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Triple-adaptive subgradient extragradient with extrapolation procedure for bilevel split variational inequality

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Abstract

This paper introduces a triple-adaptive subgradient extragradient process with extrapolation to solve a *bilevel split pseudomonotone variational inequality problem* (BSPVIP) with the common fixed point problem constraint of finitely many nonexpansive mappings. The problem under consideration is in real Hilbert spaces, where the BSPVIP involves a fixed point problem of demimetric mapping. The proposed rule exploits the strong monotonicity of one operator at the upper level and the pseudomonotonicity of another mapping at the lower level. The strong convergence result for the proposed algorithm is established under some suitable assumptions. In addition, a numerical example is given to demonstrate the viability of the proposed rule. Our results improve and extend some recent developments to a great extent.

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Keywords: Subgradient extragradient process; Bilevel split pseudomonotone variational inequality problem; Extrapolation step; Demimetric mapping; Fixed point; Nonexpansive mapping

1 Introduction

Suppose that $\emptyset \neq C \subset \mathcal{H}$ with C being a closed convex set in a real Hilbert space $\mathcal{H}$, and $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and the induced norm in $\mathcal{H}$, respectively. Let P_C be the metric projection of $\mathcal{H}$ onto C, and for a given mapping $S : C \to \mathcal{H}$, let its set of fixed points be denoted by $\mathrm{Fix}(S)$.

Let $A : \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous mapping with Lipschitz constant L, and consider the classical variational inequality problem (VIP) of finding $x^* \in C$ such that $\langle Ax^*, x - x^*\rangle \geq 0\ \forall x \in C$. We denote the solution set of the VIP by $\mathrm{VI}(C, A)$. One of the most popular approaches for settling the VIP is the extragradient method invented by Korpelevich [1] in 1976. For any given initial point $p_0 \in C$, the method of Korpelevich [1] generates a sequence $\{p_t\}$ as fabricated below:

$$\begin{cases} q_t = P_C(p_t - \ell A p_t), \\ p_{t+1} = P_C(p_t - \ell A q_t), \quad t = 0, 1, 2, \dots, \end{cases}$$

where the constant ℓ lies in $(0, \frac{1}{L})$. The literature on the VIP is numerous, and Korpelevich's extragradient method has received extensive attention of many scholars, who intensely enhanced it in various aspects; for example, please see [2–26] and the references therein, to name but a few.

Thong and Hieu [26] put forward subgradient extragradient process with extrapolation, which generates a sequence $\{p_t\}$ for any given $p_1, p_0 \in \mathcal{H}$ as follows:

$$\begin{cases} w_t = p_t + \alpha_t(p_t - p_{t-1}), \\ y_t = P_C(w_t - \zeta A w_t), \\ C_t = \{p \in \mathcal{H} : \langle w_t - \zeta A w_t - y_t, y_t - p\rangle \geq 0\}, \\ p_{t+1} = P_{C_t}(w_t - \ell A y_t), \quad t = 1, 2, 3, \ldots, \end{cases}$$

where $\zeta \in (0, \frac{1}{L})$ and weak convergence is obtained. Given nonexpansive mappings $S_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, \ldots, N$, Ceng and Shang [16] presented a subgradient extragradient-type process for computing a common element of the common fixed point set and $\mathrm{VI}(C, A)$ when

$$\Omega := \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A) \neq \emptyset.$$

Furthermore, the following strongly convergent algorithm was studied in [21] when $\Omega := \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A)$ is nonempty.

Algorithm 1.1 (See [21, Algorithm 3.1]) *Modified inertial subgradient extragradient method.*

Initialization

Let $\lambda_1 > 0$, $\alpha > 0$, $\mu \in (0, 1)$, and $x_1, x_0 \in \mathcal{H}$ be arbitrary.

Iterative steps

Calculate x_{t+1} as follows:

Step 1. Given the iterates x_t and x_{t-1} ($t \geq 1$), choose α_t such that $0 \leq \alpha_t \leq \bar{\alpha}_t$, where

$$\bar{\alpha}_t = \begin{cases} \min\{\alpha, \frac{\epsilon_t}{\|x_t - x_{t-1}\|}\} & \text{if } x_t \neq x_{t-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

Step 2. Compute $w_t = S_t x_t + \alpha_t(S_t x_t - S_t x_{t-1})$ and $y_t = P_C(w_t - \lambda_t A w_t)$.

Step 3. Identify $C_t = \{y \in \mathcal{H} : \langle w_t - \lambda_t A w_t - y_t, y_t - y\rangle \geq 0\}$, then calculate

$$z_t = P_{C_t}(w_t - \lambda_t A y_t).$$

Step 4. Update $x_{t+1} = \beta_t f(x_t) + \gamma_t x_t + ((1 - \gamma_t)I - \beta_t \rho F) z_t$, where $\rho \in (0, \frac{2\eta}{\kappa^2})$ and update

$$\lambda_{t+1} = \begin{cases} \min\{\mu \frac{\|w_t - y_t\|^2 + \|z_t - y_t\|^2}{2\langle A w_t - A y_t, z_t - y_t\rangle}, \lambda_t\} & \text{if } \langle A w_t - A y_t, z_t - y_t\rangle > 0, \\ \lambda_t & \text{otherwise.} \end{cases}$$

Set $t := t + 1$ and return to Step 1, where f is a contraction ($f : \mathcal{H} \to \mathcal{H}$ is a contraction if there exists $\nu \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \nu\|x - y\|$, $\forall x, y \in \mathcal{H}$), F is η-strongly monotone and κ-Lipschitz continuous (kindly see Sect. 2 for its definition) with $\{\beta_t\}, \{\gamma_t\}, \{\varepsilon_t\} \subset (0,1)$ fulfilling some conditions.

Next, suppose that C and Q are nonempty, closed, and convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ denote a bounded linear operator and $A, F : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings. Then, the *bilevel split variational inequality problem* (BSVIP) (see [27]) is as specified below:

$$\text{Seek } q^* \in \Omega \text{ such that } \quad \langle Fq^*, z - q^* \rangle \geq 0 \quad \forall z \in \Lambda, \tag{1.1}$$

where $\Lambda := \{z \in \mathrm{VI}(C,A) : Tz \in \mathrm{VI}(Q,B)\}$ is the solution set of the *split variational inequality problem* (SVIP), which was introduced by Censor et al. [28] and formulated as follows:

$$\text{Find } x^* \in C \text{ such that } \quad \langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in C \tag{1.2}$$

and

$$y^* = Tx^* \in Q \text{ such that } \quad \langle By^*, y - y^* \rangle \geq 0 \quad \forall y \in Q \tag{1.3}$$

with $\mathrm{VI}(C,A)$ and $\mathrm{VI}(Q,B)$ representing the solution sets of variational inequalities (1.2) and (1.3), respectively. Note that the SVIP involves finding $x^* \in \mathrm{VI}(C,A)$ such that $Tx^* \in \mathrm{VI}(Q,B)$. Censor et al. [28] proposed a weakly convergent method for approximating the solution of (1.2)–(1.3): for any given initial $x_1 \in \mathcal{H}_1$, identify the sequence $\{x_t\}$ generated by

$$x_{t+1} = P_C(I - \lambda A)\big(x_t + \gamma T^*\big(P_Q(I - \lambda B) - I\big)Tx_t\big), \quad t = 1, 2, 3, \ldots, \tag{1.4}$$

where A and B both are inverse-strongly monotone and T is a bounded linear operator. Under appropriate assumptions, it was proven in [28] that the sequence $\{x_t\}$ converges weakly to a solution of (1.2)–(1.3).

We note that the VIP can be expressed as the FPP: $Sz = P_Q(z - \mu Bz)$, $\mu > 0$, with $\mathrm{VI}(Q,B) = \mathrm{Fix}(S)$. Consequently, we can reformulate the BSVIP in (1.1) as follows: Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ be quasimonotone and L-Lipschitz continuous, $F : \mathcal{H}_1 \to \mathcal{H}_1$ be κ-Lipschitzian and η-strongly monotone, $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a nonzero bounded linear operator, and $S : \mathcal{H}_2 \to \mathcal{H}_2$ be a τ-demimetric mapping with $\tau \in (-\infty, 1)$; then,

$$\text{Find } q^* \in \Omega \text{ such that } \quad \langle Fq^*, z - q^* \rangle \geq 0 \quad \forall z \in \Omega, \tag{1.5}$$

where $\Omega := \{z \in \mathrm{VI}(C,A) : Tz \in \mathrm{Fix}(S)\}$. In this case, such a problem is referred to as a *bilevel split quasimonotone variational inequality problem* (BSQVIP) and its strong convergence results are obtained in [18].

Assume that $f : \mathcal{H}_1 \to \mathcal{H}_1$ is a contractive mapping with $\nu \in [0,1)$ with $\nu < \zeta := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$, $A : \mathcal{H}_1 \to \mathcal{H}_1$ is pseudomonotone and L-Lipschitz continuous with $\|Au\| \leq \liminf_{t\to\infty} \|Au_t\|$ for each $\{u_t\} \subset C$ with $u_t \rightharpoonup u$, $\{S_i\}_{i=1}^N$ is finitely many nonexpansive mappings on $\mathcal{H}_1$ and $\Xi := \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$. Then, the bilevel split pseudomonotone variational inequality problem (BSPVIP) with the common fixed point problem (CFPP) constraint is formulated as follows:

$$\text{Seek } q^* \in \Xi \text{ such that } \quad \langle (\rho F - f)q^*, p - q^* \rangle \geq 0 \quad \forall p \in \Xi. \tag{1.6}$$

We propose triple-adaptive subgradient extragradient-type rule with inertial extrapolation to solve (1.6) in real Hilbert spaces, where the BSPVIP involves the FPP of demimetric mapping S. The rule exploits the strong monotonicity of the operator F at the upper-level problem and the pseudomonotonicity of the mapping A at the lower level. Consequently, we obtain strong convergence result. In addition, a numerical test is provided to show the viability of the suggested rule.

The article is organized as follows: In Sect. 2, we provide some concepts and basic tools for further use. Section 3 gives the convergence analysis of the suggested algorithm. Lastly, Sect. 4 gives a numerical illustration. Our results improve and extend the corresponding ones in [21, 29], and the relevant explanatory argument is given after the main proof of convergence result in Sect. 3.

