constant $C > 0$ independent of (x,y), (x_0,y_0) such that

$$\begin{aligned}
&\big\| x(t) - x_0(t) \big\| \le C\Phi(t), \quad \forall t \in J, \\
&\big\| {}^H D_{a^+}^{u,v;\psi} x(t) - {}^H D_{a^+}^{u,v;\psi} x_0(t) \big\| \le C\Phi(t), \quad \forall t \in J.
\end{aligned}$$

$$\left\|y(t) - y_0(t)\right\| \le C\Phi(t), \quad \forall t \in J,$$
$$\left\|{}^H D_{a^+}^{u,v;\psi} y(t) - {}^H D_{a^+}^{u,v;\psi} y_0(t)\right\| \le C\Phi(t), \quad \forall t \in J.$$

Then, one can say that the fractional integrodifferential equation (1.1) has Ulam–Hyers–Rassias stability.

Remark 2.1 (Ulam–Hyers stability, [26]) If we replace the nonnegative continuous function $\Phi(t)$ with $\theta \ge 0$ in Definition 2.3, then the fractional integrodifferential equation (1.1) has Ulam–Hyers stability.

Definition 2.4 (Semi-Ulam–Hyers–Rassias stability, [26]) If for all continuously differentiable functions $x : J \to X$ satisfying

$$\left\|x(t) - I_{a^+}^{\alpha;\psi}\left[f\left(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\right) + \int_a^t K\left(t, \tau, y(\tau), y\left(\delta(\tau)\right)\right) d\tau\right]\right\|$$
$$\le I_{a^+}^{\alpha;\psi}\theta, \quad \forall t \in J,$$
$$\left\|{}^H D_{a^+}^{u,v;\psi} x(t) - I_{a^+}^{\alpha-u;\psi}\left[f\left(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\right) + \int_a^t K\left(t, \tau, y(\tau), y\left(\delta(\tau)\right)\right) d\tau\right]\right\|$$
$$\le I_{a^+}^{\alpha-u;\psi}\theta, \quad \forall t \in J,$$
$$\left\|y(t) - I_{a^+}^{\alpha';\psi}\left[g\left(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\right) + \int_a^t K\left(t, \tau, x(\tau), x\left(\delta(\tau)\right)\right) d\tau\right]\right\|$$
$$\le I_{a^+}^{\alpha';\psi}\theta, \quad \forall t \in J,$$
$$\left\|{}^H D_{a^+}^{u,v;\psi} y(t) - I_{a^+}^{\alpha'-u;\psi}\left[g\left(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\right) + \int_a^t K\left(t, \tau, x(\tau), x\left(\delta(\tau)\right)\right) d\tau\right]\right\|$$
$$\le I_{a^+}^{\alpha'-u;\psi}\theta, \quad \forall t \in J,$$

where $\theta \ge 0$, there is a solution (x_0, y_0) of system (1.1) and a constant $C > 0$ independent of (x, y), (x_0, y_0), for some positive, nonincreasing continuous function $\Phi(t)$ defined on $[a, b]$, such that

$$\left\|x(t) - x_0(t)\right\| \le C\Phi(t), \quad \forall t \in J,$$
$$\left\|{}^H D_{a^+}^{u,v;\psi} x(t) - {}^H D_{a^+}^{u,v;\psi} x_0(t)\right\| \le C\Phi(t), \quad \forall t \in J.$$
$$\left\|y(t) - y_0(t)\right\| \le C\Phi(t), \quad \forall t \in J,$$
$$\left\|{}^H D_{a^+}^{u,v;\psi} y(t) - {}^H D_{a^+}^{u,v;\psi} y_0(t)\right\| \le C\Phi(t), \quad \forall t \in J,$$

then one can say that the fractional integrodifferential equation (1.1) has semi-Ulam–Hyers–Rassias stability.

Definition 2.5 ([34]) If $\mathbb{X}$ is a nonempty set, we say that $d : \mathbb{X} \times \mathbb{X} \to [0, \infty]$ is a generalized metric on $\mathbb{X}$, if

(1) $d(x, y) = 0$, if and only if $x = y$;
(2) $d(x, y) = d(y, x)$, for all $x, y \in \mathbb{X}$;
(3) $d(x, y) \le d(x, z) + d(z, y)$, for all $x, y, z \in \mathbb{X}$.

