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The study of coefficient estimates and Fekete–Szegö inequalities for the new classes of m-fold symmetric bi-univalent functions defined using an operator

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Abstract
The objective of this paper is to introduce new classes of m-fold symmetric bi-univalent functions. We discuss estimates on the Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$, and the Fekete–Szegő problem is also considered for the new classes of functions introduced. We denote these classes by $MF - S^{p,q}_{\Sigma,m}(h)$, $MF - S^{p,q}_{\Sigma,m}(s)$, and $MF - S^{b,d}_{\Sigma,m}$. Quantum calculus aspects are also considered in this study to enhance its novelty and to obtain more interesting results.

MSC: 30C45; 30C50

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1 Introduction and preliminary results

Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$.

The subclass $S \subset \mathcal{A}$ is formed of all functions in the class $\mathcal{A}$ that are univalent in U (see[14]).

The Koebe one-quarter theorem ensures that the image of the unit disk under every $f \in S$ function contains a disk of radius 1/4, see [14].

If the function $f \in S$, then it has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

and

$$f\big(f^{-1}(w)\big) = w, \quad |w| < r_0(f), r_0(f) \geq 1/4,$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \big(2a_2^2 - a_3\big)w^3 - \big(5a_2^3 - 5a_2 a_3 + a_4\big)w^4 + \cdots. \tag{2}$$

We say that a function $f \in \mathcal{A}$ is bi-univalent in U if both f and f^{-1} are univalent in U.

We denote by Σ the class of all bi-univalent functions in U given by (1).

The study on bi-univalent functions has its origins in the article published by Lewin in [25], where it was shown that $|a_2| < 1.51$.

The domain D is m-fold symmetric if a rotation of D about the origin through an angle $2\pi/m$ carries D on itself.

The holomorphic function f in the domain D is m-fold symmetric if the following condition is true: $f(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f(z)$.

Definition 1 ([36]) A function f is said to be m-fold symmetric if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in U, m \in \mathbb{N} \cup \{0\}. \tag{3}$$

The normalized form of f is given as in (3), and the series expansion for $f^{-1}(z)$ is given below (see[4]):

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_{m+1} w^{m+1} \\ &\quad + \big[(m+1)a_{m-1}^2 - a_{2m+1}\big] w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots. \end{aligned} \tag{4}$$

Examples of m-fold symmetric bi-univalent functions are:

$$\big[-\log\big(1-z^m\big)\big]^{\frac{1}{m}}; \qquad \left\{\frac{z^m}{1-z^m}\right\}^{\frac{1}{m}}; \qquad \frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}.$$

Srivastava et al. in the paper [36] defined m-fold symmetric bi-univalent functions following the concept of m-fold symmetric univalent functions.

The interest in bi-univalent functions resurfaced in 2010 when a paper authored by Srivastava et al. in [35] was published. It opened the door for many interesting developments on the topic. Soon other new subclasses of bi-univalent functions were introduced [19–21] and special classes of bi-univalent functions were investigated such as Ma–Minda starlike and convex functions [3], analytic bi-Bazilevic functions [23], and recently a family of bi-univalent functions associated with Bazilevic functions and the λ-pseudo-starlike functions [38]. Brannan and Clunie's conjecture [8] was further investigated [32] and subordination properties were also obtained for certain subclasses of bi-univalent functions

[11]. New results continued to emerge in the recent years such as coefficient estimates for some general subclasses of analytic and bi-univalent functions [13, 27, 34]. Horadam polynomials were used for applications on Bazilevic bi-univalent functions satisfying subordination conditions [40] and for introducing certain classes of bi-univalent functions [1]. Operators were also included in the study as it can be seen in earlier publications [9] and in very recent ones [28]. Interesting results regarding m-fold symmetric bi-univalent functions were published in the same year when this notion was introduced [21]. This continued to appear in the following years [4, 16, 31, 33] and is still researched today [10, 37], proving that the topic remains in development.

The Fekete–Szegö problem is the problem of maximizing the absolute value of the functional $|a_3 - \mu a_2^2|$.

The Fekete–Szegö inequalities introduced in 1933, see [18], preoccupied researchers regarding different classes of univalent functions [15, 24]. Hence it is obvious that such inequalities were obtained regarding bi-univalent functions too and very recently published papers can be cited to support the assertion that the topic still provides interesting results [2, 6, 41]. Inspiring new results emerged when quantum calculus was involved in the studies, as can be seen in many papers [30] and in studies published very recently [5, 12, 17, 39]. Some elements of the (p,q)-calculus must be used for obtaining the original results contained in this paper. Further information can be found in [22, 30]. The tremendous impact quantum calculus has had when associated with univalent functions theory is nicely highlighted in the recent review paper [22, 30].

For obtaining the original results contained in this paper, some elements of the (p,q)-calculus must be used.

Definition 2 ([22], p. 2) Let $f \in \mathcal{A}$ given by (1) and $0 < q < p \leq 1$. Then the (p,q)-derivative operator for the function f of the form (1) is defined by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \in U^* = U - \{0\} \tag{5}$$

and

$$(D_{p,q}f)(0) = f'(0), \tag{6}$$

it follows that the function f is differentiable at 0.