2 Preliminaries

A mapping $S : C \to \mathcal{H}$ is (see [30]):

(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ such that $\|S\tilde{u} - S\bar{y}\| \leq L\|\tilde{u} - \bar{y}\|$ $\forall \tilde{u}, \bar{y} \in C$. If $L = 1$, then S is nonexpansive;

(ii) ς-strongly monotone if $\exists \varsigma > 0$ such that $\langle S\tilde{u} - S\bar{y}, \tilde{u} - \bar{y}\rangle \geq \varsigma\|\tilde{u} - \bar{y}\|^2$ $\forall \tilde{u}, \bar{y} \in C$;

(iii) monotone if $\langle S\tilde{u} - S\bar{y}, \tilde{u} - \bar{y}\rangle \geq 0$ $\forall \tilde{u}, \bar{y} \in C$;

(iv) pseudomonotone if $\langle S\tilde{u}, \bar{y} - \tilde{u}\rangle \geq 0 \Longrightarrow \langle S\bar{y}, \bar{y} - \tilde{u}\rangle \geq 0$ $\forall \tilde{u}, \bar{y} \in C$;

(v) quasimonotone if $\langle S\tilde{u}, \bar{y} - \tilde{u}\rangle > 0 \Longrightarrow \langle S\bar{y}, \bar{y} - \tilde{u}\rangle \geq 0$ $\forall \tilde{u}, \bar{y} \in C$;

(vi) τ-demicontractive if $\exists \tau \in (0, 1)$ such that

$$\|S\tilde{u} - p\|^2 \leq \|\tilde{u} - p\|^2 + \tau\|\tilde{u} - S\tilde{u}\|^2 \quad \forall \tilde{u} \in C, p \in \mathrm{Fix}(S) \neq \emptyset;$$

(vii) τ-demimetric if $\exists \tau \in (-\infty, 1)$ such that

$$\langle \tilde{u} - S\tilde{u}, \tilde{u} - p\rangle \geq \frac{1 - \tau}{2}\|\tilde{u} - S\tilde{u}\|^2 \quad \forall \tilde{u} \in C, p \in \mathrm{Fix}(S) \neq \emptyset;$$

(viii) sequentially weakly continuous if $\forall \{x_t\} \subset C$, $x_t \rightharpoonup x \Longrightarrow Sx_t \rightharpoonup Sx$.

Given $\grave{u} \in \mathcal{H}$, there exists unique $P_C\grave{u} \in C$ with the following properties.

Lemma 2.1 (See [31]) *The following hold*:

(i) $\langle \check{u} - \grave{v}, P_C\check{u} - P_C\grave{v}\rangle \geq \|P_C\check{u} - P_C\grave{v}\|^2$ $\forall \check{u}, \grave{v} \in \mathcal{H}$;

(ii) $w = P_C\check{u} \Longleftrightarrow \langle \check{u} - w, \grave{v} - w\rangle \leq 0$ $\forall \check{u} \in \mathcal{H}, \grave{v} \in C$;

(iii) $\|\check{u} - \grave{v}\|^2 \geq \|\check{u} - P_C\check{u}\|^2 + \|\grave{v} - P_C\check{u}\|^2$ $\forall \check{u} \in \mathcal{H}, \grave{v} \in C$;

(iv) $\|\check{u} - \grave{v}\|^2 = \|\check{u}\|^2 - \|\grave{v}\|^2 - 2\langle \check{u} - \grave{v}, \grave{v}\rangle$ $\forall \check{u}, \grave{v} \in \mathcal{H}$;

(v) $\|\vartheta\check{u} + (1 - \vartheta)\grave{v}\|^2 = \vartheta\|\check{u}\|^2 + (1 - \vartheta)\|\grave{v}\|^2 - \vartheta(1 - \vartheta)\|\check{u} - \grave{v}\|^2$ $\forall \check{u}, \grave{v} \in \mathcal{H}, \vartheta \in \mathbb{R}$.

Clearly, (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v). However, the converse is not generally true.

Lemma 2.2 (See [32]) *Let* $\varpi \in (0, 1]$, $S : C \to \mathcal{H}$ *be nonexpansive and* $S^{\varpi} : C \to \mathcal{H}$ *be defined by* $S^{\varpi}\acute{x} := S\acute{x} - \varpi\rho F(S\acute{x})$ $\forall \acute{x} \in C$, *where* F *is* ϱ-*Lipschitz continuous and* ς-*strongly* *monotone. Then* S^{ϖ} *is a contraction provided* $0 < \rho < \frac{2\varsigma}{\varrho^2}$, *i.e.*, $\|S^{\varpi}\acute{x} - S^{\varpi}\acute{y}\| \le (1 - \varpi\zeta)\|\acute{x} - \acute{y}\|$ $\forall \acute{x}, \acute{y} \in C$, *where* $\zeta = 1 - \sqrt{1 - \rho(2\varsigma - \rho\varrho^2)} \in (0, 1]$.

Lemma 2.3 *If* $A : C \to \mathcal{H}$ *is pseudomonotone and continuous, then* $u^* \in C$ *solves VIP* $\Leftrightarrow$ $\langle Av, v - u^*\rangle \ge 0$ $\forall v \in C$.

Proof The proof is straightforward and thus we skip it. □

Lemma 2.4 (See [32]) *Let* $\{a_t\} \subset (0, \infty)$ *satisfying the condition* $a_{t+1} \le (1 - \lambda_t)a_t + \lambda_t\gamma_t$ $\forall t \ge 1$, *where* $\{\lambda_t\}, \{\gamma_t\} \subset \mathbb{R}$ *and* (i) $\{\lambda_t\} \subset [0, 1]$ *and* $\sum_{t=1}^{\infty} \lambda_t = \infty$, *and* (ii) $\limsup_{t\to\infty} \gamma_t \le 0$ *or* $\sum_{t=1}^{\infty} |\lambda_t\gamma_t| < \infty$. *Then* $\lim_{t\to\infty} a_t = 0$.

Lemma 2.5 (See [31, demiclosedness principle]) *If S is nonexpansive with* $\mathrm{Fix}(S) \ne \emptyset$, *then* $I - S$ *is demiclosed at zero, i.e., if* $\{x_t\}$ *is a sequence in C such that* $x_t \rightharpoonup x \in C$ *and* $(I - S)x_t \to 0$, *then* $(I - S)x = 0$, *where I is the identity mapping of* $\mathcal{H}$.

Lemma 2.6 (See [6]) *Let* $\{\mathbf{\Gamma}_s\} \subset \mathbb{R}$ *with* $\exists\{\mathbf{\Gamma}_{s_k}\} \subset \{\mathbf{\Gamma}_s\}$ *such that* $\mathbf{\Gamma}_{s_k} < \mathbf{\Gamma}_{s_k+1}$ $\forall k \ge 1$. *Let* $\{\phi(s)\}_{s \ge s_0}$ *be formulated as*

$$\phi(s) = \max\{k \le s : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\}$$

with $s_0 \ge 1$ *satisfying* $\{k \le s_0 : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\} \ne \emptyset$. *Then*:
(i) $\phi(s_0) \le \phi(s_0 + 1) \le \cdots$ *and* $\phi(s) \to \infty$;
(ii) $\mathbf{\Gamma}_{\phi(s)} \le \mathbf{\Gamma}_{\phi(s)+1}$ *and* $\mathbf{\Gamma}_s \le \mathbf{\Gamma}_{\phi(s)+1}$ $\forall s \ge s_0$.

3 Convergence analysis

For the convergence analysis of our proposed rule for treating BSPVIP (1.6) with the CFPP constraint, we assume throughout that

- $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a nonzero bounded linear operator with the adjoint T^*, and $S : \mathcal{H}_2 \to \mathcal{H}_2$ is τ-demimetric with $I - S$ being demiclosed at zero, where $\tau \in (-\infty, 1)$.
- $A : \mathcal{H}_1 \to \mathcal{H}_1$ is a pseudomonotone and L-Lipschitz continuous mapping satisfying the condition: $\|Au\| \le \liminf_{t\to\infty} \|Au_t\|$ for each $\{u_t\} \subset C$ with $u_t \rightharpoonup u$.
- $\{S_i\}_{i=1}^N$ is finitely many nonexpansive self-mappings on $\mathcal{H}_1$ such that $\Xi := \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega \ne \emptyset$ with $\Omega := \{z \in \mathrm{VI}(C, A) : Tz \in \mathrm{Fix}(S)\}$. In addition, when required, we write $S_t := S_{t \bmod N}$, $t = 1, 2, 3, \ldots$.
- $f : \mathcal{H}_1 \to \mathcal{H}_1$ is a contraction with constant $\nu \in [0, 1)$, and $F : \mathcal{H}_1 \to \mathcal{H}_1$ is η-strongly monotone and κ-Lipschitzian such that $\nu < \zeta := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$.
- $\{\beta_t\}, \{\gamma_t\}, \{\varepsilon_t\} \subset (0, \infty)$ such that $\beta_t + \gamma_t < 1$, $\sum_{t=1}^{\infty} \beta_t = \infty$, $\lim_{t\to\infty} \beta_t = 0$, $0 < \liminf_{t\to\infty} \gamma_t \le \limsup_{t\to\infty} \gamma_t < 1$ and $\varepsilon_t = o(\beta_t)$.

Algorithm 3.1 (*Triple-adaptive inertial subgradient extragradient rule*)
Initialization: Let $\lambda_1 > 0$, $\epsilon > 0$, $\sigma \ge 0$, $\mu \in (0, 1)$, $\alpha \in [0, 1)$, and $x_0, x_1 \in \mathcal{H}_1$ be arbitrary.
Iterative steps: Calculate x_{t+1} as follows:
Step 1. Given the iterates x_{t-1} and x_t ($t \ge 1$), choose α_t such that $0 \le \alpha_t \le \bar{\alpha}_t$, where

$$\bar{\alpha}_t = \begin{cases} \min\{\alpha, \frac{\varepsilon_t}{\|x_t - x_{t-1}\|}\} & \text{if } x_t \ne x_{t-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $w_t = S_t x_t + \alpha_t(S_t x_t - S_t x_{t-1})$ and $y_t = P_C(w_t - \lambda_t A w_t)$.

Step 3. Construct $C_t := \{y \in \mathcal{H}_1 : \langle w_t - \lambda_t A w_t - y_t, y_t - y\rangle \geq 0\}$, and compute $v_t = P_{C_t}(w_t - \lambda_t A y_t)$ and $z_t = v_t - \sigma_t T^*(I - S)Tv_t$.

Step 4. Calculate $x_{t+1} = \beta_t f(x_t) + \gamma_t x_t + ((1-\gamma_t)I - \beta_t \rho F)z_t$ and update

$$\lambda_{t+1} = \begin{cases} \min\{\mu \frac{\|w_t - y_t\|^2 + \|v_t - y_t\|^2}{2\langle Aw_t - Ay_t, v_t - y_t\rangle}, \lambda_t\} & \text{if } \langle Aw_t - Ay_t, v_t - y_t\rangle > 0, \\ \lambda_t & \text{otherwise,} \end{cases} \tag{3.2}$$

and for any fixed $\epsilon > 0$, σ_t is chosen to be the bounded sequence satisfying

$$0 < \epsilon \leq \sigma_t \leq \frac{(1-\tau)\|Tv_t - STv_t\|^2}{\|T^*(Tv_t - STv_t)\|^2} - \epsilon \quad \text{if } Tv_t \neq STv_t, \tag{3.3}$$

otherwise set $\sigma_t = \sigma \geq 0$.