Theorem 2.1 ([34]) *Let $(\mathbb{X}, d)$ be a generalized complete metric space. Assume that $T : \mathbb{X} \to \mathbb{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in \mathbb{X}$, then the following three propositions hold true*:

(1) *The sequence $\{T^k x_0\}$ converges to a fixed point x^* of T for an initial point $x_0 \in \mathbb{X}$;*
(2) *x^* is the unique fixed point of T in $\mathbb{X}^* = \{y \in \mathbb{X} | d(T^k x, y) < \infty\}$;*
(3) *if $y \in \mathbb{X}^*$, then $d(y, x^*) \le \frac{1}{1-L} d(Ty, y)$.*

To prove the general existence and stability results, we impose special growth conditions on functions f and K extending the works given by [34, 35].

(H_1) Let $L_1(\cdot), L_2(\cdot), L_3(\cdot), L_4(\cdot), L_5(\cdot), L_6(\cdot), L_7(\cdot), L_8(\cdot)$ be nonnegative continuous functions. The continuously differentiable function $f, g : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ satisfies the following growth condition:

$$\begin{aligned}
&\|f(t,x,y) - f(t,x',y')\| \le L_1(t)\|x(t) - x'(t)\| + L_2(t)\|y(t) - y'(t)\|, \quad \forall t \in J, \\
&\|g(t,x,y) - g(t,x',y'\| \le L_3(t)\|x(t) - x'(t)\| + L_4(t)\|y(t) - y'(t)\|, \quad \forall t \in J.
\end{aligned}$$

$F, G : J \times J \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ are continuous functions satisfying the following growth condition:

$$\begin{aligned}
&\|F(t,s,x,q) - F(t,s,x',h)\| \\
&\quad \le L_5(s)\|x(s) - x'(s)\| + L_6(s)\|q(s) - h(s)\|, \quad \forall t,s \in J, \\
&\|G(t,s,x,q) - G(t,s,x',h)\| \\
&\quad \le L_7(s)\|x(s) - x'(s)\| + L_8(s)\|q(s) - h(s)\|, \quad \forall t,s \in J.
\end{aligned}$$

(H_2) Let positive constant $0 < L < 1$ and the nonnegative continuous functions $L_1(t) - L_8(t)$ satisfy the following conditions, respectively:

$$P(\eta) \max_{t \in [a,b]} \{L_{i_1}(t) + L_{i_2}(t) + b(L_{i_3} + L_{i_4})(t)\} \le \frac{L}{b-a}, \quad L \in (0,1), \tag{2.1}$$

where $i_1 = 1, 3$, $i_2 = 2, 4$, $i_3 = 5, 7$ and $i_4 = 6, 8$,

$$P(\eta) = \frac{(\psi(b) - \psi(a))^\eta}{\Gamma(\eta + 1)}, \quad \eta = \{\alpha, \alpha', \alpha - u, \alpha' - u\}.$$

For computational facilitation, we denote $P = \min\{P(\alpha), P(\alpha - u), P(\alpha'), P(\alpha' - u)\}$.

3 Main results

In this section, we will verify the existence and uniqueness of the solution for the system (1.1) via the Banach contraction mapping principle. Moreover, Ulam–Hyers–Rassias, Ulam–Hyers, and semi-Ulam–Hyers–Rassias stabilities are established on the infinite interval $[a, b]$.

3.1 Existence of a solution

Now, we present our first main result dealing with the uniqueness of solutions for the system (1.1), which relies on the Banach contraction mapping principle.

Theorem 3.1 *Assume that conditions* (H_1) *and* (H_2) *are satisfied, then the system* (1.1) *has a unique solution on* $[a,b]$.