We can deduce from relation (2) that

$$D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}, \tag{7}$$

where the (p,q)-bracket number is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p-q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \cdots + pq^{k-2} + q^{k-1}, \quad p \neq q$$

which is a natural generalization of the q-number.

We can see that $\lim_{p\to 1^-} [k]_{p,q} = [k]_q = \frac{1-q^k}{1-q}$, see the papers [17, 22].

Definition 3 ([7], p.137) Let the function $f \in \mathcal{A}$, where $0 \leq d < 1$, $s \geq 1$ is real. The function $f \in L_s(d)$ of an s-pseudo-starlike function of order d in the unit disk U if and only if

$$\operatorname{Re}\left(\frac{z[f'(z)]^s}{f(z)}\right) > d.$$

Lemma 4 [14, 29] *Let the function $w \in \mathcal{P}$ be given by the following series $w(z) = 1 + w_1 z + w_2 z^2 + \cdots$, $z \in U$, where we denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the open disk U,*

$$\mathcal{P} = \left\{w \in \mathcal{A} | w(0) = 1, \operatorname{Re}\big(w(z)\big) > 0, z \in U\right\}.$$

The sharp estimate given by $|w_n| \leq 2$, $n \in \mathbb{N}^*$ holds true.

In the next section of the paper, the original results obtained are presented in three definitions of new subclasses of m-fold symmetric bi-univalent functions and theorems concerning coefficient estimates and Fekete–Szegő problem for the newly defined classes.

2 Main results

Definition 5 The function class $M - FS^{p,q}_{\Sigma,m}(h)$, $(m \in \mathbb{N}, 0 < q < p \leq 1, 0 < h \leq 1, (z,w) \in U)$, contains all the functions f given by relation (3) that satisfy the following conditions:

$$\begin{cases} f \in \Sigma_m \\ |\arg\{D_{p,q}f(z) + z(D_{p,q}f(z))'\}| < \frac{h\pi}{2}, \quad (z \in U) \end{cases} \tag{8}$$

and

$$\left|\arg\left\{D_{p,q}g(w) + w\big(D_{p,q}g(w)\big)'\right\}\right| < \frac{h\pi}{2}, \tag{9}$$

where g is given by relation (4).

The coefficient bounds for the functions class $MF - S^{p,q}_{\Sigma,m}(h)$ are obtained in the next theorem.

Theorem 6 *If the function f, given by relation (3), is in the function class $MF - S^{p,q}_{\Sigma,m}(h)$, $(m \in \mathbb{N}, 0 < q < p \leq 1, 0 < h \leq 1, (z,w) \in U)$, then the following inequalities are true*:

$$|a_{m+1}| \leq \frac{2h}{\sqrt{(m+1)h[2m+1]_{p,q}(1+2m) - (h-1)(1+m)^2[m+1]^2_{p,q}}} \tag{10}$$

and

$$|a_{2m+1}| \leq \frac{2h}{(1+2m)[2m+1]_{p,q}} + \frac{2h^2}{(1+m)[m+1]^2_{p,q}}. \tag{11}$$

Proof If we use (8) and (9), we obtain

$$D_{p,q}f(z) + z\big(D_{p,q}f(z)\big)' = \big[\alpha(z)\big]^h, \quad z \in U \tag{12}$$

and

$$D_{p,q}g(w) + w\big(D_{p,q}g(w)\big)' = \big[\beta(w)\big]^h, \quad w \in \mathcal{U}, \tag{13}$$

where $\alpha(z)$ and $\beta(w)$ in $\mathcal{P}$ are given by

$$\alpha(z) = 1 + \alpha_m z^m + \alpha_{2m} z^{2m} + \alpha_{3m} z^{3m} + \cdots \tag{14}$$

and

$$\beta(w) = 1 + \beta_m w^m + \beta_{2m} w^{2m} + \beta_{3m} w^{3m} + \cdots . \tag{15}$$

Comparing the coefficients in (12) and (13), we obtain

$$(1+m)[m+1]_{p,q} a_{m+1} = h\alpha_m, \tag{16}$$

$$(1+2m)[2m+1]_{p,q} a_{2m+1} = h\alpha_{2m} + \frac{h(h-1)}{2}\alpha_m^2, \tag{17}$$

$$-(1+m)[m+1]_{p,q} a_{m+1} = h\beta_m, \tag{18}$$

$$(1+2m)[2m+1]_{p,q}\big((m+1)a_{m+1}^2 - a_{2m+1}\big) = h\beta_{2m} + \frac{h(h-1)}{2}\beta_m^2. \tag{19}$$

From (16) and (18) we obtain

$$\alpha_m = -\beta_m \tag{20}$$

and

$$2(1+m)^2[m+1]_{p,q}^2 a_{m+1}^2 = h^2\big(\alpha_m^2 + \beta_m^2\big). \tag{21}$$