Set $t := t + 1$ and go to Step 1.

Remark 3.1 We have from (3.1) that $\lim_{t\to\infty} \frac{\alpha_t}{\beta_t}\|x_t - x_{t-1}\| = 0$. Indeed, we have $\alpha_t\|x_t - x_{t-1}\| \leq \varepsilon_t$ $\forall t \geq 1$, which together with $\lim_{t\to\infty} \frac{\varepsilon_t}{\beta_t} = 0$ implies that $\frac{\alpha_t}{\beta_t}\|x_t - x_{t-1}\| \leq \frac{\varepsilon_t}{\beta_t} \to 0$. It is easy to see that C_t is closed and convex. Furthermore, $C_t \neq \emptyset$ since $C \subset C_t$ and $C \neq \emptyset$. Hence, $\{v_t\}$ is well defined.

Lemma 3.1 *The step size* $\{\lambda_t\}$ *is nonincreasing with* $\lambda_t \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$ $\forall t \geq 1$, *and* $\lim_{t\to\infty} \lambda_t \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$.

Proof By (3.2), we get $\lambda_t \geq \lambda_{t+1}$ $\forall t \geq 1$. Now, observe that

$$\left.\begin{array}{l} \frac{1}{2}(\|w_t - y_t\|^2 + \|v_t - y_t\|^2) \geq \|w_t - y_t\|\|v_t - y_t\| \\ \langle Aw_t - Ay_t, v_t - y_t\rangle \leq L\|w_t - y_t\|\|v_t - y_t\| \end{array}\right\} \Longrightarrow \lambda_{t+1} \geq \min\left\{\lambda_t, \frac{\mu}{L}\right\}. \qquad \square$$

We prove the following lemmas.

Lemma 3.2 *The step size* σ_t *formulated in* (3.3) *is well defined.*

Proof It suffices to show that $\|T^*(Tv_t - STv_t)\|^2 \neq 0$. Take $p \in \Xi$ arbitrarily. Since S is a τ-demimetric mapping, we obtain

$$\begin{aligned} \|v_t - p\| \big\|T^*(Tv_t - STv_t)\big\| &\geq \big\langle v_t - p, T^*(Tv_t - STv_t)\big\rangle \\ &= \langle Tv_t - Tp, Tv_t - STv_t\rangle \\ &\geq \frac{1-\tau}{2}\|Tv_t - STv_t\|^2. \end{aligned} \tag{3.4}$$

If $Tv_t \neq STv_t$, then $\|Tv_t - STv_t\|^2 > 0$. Thus, $\|T^*(Tv_t - STv_t)\|^2 > 0$. $\square$

Lemma 3.3 *The sequences* $\{w_t\}$, $\{y_t\}$, $\{v_t\}$ *satisfy*

$$\|v_t - p\|^2 \leq \|w_t - p\|^2 - \left(1 - \mu\frac{\lambda_t}{\lambda_{t+1}}\right)\|w_t - y_t\|^2 - \left(1 - \mu\frac{\lambda_t}{\lambda_{t+1}}\right)\|v_t - y_t\|^2 \quad \forall p \in \Xi.$$

Proof Observe that

$$2\langle Aw_t - Ay_t, v_t - y_t\rangle \leq \frac{\mu}{\lambda_{t+1}}\|w_t - y_t\|^2 + \frac{\mu}{\lambda_{t+1}}\|v_t - y_t\|^2 \quad \forall t \geq 1. \tag{3.5}$$

Note that (3.5) holds when $\langle Aw_t - Ay_t, v_t - y_t\rangle \leq 0$. Conversely, we have (3.5) by (3.2). Also, $\forall \hat{p} \in \Xi \subset C \subset C_t$,

$$\begin{aligned}\|v_t - \hat{p}\|^2 &= \left\|P_{C_t}(w_t - \lambda_t Ay_t) - P_{C_t}\hat{p}\right\|^2 \\ &\leq \langle v_t - \hat{p}, w_t - \lambda_t Ay_t - \hat{p}\rangle \\ &= \frac{1}{2}\|v_t - \hat{p}\|^2 + \frac{1}{2}\|w_t - \hat{p}\|^2 - \frac{1}{2}\|v_t - w_t\|^2 - \langle v_t - \hat{p}, \lambda_t Ay_t\rangle,\end{aligned}$$

which hence yields

$$\|v_t - \hat{p}\|^2 \leq \|w_t - \hat{p}\|^2 - \|v_t - w_t\|^2 - 2\langle v_t - \hat{p}, \lambda_t Ay_t\rangle. \tag{3.6}$$

Since $\hat{p} \in \mathrm{VI}(C, A)$, we get $\langle A\hat{p}, \breve{x} - \hat{p}\rangle \geq 0\ \forall \breve{x} \in C$. Pseudomonotonicity of A implies $\langle Au, u - \hat{p}\rangle \geq 0\ \forall u \in C$. Letting $u := y_t \in C$ gives $\langle Ay_t, \hat{p} - y_t\rangle \leq 0$. Thus,

$$\langle Ay_t, \hat{p} - v_t\rangle = \langle Ay_t, \hat{p} - y_t\rangle + \langle Ay_t, y_t - v_t\rangle \leq \langle Ay_t, y_t - v_t\rangle. \tag{3.7}$$

Substituting (3.7) for (3.6), we obtain

$$\|v_t - \hat{p}\|^2 \leq \|w_t - \hat{p}\|^2 - \|v_t - y_t\|^2 - \|y_t - w_t\|^2 + 2\langle w_t - \lambda_t Ay_t - y_t, v_t - y_t\rangle. \tag{3.8}$$

Since $v_t = P_{C_t}(w_t - \lambda_t Ay_t)$, we have that $v_t \in C_t$, and hence

$$\begin{aligned}2\langle w_t - \lambda_t Ay_t - y_t, v_t - y_t\rangle &= 2\langle w_t - \lambda_t Aw_t - y_t, v_t - y_t\rangle \\ &\quad + 2\lambda_t\langle Aw_t - Ay_t, v_t - y_t\rangle \\ &\leq 2\lambda_t\langle Aw_t - Ay_t, v_t - y_t\rangle,\end{aligned}$$

which together with (3.5) implies that

$$2\langle w_t - \lambda_t Ay_t - y_t, v_t - y_t\rangle \leq \mu\frac{\lambda_t}{\lambda_{t+1}}\|w_t - y_t\|^2 + \mu\frac{\lambda_t}{\lambda_{t+1}}\|v_t - y_t\|^2. \tag{3.9}$$

Therefore, substituting (3.9) for (3.8), the result follows. □

Lemma 3.4 *$\{x_t\}$ is bounded.*

Proof First of all, we show that $P_\Xi(f + I - \rho F)$ is a contraction. Indeed, for any $x, y \in \mathcal{H}_1$, by Lemma 2.2, we have

$$\begin{aligned}&\left\|P_\Xi(f + I - \rho F)x - P_\Xi(f + I - \rho F)y\right\| \\ &\quad \leq \left\|f(x) - f(y)\right\| + \left\|(I - \rho F)x - (I - \rho F)y\right\|\end{aligned}$$

$$\leq \nu\|x-y\| + (1-\zeta)\|x-y\| = \big[1-(\zeta-\nu)\big]\|x-y\|,$$

which implies that $P_\Xi(f+I-\rho F)$ is a contraction. Banach's contraction mapping principle guarantees that $P_\Xi(f+I-\rho F)$ has a unique fixed point. Say $q^* \in \mathcal{H}_1$, i.e., $q^* = P_\Xi(f+I-\rho F)q^*$. Hence, there exists unique $q^* \in \Xi$ that solves

$$\big\langle(\rho F-f)q^*, p-q^*\big\rangle \geq 0 \quad \forall p \in \Xi. \tag{3.10}$$

This also means that there exists a unique solution $q^* \in \Xi$ to BSPVIP (1.6) with the CFPP constraint.

Now, by the definition of w_t in Algorithm 3.1, we have

$$\begin{aligned}\|w_t-q^*\| &= \|S_t x_t + \alpha_t(S_t x_t - S_t x_{t-1}) - q^*\| \\ &\leq \|x_t-q^*\| + \beta_t \frac{\alpha_t}{\beta_t}\|x_t-x_{t-1}\|.\end{aligned}$$

From Remark 3.1, we know that $\lim_{t\to\infty}\frac{\alpha_t}{\beta_t}\|x_t-x_{t-1}\| = 0$. This means that $\{\frac{\alpha_t}{\beta_t}\|x_t-x_{t-1}\|\}$ is bounded. Thus, $\exists M_1 > 0$ such that $\frac{\alpha_t}{\beta_t}\|x_t-x_{t-1}\| \leq M_1$ $\forall t \geq 1$. Hence,

$$\|w_t-q^*\| \leq \|x_t-q^*\| + \beta_t M_1 \quad \forall t \geq 1. \tag{3.11}$$

From Step 3 of Algorithm 3.1, using the definition of z_t, we get

$$\begin{aligned}\|z_t-q^*\|^2 &= \|v_t-\sigma_t T^*(I-S)Tv_t-q^*\|^2 \\ &= \|v_t-q^*\|^2 - 2\sigma_t\langle v_t-q^*, T^*(I-S)Tv_t\rangle \\ &\quad + \sigma_t^2\|T^*(I-S)Tv_t\|^2 \\ &= \|v_t-q^*\|^2 - 2\sigma_t\langle T(v_t-q^*), (I-S)Tv_t\rangle \\ &\quad + \sigma_t^2\|T^*(I-S)Tv_t\|^2.\end{aligned} \tag{3.12}$$

Since the operator S is τ-demimetric, from (3.12), we get

$$\begin{aligned}\|z_t-q^*\|^2 &\leq \|v_t-q^*\|^2 - \sigma_t(1-\tau)\|(I-S)Tv_t\|^2 + \sigma_t^2\|T^*(I-S)Tv_t\|^2 \\ &= \|v_t-q^*\|^2 + \sigma_t\big[\sigma_t\|T^*(I-S)Tv_t\|^2 - (1-\tau)\|(I-S)Tv_t\|^2\big].\end{aligned} \tag{3.13}$$

However, from the step size σ_t in (3.3), we get

$$\sigma_t + \epsilon \leq \frac{(1-\tau)\|Tv_t - STv_t\|^2}{\|T^*(I-S)Tv_t\|^2}$$

if and only if

$$\sigma_t\big(\sigma_t\|T^*(I-S)Tv_t\|^2 - (1-\tau)\|Tv_t-STv_t\|^2\big) \leq -\sigma_t\epsilon\|T^*(I-S)Tv_t\|^2. \tag{3.14}$$

Using $0 < \epsilon \leq \sigma_t$ in (3.3), we have that $-\epsilon^2 \geq -\sigma_t\epsilon$, and hence

$$-\sigma_t\epsilon\|T^*(I-S)Tv_t\|^2 \leq -\epsilon^2\|T^*(I-S)Tv_t\|^2. \tag{3.15}$$