Proof Similarly to the proof of Lemma 3.1 in [35], one defines function $T:\mathbb{X}\times\mathbb{X}\to\mathbb{X}\times\mathbb{X}$ as follows:

$$\begin{aligned}T(x,y)(t) &= \biggl(I_{a^+}^{\alpha;\psi} f\bigl(t,y(t),{}^H D_{a^+}^{u,v;\psi} y(t)\bigr) + I_{a^+}^{\alpha;\psi}\int_a^t F\bigl(t,\tau,y(\tau),y\bigl(\delta(\tau)\bigr)\bigr)\,d\tau,\\ &\quad I_{a^+}^{\alpha';\psi} g\bigl(t,x(t),{}^H D_{a^+}^{u,v;\psi} x(t)\bigr) + I_{a^+}^{\alpha';\psi}\int_a^t G\bigl(t,\tau,x(\tau),x\bigl(\delta(\tau)\bigr)\bigr)\,d\tau\biggr)\\ &\triangleq \bigl(T_1y(t),T_2x(t)\bigr),\quad \forall t\in J.\end{aligned}\tag{3.1}$$

We will prove that the functional T has a unique fixed point and the fixed point is the solution of the system (1.1) by using the Banach contraction mapping principle.

First, one can conclude that $T:\mathbb{X}\times\mathbb{X}\to\mathbb{X}\times\mathbb{X}$. In fact, it is obvious that $y(t)\in\mathbb{C}^1(J,\mathbb{Y})$ and ${}^H D_{a^+}^{u,v;\psi} y(t)\in C^1(J,\mathbb{Y})$ for any $y\in\mathbb{X}$. From the definitions of the ψ-Hilfer fractional derivative and the ψ-Riemann–Liouville fractional integral, one can obtain $T_1y(t)\in C^1(J,\mathbb{Y})$ and ${}^H D_{a^+}^{u,v;\psi} T_1y(t)\in C^1(J,\mathbb{Y})$. This implies that $T_1y(t)\in\mathbb{X}$. Similarly, it holds that $T_2x(t)\in\mathbb{X}$.

Next, we derive that the operator $T:\mathbb{X}\times\mathbb{X}\to\mathbb{X}\times\mathbb{X}$ is strictly contractive on $\mathbb{X}\times\mathbb{X}$ for any $x_1,x_2,y_1,y_2\in\mathbb{X}$ and for each $t\in[a,b]$. From the Definitions 2.1, 2.2, together with (3.1), one has

$$\begin{aligned}&\bigl\|T_1y_1(t)-T_1y_2(t)\bigr\|\\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1}\bigl\|f\bigl(\xi,y_1(\xi),{}^H D_{a^+}^{u,v;\psi} y_1(\xi)\bigr)\\ &\qquad -f\bigl(\xi,y_2(\xi),{}^H D_{a^+}^{u,v;\psi} y_2(\xi)\bigr)\bigr\|\,d\xi\\ &\qquad +\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1}\int_a^\xi \bigl\|\bigl[F\bigl(t,\tau,y_1(\tau),y_1\bigl(\delta(\tau)\bigr)\bigr)\\ &\qquad -F\bigl(t,\tau,y_2(\tau),y_2\bigl(\delta(\tau)\bigr)\bigr)\bigr]\,d\tau\bigr\|\,d\xi\\ &\quad\le \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\int_a^t \bigl[L_1(\xi)\bigl\|y_1(\xi)-y_2(\xi)\bigr\|\\ &\qquad +L_2(\xi)\bigl\|{}^H D_{a^+}^{u,v;\psi} y_1(\xi)-{}^H D_{a^+}^{u,v;\psi} y_2(\xi)\bigr\|\bigr]\,d\xi\\ &\qquad +\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\int_a^t\int_a^\xi \bigl[L_5(\tau)\bigl\|y_1(\tau)-y_2(\tau)\bigr\|\\ &\qquad +L_6(\tau)\bigl\|y_1\bigl(\delta(\tau)\bigr)-y_2\bigl(\delta(\tau)\bigr)\bigr\|\bigr]\,d\tau\,d\xi\\ &\quad\le \frac{(\psi(b)-\psi(a))^\alpha\|y_1-y_2\|_{\mathbb{X}}}{\Gamma(\alpha+1)}\int_a^t\biggl[\max_{a\le\xi\le b}(L_1+L_2)(\xi)+\int_a^b\max_{a\le s\le b}(L_5+L_6)(s)\,d\tau\biggr]\,d\xi\\ &\quad\le \frac{(\psi(b)-\psi(a))^\alpha\|y_1-y_2\|_{\mathbb{X}}}{\Gamma(\alpha+1)}(b-a)\max_{t\in[a,b]}\bigl\{L_1(t)+L_2(t)+b(L_5+L_6)(t)\bigr\}\\ &\quad\le L\|y_1-y_2\|_{\mathbb{X}},\quad L\in(0,1),\end{aligned}$$

the last inequality is obtained from the assumption (H_2).