Combining (3.13), (3.14), and (3.15), we obtain

$$\begin{aligned} \|z_t - q^*\|^2 &\le \|v_t - q^*\|^2 - \sigma_t \epsilon \|T^*(I-S)Tv_t\|^2 \\ &\le \|v_t - q^*\|^2 - \epsilon^2 \|T^*(I-S)Tv_t\|^2 \\ &\le \|v_t - q^*\|^2. \end{aligned} \tag{3.16}$$

In addition, by Lemma 3.1, we have $\lim_{t\to\infty} \lambda_t \ge \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$, which leads to $\lim_{t\to\infty}(1 - \mu\frac{\lambda_t}{\lambda_{t+1}}) = 1 - \mu > 0$. Without loss of generality, we may assume that $1 - \mu\frac{\lambda_t}{\lambda_{t+1}} > 0\ \forall t \ge 1$. Thus, by Lemma 3.3, we get

$$\begin{aligned} \|v_t - q^*\|^2 &\le \|w_t - q^*\|^2 - \left(1 - \mu\frac{\lambda_t}{\lambda_{t+1}}\right)\|w_t - y_t\|^2 \\ &\quad - \left(1 - \mu\frac{\lambda_t}{\lambda_{t+1}}\right)\|v_t - y_t\|^2 \\ &\le \|w_t - q^*\|^2. \end{aligned} \tag{3.17}$$

Combining (3.11), (3.16), and (3.17), we obtain

$$\|z_t - q^*\| \le \|v_t - q^*\| \le \|w_t - q^*\| \le \|x_t - q^*\| + \beta_t M_1 \quad \forall t \ge 1. \tag{3.18}$$

Since $\beta_t + \gamma_t < 1\ \forall t \ge 1$, we get $\frac{\beta_t}{1-\gamma_t} < 1\ \forall t \ge 1$. So, from Lemma 2.2 and (3.18) it follows that

$$\begin{aligned} \|x_{t+1} - q^*\| &= \|\beta_t f(x_t) + \gamma_t x_t + ((1-\gamma_t)I - \beta_t \rho F)z_t - q^*\| \\ &\le \beta_t \|f(x_t) - q^*\| + \gamma_t \|x_t - q^*\| \\ &\quad + (1-\beta_t-\gamma_t)\left\|\left(\frac{1-\gamma_t}{1-\beta_t-\gamma_t}I - \frac{\beta_t}{1-\beta_t-\gamma_t}\rho F\right)z_t - q^*\right\| \\ &\le \beta_t(\|f(x_t) - f(q^*)\| + \|f(q^*) - q^*\|) + \gamma_t\|x_t - q^*\| \\ &\quad + (1-\beta_t-\gamma_t)\left\|\left(\frac{1-\gamma_t}{1-\beta_t-\gamma_t}I - \frac{\beta_t}{1-\beta_t-\gamma_t}\rho F\right)z_t - q^*\right\| \\ &\le \beta_t(\nu\|x_t - q^*\| + \|f(q^*) - q^*\|) + \gamma_t\|x_t - q^*\| \\ &\quad + (1-\gamma_t)\left\|\left(I - \frac{\beta_t}{1-\gamma_t}\rho F\right)z_t - \left(1 - \frac{\beta_t}{1-\gamma_t}\right)q^*\right\| \\ &= \beta_t(\nu\|x_t - q^*\| + \|f(q^*) - q^*\|) + \gamma_t\|x_t - q^*\| \\ &\quad + (1-\gamma_t)\left\|\left(I - \frac{\beta_t}{1-\gamma_t}\rho F\right)z_t - \left(I - \frac{\beta_t}{1-\gamma_t}\rho F\right)q^* + \frac{\beta_t}{1-\gamma_t}(I-\rho F)q^*\right\| \\ &\le \beta_t(\nu\|x_t - q^*\| + \|f(q^*) - q^*\|) + \gamma_t\|x_t - q^*\| \\ &\quad + (1-\gamma_t)\left[\left(1 - \frac{\beta_t}{1-\gamma_t}\zeta\right)\|z_t - q^*\| + \frac{\beta_t}{1-\gamma_t}\|(I-\rho F)q^*\|\right] \\ &= \beta_t(\nu\|x_t - q^*\| + \|f(q^*) - q^*\|) + \gamma_t\|x_t - q^*\| \\ &\quad + (1-\gamma_t-\beta_t\zeta)\|z_t - q^*\| + \beta_t\|(I-\rho F)q^*\| \end{aligned}$$

$$\begin{aligned}
&\leq \beta_t\bigl(\nu\|x_t-q^*\|+\|f(q^*)-q^*\|\bigr)+\gamma_t\|x_t-q^*\| \\
&\quad +(1-\gamma_t-\beta_t\zeta)\bigl(\|x_t-q^*\|+\beta_tM_1\bigr)+\beta_t\|(I-\rho F)q^*\| \\
&\leq \bigl[1-\beta_t(\zeta-\nu)\bigr]\|x_t-q^*\|+\beta_t\bigl(M_1+\|f(q^*)-q^*\|+\|(I-\rho F)q^*\|\bigr) \\
&= \bigl[1-\beta_t(\zeta-\nu)\bigr]\|x_t-q^*\|+\beta_t(\zeta-\nu)\frac{M_1+\|f(q^*)-q^*\|+\|(I-\rho F)q^*\|}{\zeta-\nu} \\
&\leq \max\biggl\{\|x_t-q^*\|,\frac{M_1+\|f(q^*)-q^*\|+\|(I-\rho F)q^*\|}{\zeta-\nu}\biggr\}.
\end{aligned}$$

Thus, $\|x_t-q^*\|\leq\max\{\|x_1-q^*\|,\frac{M_1+\|f(q^*)-q^*\|+\|(I-\rho F)q^*\|}{\zeta-\nu}\}$ for all $t\geq 1$. Thus, $\{x_t\}$ is bounded, and so are the sequences $\{v_t\}$, $\{w_t\}$, $\{y_t\}$, $\{z_t\}$, $\{f(x_t)\}$, $\{Fz_t\}$, $\{S_tx_t\}$. □

Lemma 3.5 *Let $\{v_t\}$, $\{w_t\}$, $\{x_t\}$, $\{y_t\}$, $\{z_t\}$ be the sequences generated by Algorithm 3.1. Suppose that $x_t-x_{t+1}\to 0$, $w_t-x_t\to 0$, $w_t-y_t\to 0$, and $v_t-z_t\to 0$. Then $\omega_w(\{x_t\})\subset\Xi$ with $\omega_w(\{x_t\})=\{z\in\mathcal{H}_1: x_{t_k}\rightharpoonup z$ for some $\{x_{t_k}\}\subset\{x_t\}\}$.*

Proof Take an arbitrary fixed $z\in\omega_w(\{x_t\})$. Then $\exists\{x_{t_k}\}\subset\{x_t\}$ such that $x_{t_k}\rightharpoonup z\in\mathcal{H}_1$. Thanks to $w_t-x_t\to 0$, by which $\exists\{w_{t_k}\}\subset\{w_t\}$ such that $w_{t_k}\rightharpoonup z\in\mathcal{H}_1$. In what follows, we claim that $z\in\Xi$. In fact, from Algorithm 3.1, we get $w_t-x_t=S_tx_t-x_t+\alpha_t(S_tx_t-S_tx_{t-1})$ $\forall t\geq 1$, and hence

$$\begin{aligned}
\|S_tx_t-x_t\| &= \bigl\|w_t-x_t-\alpha_t(S_tx_t-S_tx_{t-1})\bigr\| \\
&\leq \|w_t-x_t\|+\alpha_t\|S_tx_t-S_tx_{t-1}\| \\
&\leq \|w_t-x_t\|+\beta_t\frac{\alpha_t}{\beta_t}\|x_t-x_{t-1}\|.
\end{aligned}$$

Using Remark 3.1 and the assumption $w_t-x_t\to 0$, we have

$$\lim_{t\to\infty}\|x_t-S_tx_t\|=0. \tag{3.19}$$

Also, from $y_t=P_C(w_t-\lambda_tAw_t)$, we have $\langle w_t-\lambda_tAw_t-y_t,y_t-y\rangle\geq 0$ $\forall y\in C$, and hence

$$\frac{1}{\lambda_t}\langle w_t-y_t,\nu-y_t\rangle+\langle Aw_t,y_t-w_t\rangle\leq\langle Aw_t,\nu-w_t\rangle\quad\forall\nu\in C. \tag{3.20}$$

Observe that $\lambda_t\geq\min\{\lambda_1,\frac{\mu}{L}\}$. So, from (3.20), we get $\liminf_{k\to\infty}\langle Aw_{t_k},y-w_{t_k}\rangle\geq 0$ $\forall y\in C$. In the meantime, observe that $\langle Ay_t,y-y_t\rangle=\langle Ay_t-Aw_t,y-w_t\rangle+\langle Aw_t,y-w_t\rangle+\langle Ay_t,w_t-y_t\rangle$. Since $w_t-y_t\to 0$, we obtain $Aw_t-Ay_t\to 0$, which together with (3.20) arrives at $\liminf_{k\to\infty}\langle Ay_{t_k},\nu-y_{t_k}\rangle\geq 0$ $\forall\nu\in C$.

For $i=1,2,\ldots,N$,

$$\begin{aligned}
\|x_t-S_{t+i}x_t\| &\leq \|x_t-x_{t+i}\|+\|x_{t+i}-S_{t+i}x_{t+i}\|+\|S_{t+i}x_{t+i}-S_{t+i}x_t\| \\
&\leq 2\|x_t-x_{t+i}\|+\|x_{t+i}-S_{t+i}x_{t+i}\|.
\end{aligned}$$

Hence, from (3.19) and the assumption $x_t-x_{t+1}\to 0$, we get $\lim_{t\to\infty}\|x_t-S_{t+i}x_t\|=0$ for $i=1,2,\ldots,N$. This immediately implies that

$$\lim_{t\to\infty}\|x_t-S_lx_t\|=0\quad\text{for } l=1,2,\ldots,N. \tag{3.21}$$

Pick $\{\varsigma_k\} \subset (0,1)$, $\varsigma_k \downarrow 0$. For all $k \geq 1$, let m_k be the smallest positive integer such that

$$\langle Ay_{t_k}, y - y_{t_k}\rangle + \varsigma_k \geq 0 \quad \forall k \geq m_k. \tag{3.22}$$

Since $\{\varsigma_k\}$ is nonincreasing, it is clear that $\{m_k\}$ is nondecreasing.

Again from the assumption on A, we know that $\liminf_{k\to\infty} \|Ay_{t_k}\| \geq \|Az\|$. If $Az = 0$, then z is a solution, i.e., $z \in \mathrm{VI}(C,A)$. Let $Az \neq 0$. Then we have $0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{t_k}\|$. Without loss of generality, we may assume that $Ay_{t_k} \neq 0$ $\forall k \geq 1$. Noticing $\{y_{m_k}\} \subset \{y_{t_k}\}$ and $Ay_{t_k} \neq 0$ $\forall k \geq 1$, set $u_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, and then $\langle Ay_{m_k}, u_{m_k}\rangle = 1$ $\forall k \geq 1$. So, from (3.22), we get $\langle Ay_{m_k}, y + \varsigma_k u_{m_k} - y_{m_k}\rangle \geq 0$ $\forall k \geq 1$. By the pseudomonotonicity of A, we obtain $\langle A(y + \varsigma_k u_{m_k}), y + \varsigma_k u_{m_k} - y_{m_k}\rangle \geq 0$ $\forall k \geq 1$. This immediately yields

$$\langle Ay, y - y_{m_k}\rangle \geq \bigl\langle Ay - A(y + \varsigma_k u_{m_k}), y + \varsigma_k u_{m_k} - y_{m_k}\bigr\rangle - \varsigma_k \langle Ay, u_{m_k}\rangle \quad \forall k \geq 1. \tag{3.23}$$

From $x_{t_k} \rightharpoonup z$ and $x_t - y_t \to 0$ (due to $w_t - x_t \to 0$ and $w_t - y_t \to 0$), we obtain $y_{t_k} \rightharpoonup z$. So, $\{y_t\} \subset C$ guarantees $z \in C$. Since $\{y_{m_k}\} \subset \{y_{t_k}\}$ and $\varsigma_k \downarrow 0$, we have $0 \leq \limsup_{k\to\infty} \|\varsigma_k u_{m_k}\| = \limsup_{k\to\infty} \frac{\varsigma_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty} \varsigma_k}{\liminf_{k\to\infty} \|Ay_{t_k}\|} = 0$. Hence, we get $\varsigma_k u_{m_k} \to 0$.

Next, we show that $z \in \Xi$. Indeed, using (3.21), we have $x_{t_k} - S_l x_{t_k} \to 0$ for $l = 1, 2, \ldots, N$. By Lemma 2.5, $I - S_l$ is demiclosed at zero for $l = 1, 2, \ldots, N$. Thus, from $x_{t_k} \rightharpoonup z$, we get $z \in \mathrm{Fix}(S_l)$. Since l is an arbitrary element in the finite set $\{1, 2, \ldots, N\}$, it follows that $z \in \bigcap_{i=1}^N \mathrm{Fix}(S_i)$. Also, letting $k \to \infty$, we have that the right-hand side of (3.23) tends to zero. Thus, $\langle A\vec{y}, \vec{y} - z\rangle = \liminf_{k\to\infty} \langle A\vec{y}, \vec{y} - y_{m_k}\rangle \geq 0$ $\forall \vec{y} \in C$. By Lemma 2.3 we have $z \in \mathrm{VI}(C,A)$. Furthermore, we claim $Tz \in \mathrm{Fix}(S)$. In fact, noticing $z_t = v_t - \sigma_t T^*(I-S)Tv_t$, from $0 < \epsilon \leq \sigma_t$ and $v_t - z_t \to 0$, we get

$$\epsilon \bigl\| T^*(I-S)Tv_t\bigr\| \leq \sigma_t \bigl\| T^*(I-S)Tv_t\bigr\| = \|v_t - z_t\| \to 0 \quad (t \to \infty),$$

which together with the τ-demimetricness of S leads to

$$\begin{aligned} \frac{1-\tau}{2} \bigl\|(I-S)Tv_t\bigr\|^2 &\leq \bigl\langle (I-S)Tv_t, T\bigl(v_t - q^*\bigr)\bigr\rangle \\ &\leq \bigl\| T^*(I-S)Tv_t\bigr\| \bigl\| v_t - q^*\bigr\| \to 0 \quad (t \to \infty). \end{aligned} \tag{3.24}$$

Noticing $x_{t+1} = \beta_t f(x_t) + \gamma_t x_t + ((1-\gamma_t)I - \beta_t \rho F)z_t$, we have

$$\begin{aligned} (1-\gamma_t)\|z_t - x_t\| &= \bigl\| x_{t+1} - x_t - \beta_t\bigl(f(x_t) - \rho F z_t\bigr)\bigr\| \\ &\leq \|x_{t+1} - x_t\| + \beta_t\bigl(\bigl\|f(x_t)\bigr\| + \|\rho F z_t\|\bigr). \end{aligned}$$

Since $0 < \liminf_{t\to\infty}(1-\gamma_t)$, $x_t - x_{t+1} \to 0$ and $\beta_t \to 0$, from the boundedness of $\{x_t\}$ and $\{z_t\}$, we get $\lim_{t\to\infty} \|z_t - x_t\| = 0$, which hence yields

$$\|v_t - x_t\| \leq \|v_t - z_t\| + \|z_t - x_t\| \to 0 \quad (t \to \infty).$$

From $x_{t_k} \rightharpoonup z$, we get $v_{t_k} \rightharpoonup z$. It follows that $Tv_{t_k} \rightharpoonup Tz$. From (3.24) one derives $Tz \in \mathrm{Fix}(S)$. Therefore, $z \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega = \Xi$. This completes the proof. □

Theorem 3.1 *$\{x_t\}$ generated by Algorithm 3.1 converges strongly to the unique solution $q^* \in \Xi$ of BSPVIP (1.6) with the CFPP constraint.*

Proof First of all, in terms of Lemma 3.4 we obtain that $\{x_t\}$ is bounded. From its proof we know that there exists a unique solution $q^* \in \Xi$ of BSPVIP (1.6) with the CFPP constraint, i.e., VIP (3.10) has a unique solution $q^* \in \Xi$.

Step 1. We claim that

$$\begin{aligned}(1-\beta_t\zeta-\gamma_t)&\left[\left(1-\mu\frac{\lambda_t}{\lambda_{t+1}}\right)\left(\|w_t-y_t\|^2+\|v_t-y_t\|^2\right)+\epsilon^2\left\|T^*(I-S)Tv_t\right\|^2\right]\\&\le\left\|x_t-q^*\right\|^2-\left\|x_{t+1}-q^*\right\|^2+\beta_tM_4\end{aligned}$$

for some $M_4 > 0$. Also

$$\begin{aligned}x_{t+1}-q^*&=\beta_t\big(f(x_t)-q^*\big)+\gamma_t\big(x_t-q^*\big)+(1-\beta_t-\gamma_t)\left\{\frac{1-\gamma_t}{1-\beta_t-\gamma_t}\left[\left(I-\frac{\beta_t}{1-\gamma_t}\rho F\right)z_t\right.\right.\\&\qquad\left.\left.-\left(I-\frac{\beta_t}{1-\gamma_t}\rho F\right)q^*\right]+\frac{\beta_t}{1-\beta_t-\gamma_t}(I-\rho F)q^*\right\}\\&=\beta_t\big(f(x_t)-f\big(q^*\big)\big)+\gamma_t\big(x_t-q^*\big)\\&\qquad+(1-\gamma_t)\left[\left(I-\frac{\beta_t}{1-\gamma_t}\rho F\right)z_t-\left(I-\frac{\beta_t}{1-\gamma_t}\rho F\right)q^*\right]+\beta_t(f-\rho F)q^*.\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}\left\|x_{t+1}-q^*\right\|^2&\le\left\|\beta_t\big(f(x_t)-f\big(q^*\big)\big)+\gamma_t\big(x_t-q^*\big)\right.\\&\qquad\left.+(1-\gamma_t)\left[\left(I-\frac{\beta_t}{1-\gamma_t}\rho F\right)z_t-\left(I-\frac{\beta_t}{1-\gamma_t}\rho F\right)q^*\right]\right\|^2\\&\qquad+2\beta_t\big\langle(f-\rho F)q^*,x_{t+1}-q^*\big\rangle\\&\le\left[\beta_t\nu\left\|x_t-q^*\right\|+\gamma_t\left\|x_t-q^*\right\|+(1-\gamma_t)\left(1-\frac{\beta_t}{1-\gamma_t}\zeta\right)\left\|z_t-q^*\right\|\right]^2\\&\qquad+2\beta_t\big\langle(f-\rho F)q^*,x_{t+1}-q^*\big\rangle\\&=\left[\beta_t\nu\left\|x_t-q^*\right\|+\gamma_t\left\|x_t-q^*\right\|+(1-\beta_t\zeta-\gamma_t)\left\|z_t-q^*\right\|\right]^2\\&\qquad+2\beta_t\big\langle(f-\rho F)q^*,x_{t+1}-q^*\big\rangle\\&\le\beta_t\nu\left\|x_t-q^*\right\|^2+\gamma_t\left\|x_t-q^*\right\|^2+(1-\beta_t\zeta-\gamma_t)\left\|z_t-q^*\right\|^2\\&\qquad+2\beta_t\big\langle(f-\rho F)q^*,x_{t+1}-q^*\big\rangle\\&\le\beta_t\nu\left\|x_t-q^*\right\|^2+\gamma_t\left\|x_t-q^*\right\|^2+(1-\beta_t\zeta-\gamma_t)\left\|z_t-q^*\right\|^2\\&\qquad+\beta_tM_2\end{aligned}\tag{3.25, 3.26}$$

(due to $\beta_t\nu+\gamma_t+(1-\beta_t\zeta-\gamma_t)=1-\beta_t(\zeta-\nu)\le 1$), where $\sup_{t\ge1}2\|(f-\rho F)q^*\|\|x_t-q^*\|\le M_2$ for some $M_2 > 0$. Substituting (3.16) for (3.25), by Lemma 3.3 we get

$$\left\|x_{t+1}-q^*\right\|^2\le\beta_t\nu\left\|x_t-q^*\right\|^2+\gamma_t\left\|x_t-q^*\right\|^2+(1-\beta_t\zeta-\gamma_t)\left[\left\|v_t-q^*\right\|^2\right.$$

$$
\begin{aligned}
&-\epsilon^2\left\|T^*(I-S)Tv_t\right\|^2\Big]+\beta_t M_2\\
&\le \beta_t\nu\left\|x_t-q^*\right\|^2+\gamma_t\left\|x_t-q^*\right\|^2+(1-\beta_t\zeta-\gamma_t)\Big[\left\|w_t-q^*\right\|^2\\
&-\left(1-\mu\frac{\lambda_t}{\lambda_{t+1}}\right)\left(\|w_t-y_t\|^2+\|v_t-y_t\|^2\right)\\
&-\epsilon^2\left\|T^*(I-S)Tv_t\right\|^2\Big]+\beta_t M_2.
\end{aligned}
\tag{3.27}
$$