On the other hand, combining Definitions 2.1, 2.2, and (2.1), one can obtain

$$\begin{aligned}
&\left\| {}^{H}D_{a^+}^{u,v;\psi} T_1 y_1(t) - {}^{H}D_{a^+}^{u,v;\psi} T_1 y_2(t) \right\| \\
&\quad \le \frac{(\psi(b)-\psi(a))^{\alpha-u}\|y_1-y_2\|_{\mathbb{X}}}{\Gamma(\alpha-u+1)}(b-a)\max_{t\in[a,b]}\bigl\{L_1(t)+L_2(t)+b(L_5+L_6)(t)\bigr\} \\
&\quad \le L\|y_1-y_2\|_{\mathbb{X}}, \quad L\in(0,1).
\end{aligned}$$

For any $x_1(t), x_2(t) \in \mathbb{X}$, by the definition of T, it is obvious that

$$\begin{aligned}
&\left\| T_2 x_1(t) - T_2 x_2(t) \right\| \\
&\quad \le \frac{1}{\Gamma(\alpha')}\int_a^t \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha'-1}\bigl\| g\bigl(\xi, x_1(\xi), {}^{H}D_{a^+}^{u,v;\psi} x_1(\xi)\bigr) \\
&\quad - g\bigl(\xi, x_2(\xi), {}^{H}D_{a^+}^{u,v;\psi} x_2(\xi)\bigr)\bigr\|\,d\xi \\
&\quad + \frac{1}{\Gamma(\alpha')}\int_a^t \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha'-1}\int_a^\xi \bigl\| G\bigl(t,\tau,x_1(\tau),x_1\bigl(\delta(\tau)\bigr)\bigr) \\
&\quad - G\bigl(t,\tau,x_2(\tau),x_2\bigl(\delta(\tau)\bigr)\bigr)\bigr\|\,d\tau\,d\xi \\
&\quad \le \frac{(\psi(b)-\psi(a))^{\alpha'}}{\Gamma(\alpha'+1)}\int_a^t \bigl[L_3(\xi)\bigl\|x_1(\xi)-x_2(\xi)\bigr\| \\
&\quad + L_4(\xi)\bigl\|{}^{H}D_{a^+}^{u,v;\psi} x_1(\xi) - {}^{H}D_{a^+}^{u,v;\psi} x_2(\xi)\bigr\|\bigr]\,d\xi \\
&\quad + \frac{(\psi(b)-\psi(a))^{\alpha'}}{\Gamma(\alpha'+1)}\int_a^t\int_a^\xi \bigl[L_7(\tau)\bigl\|x_1(\tau)-x_2(\tau)\bigr\| \\
&\quad + L_8(\tau)\bigl\|x_1\bigl(\delta(\tau)\bigr)-x_2\bigl(\delta(\tau)\bigr)\bigr\|\bigr]\,d\tau\,d\xi \\
&\quad \le \frac{(\psi(b)-\psi(a))^{\alpha'}\|x_1-x_2\|_{\mathbb{X}}}{\Gamma(\alpha'+1)}(b-a)\max_{t\in[a,b]}\bigl\{L_3(t)+L_4(t)+b(L_7+L_8)(t)\bigr\} \\
&\quad \le L\|x_1-x_2\|_{\mathbb{X}}, \quad L\in(0,1).
\end{aligned}$$

Similarly, it is not difficult to obtain the following embedding inequality

$$\begin{aligned}
&\left\| {}^{H}D_{a^+}^{u,v;\psi} T_2 x_1(t) - {}^{H}D_{a^+}^{u,v;\psi} T_2 x_2(t) \right\| \\
&\quad \le \frac{(\psi(b)-\psi(a))^{\alpha'-u}\|x_1-x_2\|_{\mathbb{X}}}{\Gamma(\alpha'-u+1)}(b-a)\max_{t\in[a,b]}\bigl\{L_3(t)+L_4(t)+b(L_7+L_8)(t)\bigr\} \\
&\quad \le L\|x_1-x_2\|_{\mathbb{X}}, \quad L\in(0,1).
\end{aligned}$$