Also, from (3.18) we have

$$
\begin{aligned}
\left\|w_t-q^*\right\|^2 &\le \left(\left\|x_t-q^*\right\|+\beta_t M_1\right)^2\\
&=\left\|x_t-q^*\right\|^2+\beta_t\left(2M_1\left\|x_t-q^*\right\|+\beta_t M_1^2\right)\\
&\le \left\|x_t-q^*\right\|^2+\beta_t M_3,
\end{aligned}
\tag{3.28}
$$

where $\sup_{t\ge 1}(2M_1\|x_t-q^*\|+\beta_t M_1^2)\le M_3$ for some $M_3>0$. Combining (3.27) and (3.28), we obtain

$$
\begin{aligned}
\left\|x_{t+1}-q^*\right\|^2 &\le \beta_t\nu\left\|x_t-q^*\right\|^2+\gamma_t\left\|x_t-q^*\right\|^2\\
&\quad+(1-\beta_t\zeta-\gamma_t)\left[\left\|x_t-q^*\right\|^2+\beta_t M_3\right]\\
&\quad-(1-\beta_t\zeta-\gamma_t)\Big[\left(1-\mu\frac{\lambda_t}{\lambda_{t+1}}\right)\left(\|w_t-y_t\|^2+\|v_t-y_t\|^2\right)\\
&\quad+\epsilon^2\left\|T^*(I-S)Tv_t\right\|^2\Big]+\beta_t M_2\\
&\le\left[1-\beta_t(\zeta-\nu)\right]\left\|x_t-q^*\right\|^2-(1-\beta_t\zeta-\gamma_t)\Big[\left(1-\mu\frac{\lambda_t}{\lambda_{t+1}}\right)\left(\|w_t-y_t\|^2\right.\\
&\quad\left.+\|v_t-y_t\|^2\right)+\epsilon^2\left\|T^*(I-S)Tv_t\right\|^2\Big]+\beta_t M_4\\
&\le\left\|x_t-q^*\right\|^2-(1-\beta_t\zeta-\gamma_t)\Big[\left(1-\mu\frac{\lambda_t}{\lambda_{t+1}}\right)\left(\|w_t-y_t\|^2+\|v_t-y_t\|^2\right)\\
&\quad+\epsilon^2\left\|T^*(I-S)Tv_t\right\|^2\Big]+\beta_t M_4,
\end{aligned}
$$

where $M_4:=M_2+M_3$. This immediately implies that

$$
\begin{aligned}
&(1-\beta_t\zeta-\gamma_t)\Big[\left(1-\mu\frac{\lambda_t}{\lambda_{t+1}}\right)\left(\|w_t-y_t\|^2+\|v_t-y_t\|^2\right)+\epsilon^2\left\|T^*(I-S)Tv_t\right\|^2\Big]\\
&\quad\le\left\|x_t-q^*\right\|^2-\left\|x_{t+1}-q^*\right\|^2+\beta_t M_4.
\end{aligned}
\tag{3.29}
$$

Step 2. We claim that

$$
\begin{aligned}
\left\|x_{t+1}-q^*\right\|^2 &\le\left[1-\beta_t(\zeta-\nu)\right]\left\|x_t-q^*\right\|^2+\beta_t(\zeta-\nu)\Big[\frac{2}{\zeta-\nu}\left\langle(f-\rho F)q^*,x_{t+1}-q^*\right\rangle\\
&\quad+\frac{M}{\zeta-\nu}\cdot\frac{\alpha_t}{\beta_t}\cdot\|x_t-x_{t-1}\|\Big]
\end{aligned}
$$

for some $M > 0$. Indeed, we have

$$\begin{aligned}\|w_t - q^*\|^2 &\le \big[\|x_t - q^*\| + \alpha_t\|x_t - x_{t-1}\|\big]^2 \\ &\le \|x_t - q^*\|^2 + \alpha_t\|x_t - x_{t-1}\|\big[2\|x_t - q^*\| + \alpha_t\|x_t - x_{t-1}\|\big].\end{aligned} \tag{3.30}$$

Combining (3.18), (3.25), and (3.30), we have

$$\begin{aligned}\|x_{t+1} - q^*\|^2 &\le \beta_t\nu\|x_t - q^*\|^2 + \gamma_t\|x_t - q^*\|^2 \\ &\quad + (1-\beta_t\zeta-\gamma_t)\|z_t - q^*\|^2 + 2\beta_t\langle(f-\rho F)q^*, x_{t+1} - q^*\rangle \\ &\le \beta_t\nu\|x_t - q^*\|^2 + \gamma_t\|x_t - q^*\|^2 + (1-\beta_t\zeta-\gamma_t)\|w_t - q^*\|^2 \\ &\quad + 2\beta_t\langle(f-\rho F)q^*, x_{t+1} - q^*\rangle \\ &\le \beta_t\nu\|x_t - q^*\|^2 + \gamma_t\|x_t - q^*\|^2 + (1-\beta_t\zeta-\gamma_t)\big\{\|x_t - q^*\|^2 \\ &\quad + \alpha_t\|x_t - x_{t-1}\|\big[2\|x_t - q^*\| + \alpha_t\|x_t - x_{t-1}\|\big]\big\} \\ &\quad + 2\beta_t\langle(f-\rho F)q^*, x_{t+1} - q^*\rangle \\ &\le \big[1-\beta_t(\zeta-\nu)\big]\|x_t - q^*\|^2 + \alpha_t\|x_t - x_{t-1}\|\big[2\|x_t - q^*\| + \alpha_t\|x_t - x_{t-1}\|\big] \\ &\quad + 2\beta_t\langle(f-\rho F)q^*, x_{t+1} - q^*\rangle \\ &\le \big[1-\beta_t(\zeta-\nu)\big]\|x_t - q^*\|^2 + \alpha_t\|x_t - x_{t-1}\|M \\ &\quad + 2\beta_t\langle(f-\rho F)q^*, x_{t+1} - q^*\rangle \\ &= \big[1-\beta_t(\zeta-\nu)\big]\|x_t - q^*\|^2 + \beta_t(\zeta-\nu)\cdot\left[\frac{2\langle(f-\rho F)q^*, x_{t+1} - q^*\rangle}{\zeta-\nu}\right. \\ &\quad \left. + \frac{M}{\zeta-\nu}\cdot\frac{\alpha_t}{\beta_t}\cdot\|x_t - x_{t-1}\|\right],\end{aligned} \tag{3.31}$$

where $\sup_{t\ge1}\{2\|x_t - q^*\| + \alpha_t\|x_t - x_{t-1}\|\} \le M$.

Step 3. We show that $\{x_t\}$ converges strongly to $q^* \in \Xi$. Put $\mathbf{\Gamma}_t = \|x_t - q^*\|^2$.

Case 1. Assume that integer $t_0 \ge 1$ with $\{\mathbf{\Gamma}_t\}_{t\ge t_0}$ is nonincreasing. Then $\lim_{t\to\infty}\mathbf{\Gamma}_t = d < +\infty$, $\lim_{t\to\infty}(\mathbf{\Gamma}_t - \mathbf{\Gamma}_{t+1}) = 0$. By (3.29), one obtains

$$\begin{aligned}&(1-\beta_t\zeta-\gamma_t)\left[\left(1-\mu\frac{\lambda_t}{\lambda_{t+1}}\right)\left(\|w_t - y_t\|^2 + \|v_t - y_t\|^2\right) + \epsilon^2\|T^*(I-S)Tv_t\|^2\right] \\ &\quad \le \|x_t - q^*\|^2 - \|x_{t+1} - q^*\|^2 + \beta_t M_4 = \mathbf{\Gamma}_t - \mathbf{\Gamma}_{t+1} + \beta_t M_4.\end{aligned}$$

Since $\lim_{t\to\infty}(1-\mu\frac{\lambda_t}{\lambda_{t+1}}) = 1-\mu > 0$, $\liminf_{t\to\infty}(1-\gamma_t) > 0$, $\beta_t \to 0$, and $\mathbf{\Gamma}_t - \mathbf{\Gamma}_{t+1} \to 0$, one has

$$\lim_{t\to\infty}\|w_t - y_t\| = \lim_{t\to\infty}\|v_t - y_t\| = \lim_{t\to\infty}\|T^*(I-S)Tv_t\| = 0. \tag{3.32}$$

Noticing $z_t = v_t - \sigma_t T^*(I-S)Tv_t$ and the boundedness of $\{\sigma_t\}$, from (3.32) we get

$$\|v_t - z_t\| = \sigma_t\|T^*(I-S)Tv_t\| \to 0 \quad (t\to\infty), \tag{3.33}$$

and hence

$$\|w_t - z_t\| \le \|w_t - y_t\| + \|y_t - v_t\| + \|v_t - z_t\| \to 0 \quad (t \to \infty). \tag{3.34}$$

Moreover, noticing $x_{t+1} - q^* = \gamma_t(x_t - q^*) + (1 - \gamma_t)(z_t - q^*) + \beta_t(f(x_t) - \rho F z_t)$, we obtain from (3.18) that

$$\begin{aligned}
\|x_{t+1} - q^*\|^2 &= \|\gamma_t(x_t - q^*) + (1 - \gamma_t)(z_t - q^*) + \beta_t(f(x_t) - \rho F z_t)\|^2 \\
&\le \|\gamma_t(x_t - q^*) + (1 - \gamma_t)(z_t - q^*)\|^2 \\
&\quad + 2\langle \beta_t(f(x_t) - \rho F z_t), x_{t+1} - q^*\rangle \\
&\le \gamma_t\|x_t - q^*\|^2 + (1 - \gamma_t)\|z_t - q^*\|^2 - \gamma_t(1 - \gamma_t)\|x_t - z_t\|^2 \\
&\quad + 2\|\beta_t(f(x_t) - \rho F z_t)\|\|x_{t+1} - q^*\| \\
&\le \gamma_t\|x_t - q^*\|^2 + (1 - \gamma_t)\|z_t - q^*\|^2 - \gamma_t(1 - \gamma_t)\|x_t - z_t\|^2 \\
&\quad + 2\beta_t(\|f(x_t)\| + \|\rho F z_t\|)\|x_{t+1} - x^*\| \\
&\le \gamma_t(\|x_t - q^*\| + \beta_t M_1)^2 + (1 - \gamma_t)(\|x_t - q^*\| + \beta_t M_1)^2 \\
&\quad - \gamma_t(1 - \gamma_t)\|x_t - z_t\|^2 + 2\beta_t(\|f(x_t)\| + \|\rho F z_t\|)\|x_{t+1} - q^*\| \\
&= (\|x_t - q^*\| + \beta_t M_1)^2 - \gamma_t(1 - \gamma_t)\|x_t - z_t\|^2 \\
&\quad + 2\beta_t(\|f(x_t)\| + \|\rho F z_t\|)\|x_{t+1} - q^*\| \\
&= \|x_t - q^*\|^2 + \beta_t M_1[2\|x_t - q^*\| + \beta_t M_1] \\
&\quad - \gamma_t(1 - \gamma_t)\|x_t - z_t\|^2 + 2\beta_t(\|f(x_t)\| + \|\rho F z_t\|)\|x_{t+1} - q^*\|,
\end{aligned}$$

which immediately leads to

$$\begin{aligned}
\gamma_t(1 - \gamma_t)\|x_t - z_t\|^2 &\le \|x_t - q^*\|^2 - \|x_{t+1} - q^*\|^2 \\
&\quad + \beta_t M_1[2\|x_t - q^*\| + \beta_t M_1] + 2\beta_t(\|f(x_t)\| \\
&\quad + \|\rho F z_t\|)\|x_{t+1} - q^*\| \\
&\le \mathbf{\Gamma}_t - \mathbf{\Gamma}_{t+1} + \beta_t M_1[2\mathbf{\Gamma}_t^{\frac{1}{2}} + \beta_t M_1] \\
&\quad + 2\beta_t(\|f(x_t)\| + \|\rho F z_t\|)\mathbf{\Gamma}_{t+1}^{\frac{1}{2}}.
\end{aligned}$$

Since $0 < \liminf_{t\to\infty} \gamma_t \le \limsup_{t\to\infty} \gamma_t < 1$, $\beta_t \to 0$, $\mathbf{\Gamma}_t - \mathbf{\Gamma}_{t+1} \to 0$, and $\lim_{t\to\infty} \mathbf{\Gamma}_t = d < +\infty$, from the boundedness of $\{x_t\}$, $\{z_t\}$, we infer that

$$\lim_{t\to\infty} \|x_t - z_t\| = 0.$$

So, it follows from (3.34) that

$$\|w_t - x_t\| \le \|w_t - z_t\| + \|z_t - x_t\| \to 0 \quad (t \to \infty). \tag{3.35}$$

Also, from Algorithm 3.1 we obtain that

$$\begin{aligned}\|x_{t+1}-x_t\| &= \left\|\beta_t f(x_t)+(1-\gamma_t)(z_t-x_t)-\beta_t\rho F z_t\right\| \\ &\le (1-\gamma_t)\|z_t-x_t\|+\beta_t\left\|f(x_t)-\rho F z_t\right\| \\ &\le \|z_t-x_t\|+\beta_t\left(\left\|f(x_t)\right\|+\|\rho F z_t\|\right)\to 0 \quad (t\to\infty).\end{aligned} \tag{3.36}$$

In addition, the boundedness of $\{x_t\}$ means there is $\{x_{t_k}\}\subset\{x_t\}$ such that

$$\limsup_{t\to\infty}\left\langle (f-\rho F)q^*, x_t-q^*\right\rangle = \lim_{k\to\infty}\left\langle (f-\rho F)q^*, x_{t_k}-q^*\right\rangle. \tag{3.37}$$

Since $\{x_t\}$ is bounded, we may assume that $x_{t_k}\rightharpoonup\tilde{z}$. We get from (3.37)

$$\begin{aligned}\limsup_{t\to\infty}\left\langle (f-\rho F)q^*, x_t-q^*\right\rangle &= \lim_{k\to\infty}\left\langle (f-\rho F)q^*, x_{t_k}-q^*\right\rangle \\ &= \left\langle (f-\rho F)q^*, \tilde{z}-q^*\right\rangle.\end{aligned} \tag{3.38}$$

Since $x_t-x_{t+1}\to 0$, $w_t-x_t\to 0$, $w_t-y_t\to 0$, and $v_t-z_t\to 0$, by Lemma 3.5 we deduce that $\tilde{z}\in\omega_w(\{x_t\})\subset\Xi$. Hence, from (3.10) and (3.38), one gets

$$\limsup_{t\to\infty}\left\langle (f-\rho F)q^*, x_t-q^*\right\rangle = \left\langle (f-\rho F)q^*, \tilde{z}-q^*\right\rangle \le 0, \tag{3.39}$$

which together with (3.36) leads to

$$\begin{aligned}&\limsup_{t\to\infty}\left\langle (f-\rho F)q^*, x_{t+1}-q^*\right\rangle \\ &\quad = \limsup_{t\to\infty}\left[\left\langle (f-\rho F)q^*, x_{t+1}-x_t\right\rangle+\left\langle (f-\rho F)q^*, x_t-q^*\right\rangle\right] \\ &\quad \le \limsup_{t\to\infty}\left[\left\|(f-\rho F)q^*\right\|\|x_{t+1}-x_t\|+\left\langle (f-\rho F)q^*, x_t-q^*\right\rangle\right]\le 0.\end{aligned} \tag{3.40}$$

Note that $\{\beta_t(\zeta-\nu)\}\subset[0,1]$, $\sum_{t=1}^{\infty}\beta_t(\zeta-\nu)=\infty$, and

$$\limsup_{t\to\infty}\left[\frac{2\langle (f-\rho F)q^*, x_{t+1}-q^*\rangle}{\zeta-\nu}+\frac{M}{\zeta-\nu}\cdot\frac{\alpha_t}{\beta_t}\cdot\|x_t-x_{t-1}\|\right]\le 0.$$

By Lemma 2.4 and (3.31), $\lim_{t\to\infty}\|x_t-q^*\|^2=0$.

Case 2. Suppose that $\exists\{\boldsymbol{\Gamma}_{t_k}\}\subset\{\boldsymbol{\Gamma}_t\}$ such that $\boldsymbol{\Gamma}_{t_k}<\boldsymbol{\Gamma}_{t_k+1}$ $\forall k\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\phi:\mathcal{N}\to\mathcal{N}$ by

$$\phi(t):=\max\{k\le t:\boldsymbol{\Gamma}_k<\boldsymbol{\Gamma}_{k+1}\}.$$

By Lemma 2.6, we get

$$\boldsymbol{\Gamma}_{\phi(t)}\le\boldsymbol{\Gamma}_{\phi(t)+1} \quad \text{and} \quad \boldsymbol{\Gamma}_t\le\boldsymbol{\Gamma}_{\phi(t)+1}.$$

From (3.29) we have

$$\begin{aligned}
&(1-\beta_{\phi(t)}\zeta-\gamma_{\phi(t)})\biggl[\biggl(1-\mu\frac{\lambda_{\phi(t)}}{\lambda_{\phi(t)+1}}\biggr)\bigl(\|w_{\phi(t)}-y_{\phi(t)}\|^2+\|v_{\phi(t)}-y_{\phi(t)}\|^2\bigr)\\
&\quad+\epsilon^2\bigl\|T^*(I-S)Tv_{\phi(t)}\bigr\|^2\biggr]\\
&\quad\le\bigl\|x_{\phi(t)}-q^*\bigr\|^2-\bigl\|x_{\phi(t)+1}-q^*\bigr\|^2+\beta_{\phi(t)}M_4\\
&\quad=\mathbf{\Gamma}_{\phi(t)}-\mathbf{\Gamma}_{\phi(t)+1}+\beta_{\phi(t)}M_4,
\end{aligned}\tag{3.41}$$

which immediately yields

$$\lim_{t\to\infty}\|w_{\phi(t)}-y_{\phi(t)}\|=\lim_{t\to\infty}\|v_{\phi(t)}-y_{\phi(t)}\|=\lim_{t\to\infty}\bigl\|T^*(I-S)Tv_{\phi(t)}\bigr\|=0.$$

Similar to Case 1,

$$\begin{aligned}
&\lim_{t\to\infty}\|v_{\phi(t)}-z_{\phi(t)}\|=\lim_{t\to\infty}\|w_{\phi(t)}-x_{\phi(t)}\|=\lim_{t\to\infty}\|x_{\phi(t)+1}-x_{\phi(t)}\|=0,\\
&\limsup_{t\to\infty}\bigl\langle(f-\rho F)q^*,x_{\phi(t)+1}-q^*\bigr\rangle\le 0.
\end{aligned}\tag{3.42}$$

By (3.31),

$$\begin{aligned}
\beta_{\phi(t)}(\zeta-\nu)\mathbf{\Gamma}_{\phi(t)}&\le\mathbf{\Gamma}_{\phi(t)}-\mathbf{\Gamma}_{\phi(t)+1}+\beta_{\phi(t)}(\zeta-\nu)\biggl[\frac{2\langle(f-\rho F)q^*,x_{\phi(t)+1}-q^*\rangle}{\zeta-\nu}\\
&\quad+\frac{M}{\zeta-\nu}\cdot\frac{\alpha_{\phi(t)}}{\beta_{\phi(t)}}\cdot\|x_{\phi(t)}-x_{\phi(t)-1}\|\biggr]\\
&\le\beta_{\phi(t)}(\zeta-\nu)\biggl[\frac{2\langle(f-\rho F)q^*,x_{\phi(t)+1}-q^*\rangle}{\zeta-\nu}\\
&\quad+\frac{M}{\zeta-\nu}\cdot\frac{\alpha_{\phi(t)}}{\beta_{\phi(t)}}\cdot\|x_{\phi(t)}-x_{\phi(t)-1}\|\biggr],
\end{aligned}$$

and so

$$\begin{aligned}
&\limsup_{t\to\infty}\mathbf{\Gamma}_{\phi(t)}\\
&\quad\le\limsup_{t\to\infty}\biggl[\frac{2\langle(f-\rho F)q^*,x_{\phi(t)+1}-q^*\rangle}{\zeta-\nu}+\frac{M}{\zeta-\nu}\cdot\frac{\alpha_{\phi(t)}}{\beta_{\phi(t)}}\cdot\|x_{\phi(t)}-x_{\phi(t)-1}\|\biggr]\\
&\quad\le 0.
\end{aligned}$$

Thus, $\lim_{t\to\infty}\|x_{\phi(t)}-q^*\|^2=0$. Also note that

$$\begin{aligned}
\bigl\|x_{\phi(t)+1}-q^*\bigr\|^2-\bigl\|x_{\phi(t)}-q^*\bigr\|^2&=2\bigl\langle x_{\phi(t)+1}-x_{\phi(t)},x_{\phi(t)}-q^*\bigr\rangle\\
&\quad+\|x_{\phi(t)+1}-x_{\phi(t)}\|^2\\
&\le 2\|x_{\phi(t)+1}-x_{\phi(t)}\|\bigl\|x_{\phi(t)}-q^*\bigr\|\\
&\quad+\|x_{\phi(t)+1}-x_{\phi(t)}\|^2.
\end{aligned}\tag{3.43}$$

Owing to $\mathbf{\Gamma}_t \le \mathbf{\Gamma}_{\phi(t)+1}$, we get

$$\begin{aligned}\|x_t - q^*\|^2 &\le \|x_{\phi(t)+1} - q^*\|^2 \\ &\le \|x_{\phi(t)} - q^*\|^2 + 2\|x_{\phi(t)+1} - x_{\phi(t)}\|\|x_{\phi(t)} - q^*\| \\ &\quad + \|x_{\phi(t)+1} - x_{\phi(t)}\|^2 \to 0,\end{aligned}$$

i.e., $x_t \to q^*$ as $t \to \infty$. □

Remark 3.2

(i) The results in [21] are extended to develop BSPVIP (1.6) with the CFPP constraint, i.e., the problem of finding $q^* \in \Xi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega$ such that $\langle (\rho F - f)q^*, p - q^*\rangle \ge 0\ \forall p \in \Xi$, where $\Omega = \{z \in \mathrm{VI}(C,A) : Tz \in \mathrm{Fix}(S)\}$ with A being pseudomonotone and Lipschitzian mapping. The results in [21] are extended to develop our triple-adaptive inertial subgradient extragradient rule for settling BSPVIP (1.6) with the CFPP constraint, which is on the basis of the subgradient extragradient method with adaptive step sizes, accelerated inertial approach, hybrid deepest-descent method, and viscosity approximation technique. In [21] the following holds:

$$x_t \to q^* \in \Omega = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{VI}(C,A) \quad \Leftrightarrow \quad \|x_t - x_{t+1}\| \to 0$$

with $q^* = P_\Omega(I - \rho F + f)q^*$. In our results, Lemma 2.6 implies that

$$x_t \to q^* \in \Xi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \{z \in \mathrm{VI}(C,A) : Tz \in \mathrm{Fix}(S)\}$$

with $q^* = P_\Xi(I - \rho F + f)q^*$.