Hence, for all $(x_1,y_1),(x_2,y_2) \in \mathbb{X}\times\mathbb{X}$, we conclude that

$$\bigl\| T(x_1,y_1) - T(x_2,y_2) \bigr\|_{\mathbb{X}\times\mathbb{X}} \le L\bigl\|(x_1,y_1)-(x_2,y_2)\bigr\|_{\mathbb{X}\times\mathbb{X}}, \quad L\in(0,1),$$

which implies T is a contraction mapping. By Banach's contraction mapping principle, the operator $T: \mathbb{X}\times\mathbb{X} \to \mathbb{X}\times\mathbb{X}$ has a fixed point $(x^*,y^*) \in \mathbb{X}\times\mathbb{X}$ that satisfies $T(x^*,y^*) = (x^*,y^*)$. In consequence, the problem (1.1) has a unique solution on $[a,b]$. The proof is completed. □

3.2 Stability

In this subsection, we will deal with Ulam–Hyers–Rassias stability, Ulam–Hyers stability, and semi-Ulam–Hyers–Rassias stability on the infinite interval $[a,b]$.

In order to achieve stability results, we list appropriate metrics $d_1(\cdot)$ and $d_2(\cdot)$ in Banach space $\mathbb{X}\times\mathbb{X}$. For any $(x_1,y_1),(x_2,y_2)\in\mathbb{X}\times\mathbb{X}$, we denote the distance $d_i((x_1,y_1),(x_2,y_2)),(i=1,2)$ as follows, respectively:

$$\begin{aligned} d_1\big((x_1,y_1),(x_2,y_2)\big) = \inf\big\{&M\in[0,\infty)\mid \big\|x_1(t)-x_2(t)\big\|\le M\Phi(t),\\ &\big\|{}^HD_{a^+}^{u,v;\psi}x_1(t)-{}^HD_{a^+}^{u,v;\psi}x_2(t)\big\|\le M\Phi(t), \big\|y_1(t)-y_2(t)\big\|\le M\Phi(t),\\ &\big\|{}^HD_{a^+}^{u,v;\psi}y_1(t)-{}^HD_{a^+}^{u,v;\psi}y_2(t)\big\|\le M\Phi(t), t\in[a,b]\big\}, \end{aligned}$$

where M is a positive constant and $\Phi(t)$ is a positive, nondecreasing continuous function,

$$\begin{aligned} d_2\big((x_1,y_1),(x_2,y_2)\big) = \sup\bigg\{&M\in[0,\infty)\;\Big|\;\frac{\|x_1(t)-x_2(t)\|}{\Psi(t)}\le M,\\ &\frac{\|{}^HD_{a^+}^{u,v;\psi}x_1(t)-{}^HD_{a^+}^{u,v;\psi}x_2(t)\|}{\Psi(t)}\le M, \frac{\|y_1(t)-y_2(t)\|}{\Psi(t)}\le M,\\ &\frac{\|{}^HD_{a^+}^{u,v;\psi}y_1(t)-{}^HD_{a^+}^{u,v;\psi}y_2(t)\|}{\Psi(t)}\le M, t\in[a,b]\bigg\}, \end{aligned}$$

where $\Psi(t)$ is a positive, nonincreasing continuous function on the finite interval $[a,b]$. In a similar manner to [36] and references therein, we can ensure $d_1(\cdot)$ and $d_2(\cdot)$ are metrics in Banach space $\mathbb{X}\times\mathbb{X}$.

Theorem 3.2 *Let (H_1) and (H_2) hold, $\Phi:[a,b]\to(\omega,\varpi)$ $(\omega,\varpi>0)$ is a positive, nondecreasing continuous function. In addition, $x,y:J\to\mathbb{X}$ are continuously differentiable functions satisfying*

$$\bigg\|x(t)-I_{a^+}^{\alpha;\psi}\bigg[f\big(t,y(t),{}^HD_{a^+}^{u,v;\psi}y(t)\big)+\int_a^t F\big(t,\tau,y(\tau),y\big(\delta(\tau)\big)\big)\,d\tau\bigg]\bigg\|\le I_{a^+}^{\alpha;\psi}\Phi(t), \tag{3.2}$$

$$\begin{aligned} &\bigg\|{}^HD_{a^+}^{u,v;\psi}x(t)-I_{a^+}^{\alpha-u;\psi}\bigg[f\big(t,y(t),{}^HD_{a^+}^{u,v;\psi}y(t)\big)+\int_a^t F\big(t,\tau,y(\tau),y\big(\delta(\tau)\big)\big)\,d\tau\bigg]\bigg\|\\ &\quad\le I_{a^+}^{\alpha-u;\psi}\Phi(t), \end{aligned}$$

$$\bigg\|y(t)-I_{a^+}^{\alpha';\psi}\bigg[g\big(t,x(t),{}^HD_{a^+}^{u,v;\psi}x(t)\big)+\int_a^t G\big(t,\tau,x(\tau),x\big(\delta(\tau)\big)\big)\,d\tau\bigg]\bigg\|\le I_{a^+}^{\alpha';\psi}\Phi(t) \tag{3.3}$$

and

$$\begin{aligned} &\bigg\|{}^HD_{a^+}^{u,v;\psi}y(t)-I_{a^+}^{\alpha'-u;\psi}\bigg[g\big(t,x(t),{}^HD_{a^+}^{u,v;\psi}x(t)\big)+\int_a^t G\big(t,\tau,x(\tau),x\big(\delta(\tau)\big)\big)\,d\tau\bigg]\bigg\|\\ &\quad\le I_{a^+}^{\alpha'-u;\psi}\Phi(t) \end{aligned}$$

for all $t\in J$, then there exists a unique solution $(x_0,y_0)\in\mathbb{X}$ such that inequalities

$$\big\|x(t)-x_0(t)\big\|\le\frac{P}{1-L}\Phi(t),$$

$$\left\|{}^{H}D_{a^+}^{u,\nu;\psi}x(t)-{}^{H}D_{a^+}^{u,\nu;\psi}x_0(t)\right\|\le\frac{P}{1-L}\Phi(t),$$
$$\left\|y(t)-y_0(t)\right\|\le\frac{P}{1-L}\Phi(t)$$

and

$$\left\|{}^{H}D_{a^+}^{u,\nu;\psi}y(t)-{}^{H}D_{a^+}^{u,\nu;\psi}y_0(t)\right\|\le\frac{P}{1-L}\Phi(t)$$

hold, where $L\in(0,1)$. *These inequalities imply that the system* (1.1) *has Ulam–Hyers–Rassias stability.*

Proof For any $t\in J$, we introduce an operator $T:\mathbb{X}\times\mathbb{X}\to\mathbb{X}\times\mathbb{X}$ as

$$Tx(t)=I_{a^+}^{\alpha;\psi}f\bigl(t,x(t),{}^{H}D_{a^+}^{u,\nu;\psi}x(t)\bigr)+I_{a^+}^{\alpha;\psi}\int_a^t K\bigl(t,\tau,x(\tau),x\bigl(\delta(\tau)\bigr)\bigr)\,d\tau.$$

For all $(x_1,y_1),(x_2,y_2)\in\mathbb{X}\times\mathbb{X}$ and $t\in[a,b]$, it is easy to derive from conditions (H_1), (H_2), and metric $d_1(\cdot)$ that

$$\begin{aligned}
&\left\|T_1y_1(t)-T_1y_2(t)\right\|\\
&\quad\le\frac{M}{\Gamma(\alpha)}\int_a^t\psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1}\bigl[L_1(\xi)\Phi(\xi)+L_2(\xi)\Phi(\xi)\bigr]\,d\xi\\
&\qquad+\frac{M}{\Gamma(\alpha)}\int_a^t\psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1}\int_a^\xi\bigl[L_5(\tau)\Phi(\tau)+L_6(\tau)\Phi\bigl(\delta(\tau)\bigr)\bigr]\,d\tau\,d\xi\\
&\quad\le\frac{M\Phi(t)}{\Gamma(\alpha)}\int_a^t\psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1}\left[(L_1+L_2)(\xi)+\int_a^\xi(L_5+L_6)(\tau)\,d\tau\right]d\xi\\
&\quad\le LM\Phi(t).
\end{aligned}\tag{3.4}$$

In a similar manner, for each $t\in J$, one can find that

$$\begin{aligned}
&\left\|{}^{H}D_{a^+}^{u,\nu;\psi}T_1y_1(t)-{}^{H}D_{a^+}^{u,\nu;\psi}T_1y_2(t)\right\|\\
&\quad\le\frac{M\Phi(t)}{\Gamma(\alpha-u)}\int_a^t\psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-u-1}\\
&\qquad\times\left[(L_1+L_2)(\xi)+\int_a^\xi(L_5+L_6)(\tau)\,d\tau\right]d\xi\\
&\quad\le LM\Phi(t).
\end{aligned}\tag{3.5}$$