(ii) BSQVIP (1.5) (i.e., the problem of finding $q^* \in \Omega$ such that $\langle Fq^*, p - q^*\rangle \ge 0\ \forall p \in \Omega$, where $\Omega = \{z \in \mathrm{VI}(C,A) : Tz \in \mathrm{Fix}(S)\}$ with A being quasimonotone and Lipschitzian mapping) in [29] is extended to develop BSPVIP (1.6) with the CFPP constraint, i.e., the problem of finding $q^* \in \Xi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega$ such that $\langle (\rho F - f)q^*, p - q^*\rangle \ge 0\ \forall p \in \Xi$, where $\Omega = \{z \in \mathrm{VI}(C,A) : Tz \in \mathrm{Fix}(S)\}$ with A being pseudomonotone and Lipschitzian mapping.

4 Numerical implementation

In this section, we compare our proposed Algorithm 3.1 with Algorithm 1 of [27] using the example below. All codes were written in MATLAB R2017a and performed on a PC Desktop Intel(R) Core(TM) i7-8700U CPU @ 3.20GHz 3.19GHz, RAM 8.00 GB.

Suppose that $H_1 = H_2 = L_2([0,1])$ is endowed with the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)\,dt$, $\forall x, y \in L_2([0,1])$ and the induced norm $\|x\| := \int_0^1 |x(t)|^2\,dt$, $\forall x, y \in L_2([0,1])$. Let $T : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Tx(s) = \int_0^1 e^{-st}x(t)\,dt, \quad \forall x \in L_2([0,1]), \forall s, t \in [0,1].$$

Then T is a bounded linear operator with adjoint

$$T^*x(s) = \int_0^1 e^{-st}x(t)\,dt, \quad \forall x \in L_2\big([0,1]\big), \forall s,t \in [0,1].$$

Let $C = \{x \in L_2([0,1]) : \langle t+1, x\rangle \leq 1\}$. Then C is a nonempty closed and convex subset. The projection P_C is given as

$$P_C(x) = \begin{cases} \frac{1-\langle t+1,x\rangle}{\|y\|^2}(t+1) + x, & \text{if } \langle t+1,x\rangle > 1, \\ x, & \text{if } \langle t+1,x\rangle \leq 1. \end{cases}$$

Also, let $Q = \{x \in L_2([0,1]) : \|x\| \leq 2\}$. Then Q is a nonempty closed and convex subset. P_Q is

$$P_Q(x) = \begin{cases} x & \text{if } x \in Q, \\ \frac{2x}{\|x\|} & \text{if otherwise.} \end{cases}$$

Let $A : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Ax(t) := e^{-\|x\|^2}\int_0^t x(s)\,ds, \quad \forall x \in L_2\big([0,1]\big), t \in [0,1].$$

Then A is pseudomonotone and Lipschitz continuous but not monotone. Also define $B : L_2([0,1]) \to L_2([0,1])$ by

$$Bx(t) := \max\big\{x(t), 0\big\}, \quad \forall t \in [0,1].$$

Take $f(x) = \frac{x}{2}$, $x \in L_2([0,1])$, $\beta_t = \frac{1}{t+1}$ and $F = I$.

To test the algorithms, we choose the following parameters for the algorithm: for our algorithm, we used $\lambda_1 = 0.06$, $\epsilon = 10^{-4}$, $\sigma = 0.5$, $\mu = 0.06$, $\alpha = 10^{-3}$, $\epsilon_t = (t+1)^{-2}$, $\beta_t = (t+1)^{-1}$, $\gamma_t = 2t(5t+9)^{-1}$, $\rho = 0.07$. For Anh's algorithm, we choose $\eta = 0.06$, $\gamma = 0.05$, $\mu = 0.07$, $\delta_t = 10^{-3}$, $\lambda_t = 2t(5t+1)^{-1}$, $\alpha_t = (t+1)^{-1}$. We used $Err = \|x_{t+1} - x_t\| < 10^{-4}$ as a stopping criterion for each algorithm. We test the algorithms using the following starting points:

Case I: $x_0 = 2t^2 + 1$, $x_1 = \exp(3t)$

Case II: $x_0 = 2t^2 - 2t + 1$, $x_1 = -4(t^3 + 2t - 3)$;

Case III: $x_0 = t^4 - 1$, $x_1 = t^5 - 9$;

Case IV: $x_0 = \frac{1}{4}t^2 + 2t$, $x_1 = \frac{1}{3}\cos(2t)$.

The numerical results are shown in Table 1 and Fig. 1.

Algorithm 4.1

Initialization: Let $\lambda_1 > 0$, $\epsilon > 0$, $\sigma \geq 0$, $\mu \in (0,1)$, $\alpha \in [0,1)$, and $x_0, x_1 \in \mathcal{H}_1$ be arbitrary.

Iterative steps: Calculate x_{t+1} as follows:

Step 1. Given the iterates x_{t-1} and x_t $(t \geq 1)$, choose α_t such that $0 \leq \alpha_t \leq \bar{\alpha}_t$, where

$$\bar{\alpha}_t = \begin{cases} \min\{\alpha, \frac{\varepsilon_t}{\|x_t - x_{t-1}\|}\} & \text{if } x_t \neq x_{t-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{4.1}$$

Table 1 Computational result

		Algorithm 4.1	Anh's algorithm
Case I	No of Iter.	8	271
	CPU time (sec)	2.1034	10.2340
Case II	No of Iter.	8	285
	CPU time (sec)	3.7897	11.7137
Case III	No of Iter.	9	291
	CPU time (sec)	3.5364	19.1699
Case IV	No of Iter.	7	133
	CPU time (sec)	1.6817	7.7101

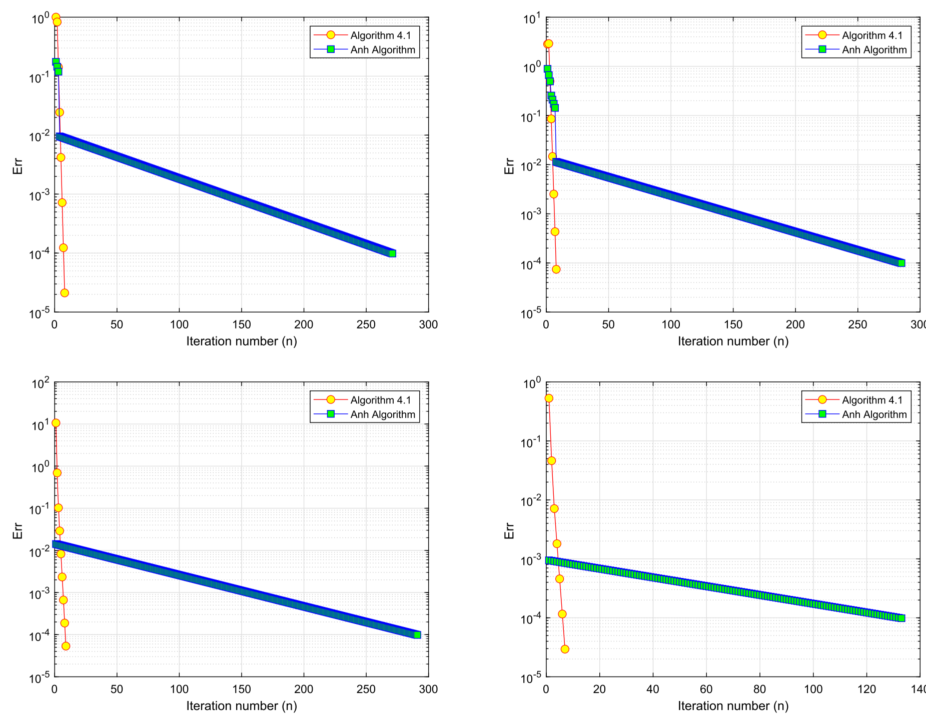

Figure 1 Numerical results, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV

Step 2. Compute $w_t = x_t + \alpha_t(x_t - x_{t-1})$ and $y_t = P_C(w_t - \lambda_t A w_t)$.

Step 3. Construct $C_t := \{y \in \mathcal{H}_1 : \langle w_t - \lambda_t A w_t - y_t, y_t - y\rangle \geq 0\}$, and compute $v_t = P_{C_t}(w_t - \lambda_t A y_t)$ and $z_t = v_t - \sigma_t T^*(I-S)Tv_t$, where $S = P_Q(I-\varphi B) - \varphi(B(P_Q(I-\varphi B)) - B)$ and $\varphi \in (0,1)$.

Step 4. Calculate $x_{t+1} = \beta_t \frac{x_t}{2} + \gamma_t x_t + ((1-\gamma_t)I - \beta_t\rho)z_t$ and update

$$\lambda_{t+1} = \begin{cases} \min\{\mu \frac{\|w_t - y_t\|^2 + \|v_t - y_t\|^2}{2\langle Aw_t - Ay_t, v_t - y_t\rangle}, \lambda_t\} & \text{if } \langle Aw_t - Ay_t, v_t - y_t\rangle > 0, \\ \lambda_t & \text{otherwise,} \end{cases} \tag{4.2}$$

and for any fixed $\epsilon > 0$, σ_t is chosen to be the bounded sequence satisfying

$$0 < \epsilon \leq \sigma_t \leq \frac{(1-\tau)\|Tv_t - STv_t\|^2}{\|T^*(Tv_t - STv_t)\|^2} - \epsilon \quad \text{if } Tv_t \neq STv_t, \tag{4.3}$$

otherwise set $\sigma_t = \sigma \geq 0$.

Set $t := t + 1$ and go to Step 1.

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Abbreviations
VIP, Variational inequality problem; BSPVIP, Bilevel split pseudomonotone variational inequality problem; CFPP, Common fixed point problem.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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