Similarly, one can obtain

$$\begin{aligned}
&\left\|T_2x_1(t)-T_2x_2(t)\right\|\\
&\quad\le\frac{M\Phi(t)}{\Gamma(\alpha')}\int_a^t\psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha'-1}\left[(L_3+L_4)(\xi)+\int_a^\xi(L_7+L_8)(\tau)\,d\tau\right]d\xi\\
&\quad\le LM\Phi(t),\quad\forall t\in J.
\end{aligned}\tag{3.6}$$

It follows from the definition of ${}^{H}D_{a^+}^{u,\nu;\psi}$ and the conditions of this theorem that

$$\begin{aligned}
&\left\|{}^{H}D_{a^+}^{u,\nu;\psi}T_2x_1(t)-{}^{H}D_{a^+}^{u,\nu;\psi}T_2x_2(t)\right\|\\
&\quad\le\frac{M\Phi(t)}{\Gamma(\alpha'-u)}\int_a^t\psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha'-u-1}\\
&\qquad\times\left[(L_3+L_4)(\xi)+\int_a^{\xi}(L_7+L_8)(\tau)\,d\tau\right]d\xi\\
&\quad\le LM\Phi(t),\quad\forall t\in J.
\end{aligned}\tag{3.7}$$

From the definition of $d_1(\cdot)$ and combining (3.4)–(3.7), we have

$$d_1\bigl(T(x_1,y_1),T(x_2,y_2)\bigr)\le ML=Ld_1\bigl((x_1,y_1),(x_2,y_2)\bigr),\quad L\in(0,1).$$

Next, we show that $d_1(x,Tx)<\infty$, so operator T has a fixed point. In fact, from (3.1), (3.2), and (3.3), we obtain

$$\bigl\|x(t)-T_1y(t)\bigr\|\le I_{a^+}^{\alpha;\psi}\Phi(t)\le P(\alpha)\Phi(t),\tag{3.8}$$

$$\left\|{}^{H}D_{a^+}^{u,\nu;\psi}x(t)-{}^{H}D_{a^+}^{u,\nu;\psi}T_1y(t)\right\|\le I_{a^+}^{\alpha-u;\psi}\Phi(t)\le P(\alpha-u)\Phi(t),\tag{3.9}$$

$$\bigl\|y(t)-T_2x(t)\bigr\|\le I_{a^+}^{\alpha';\psi}\Phi(t)\le P\bigl(\alpha'\bigr)\Phi(t),\tag{3.10}$$

$$\left\|{}^{H}D_{a^+}^{u,\nu;\psi}y(t)-{}^{H}D_{a^+}^{u,\nu;\psi}T_2x(t)\right\|\le I_{a^+}^{\alpha'-u;\phi}\psi(t)\le P\bigl(\alpha'-u\bigr)\Phi(t)\tag{3.11}$$

for all $t\in J$. Based on the above results, we have

$$d_1\bigl((x,y),T(x,y)\bigr)\le P<\infty,$$

where $P=\min\{P(\alpha),P(\alpha-u),P(\alpha'),P(\alpha'-u)\}$. Hence, it follows by items (1) and (2) of Theorem 2.1 that there exists a unique fixed point (x_0,y_0) such that $T(x_0,y_0)=(x_0,y_0)$. According to item (3) of Theorem 2.1, we can obtain

$$d_1\bigl((x,y),(x_0,y_0)\bigr)\le\frac{1}{1-L}d_1\bigl((x,y),T(x,y)\bigr)\le\frac{P}{1-L},\quad L\in(0,1).$$

Based on the above facts, we conclude the system (1.1) has Ulam–Hyers–Rassias stability and the proof is completed. □

Remark 3.1 (Ulam–Hyers stability) Assume a positive, nondecreasing continuous function $\Phi(t)=1$ in Theorem 3.2, then the fractional integrodifferential equation (1.1) has Ulam–Hyers stability.

In the following theorem, we will show that the solution of system (1.1) has semi-Ulam–Hyers–Rassias stability.

Theorem 3.3 *Assume that* (H_1) *and* (H_2) *hold and* $\Phi(t)$ *is a positive, nonincreasing continuous function. Moreover, for* $t\in J$, *continuously differentiable functions* $x,y:J\to\mathbb{X}$ *satisfy*

$$\left\|x(t)-I_{a^+}^{\alpha;\psi}\left[f\bigl(t,y(t),{}^{H}D_{a^+}^{u,\nu;\psi}y(t)\bigr)+\int_a^tF\bigl(t,\tau,y(\tau),y\bigl(\delta(\tau)\bigr)\bigr)\,d\tau\right]\right\|\le I_{a^+}^{\alpha;\psi}\theta,\tag{3.12}$$

$$\begin{aligned}&\left\|{}^{H}D_{a^+}^{u,v;\psi}x(t)-I_{a^+}^{\alpha-u;\psi}\left[f\big(t,y(t),{}^{H}D_{a^+}^{u,v;\psi}y(t)\big)+\int_a^t F\big(t,\tau,y(\tau),y\big(\delta(\tau)\big)\big)\,d\tau\right]\right\|\\&\quad\le I_{a^+}^{\alpha-u;\psi}\theta,\end{aligned}\tag{3.13}$$

$$\left\|y(t)-I_{a^+}^{\alpha';\psi}\left[g\big(t,x(t),{}^{H}D_{a^+}^{u,v;\psi}x(t)\big)+\int_a^t G\big(t,\tau,x(\tau),x\big(\delta(\tau)\big)\big)\,d\tau\right]\right\|\le I_{a^+}^{\alpha';\psi}\theta,\tag{3.14}$$

$$\begin{aligned}&\left\|{}^{H}D_{a^+}^{u,v;\psi}y(t)-I_{a^+}^{\alpha'-u;\psi}\left[g\big(t,x(t),{}^{H}D_{a^+}^{u,v;\psi}x(t)\big)+\int_a^t G\big(t,\tau,x(\tau),x\big(\delta(\tau)\big)\big)\,d\tau\right]\right\|\\&\quad\le I_{a^+}^{\alpha'-u;\psi}\theta.\end{aligned}\tag{3.15}$$

Then, for any $t\in J$, there exists a unique solution $(x_0,y_0)\in\mathbb{X}$ such that

$$\begin{aligned}&\big\|x(t)-x_0(t)\big\|\le\frac{\theta PM}{1-L}\Phi(t),\\&\big\|{}^{H}D_{a^+}^{u,v;\psi}x(t)-{}^{H}D_{a^+}^{u,v;\psi}x_0(t)\big\|\le\frac{\theta PM}{1-L}\Phi(t),\\&\big\|y(t)-y_0(t)\big\|\le\frac{\theta PM}{1-L}\Phi(t),\\&\big\|{}^{H}D_{a^+}^{u,v;\psi}y(t)-{}^{H}D_{a^+}^{u,v;\psi}y_0(t)\big\|\le\frac{\theta PM}{1-L}\Phi(t).\end{aligned}$$

Proof Similar to the proof of Theorem 3.2, for all $(x_1,y_1),(x_2,y_2)\in\mathbb{X}\times\mathbb{X}$ and $t\in[a,b]$, one can derive from conditions (H_1), (H_2), and metric $d_2(\cdot)$ that

$$d_2\big(T(x_1,y_1),T(x_2,y_2)\big)\le Ld_2\big((x_1,y_1),(x_2,y_2)\big),\quad L\in(0,1).$$

From the definition of $d_2(\cdot)$ and combining (3.12)–(3.15), we have

$$d_2\big((x,y),T(x,y)\big)\le\theta PM<\infty.$$

It follows from items (1) and (2) of Theorem 2.1 that there exists a unique fixed point (x_0,y_0) such that $T(x_0,y_0)=(x_0,y_0)$. According to item (3) of Theorem 2.1, one can obtain

$$d_2\big((x,y),(x_0,y_0)\big)\le\frac{1}{1-L}d_2\big((x,y),T(x,y)\big)\le\frac{\theta PM}{1-L},\quad L\in(0,1),$$

so its conclusion implies that the solution of system (1.1) has semi-Ulam–Hyers–Rassias stability. This finishes the proof. □

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