Now, from (17), (19), and (21) we obtain that

$$\begin{aligned}&(m+1)(1+2m)[2m+1]_{p,q} a_{m+1}^2 \\ &\quad = h(\alpha_{2m} + \beta_{2m}) + (h-1)\left[\frac{(1+m)^2[m+1]_{p,q}^2}{h}\right] a_{m+1}^2.\end{aligned}$$

Therefore, we obtain that

$$a_{m+1}^2 = \frac{h^2(\alpha_{2m} + \beta_{2m})}{(m+1)(1+2m)[2m+1]_{p,q}h - (h-1)(1+m)^2[m+1]_{p,q}^2}.$$

Now, for the coefficients α_{2m} and β_{2m}, if we apply Lemma 4, we obtain

$$|a_{m+1}| \le \frac{2h}{\sqrt{h(m+1)(1+2m)[2m+1]_{p,q} - (h-1)(1+m)^2[m+1]_{p,q}^2}}.$$

If we use (17) and (19), then we obtain

$$\begin{aligned}&2(1+2m)[2m+1]_{p,q}a_{2m+1}-(m+1)(1+2m)[2m+1]_{p,q}a_{m+1}^2\\&\quad=h(\alpha_{2m}-\beta_{2m})+\frac{h(h-1)}{2}\left(\alpha_m^2-\beta_m^2\right).\end{aligned}\tag{22}$$

From (20), (21), and (22), we obtain

$$a_{2m+1}=\frac{h(\alpha_{2m}-\beta_{2m})}{2(1+2m)[2m+1]_{p,q}}+\frac{h^2(\alpha_m^2+\beta_m^2)}{4(1+m)[m+1]_{p,q}^2}.\tag{23}$$

If we apply Lemma 4 for the coefficients α_m, α_{2m}, β_m, β_{2m}, we obtain

$$|a_{2m+1}|\le\frac{2h}{(1+2m)[2m+1]_{p,q}}+\frac{2h^2}{(1+m)[m+1]_{p,q}^2}.$$ □

The Fekete–Szegö functional for the class $MF-S_{\Sigma,m}^{p,q}(h)$ is given in the next theorem.

Theorem 7 *Let f be a function of the form (3) in the class $MF-S_{\Sigma,m}^{p,q}(h)$. Then*

$$\left|a_{2m+1}-\rho a_{m+1}^2\right|\le\begin{cases}\frac{2h}{(1+2m)[2m+1]_{p,q}}, & |l(\rho)|\le\frac{1}{(1+2m)[2m+1]_{p,q}},\\ 4h(1+2m)[2m+1]_{p,q}^2|l(\rho)|, & |l(\rho)|\ge\frac{1}{(1+2m)[2m+1]_{p,q}},\end{cases}\tag{24}$$

where we denote

$$l(\rho)=\frac{h\{m+1-2\rho\}}{2\{h[2m+1]_{p,q}(1+2m)-[m+1]_{p,q}^2(h-1)(1+m)\}}.$$

Proof The values of the coefficients a_{m+1}^2 and a_{2m+1} are given in the proof of Theorem 6 as follows:

$$\begin{aligned}a_{2m+1}&=\frac{h(\alpha_{2m}-\beta_{2m})}{2(1+2m)[2m+1]_{p,q}}+\frac{h^2(\alpha_m^2+\beta_m^2)}{4(1+m)[m+1]_{p,q}^2},\\ a_{m+1}^2&=\frac{h^2(\alpha_{2m}+\beta_{2m})}{h(m+1)(1+2m)[2m+1]_{p,q}-(h-1)(1+m)^2[m+1]_{p,q}^2}.\end{aligned}$$

We start to compute $a_{2m+1}-\rho a_{m+1}^2$.

It follows that

$$\begin{aligned}&a_{2m+1}-\rho a_{m+1}^2\\&\quad=h\left\{\alpha_{2m}\left[\frac{1}{2(1+2m)[2m+1]_{p,q}}\right.\right.\\&\qquad\left.+\frac{h(m+1-2\rho)}{2\{[2m+1]_{p,q}(1+2m)h-(h-1)(1+m)[m+1]_{p,q}^2\}}\right]\\&\qquad+\beta_{2m}\left[-\frac{1}{2(1+2m)[2m+1]_{p,q}}\right.\\&\qquad\left.+\frac{h(m+1-2\rho)}{2[h(1+2m)[2m+1]_{p,q}-(h-1)(1+m)[m+1]_{p,q}^2]}\right].\end{aligned}$$

After some computations and according to Lemma 4, we obtain

$$\left|a_{2m+1}-\rho a_{m+1}^2\right| \le \begin{cases} \frac{2h}{(1+2m)[2m+1]_{p,q}}, & |l(\rho)| \le \frac{1}{(1+2m)[2m+1]_{p,q}}, \\ 4h(1+2m)[2m+1]_{p,q}^2|l(\rho)|, & |l(\rho)| \ge \frac{1}{(1+2m)[2m+1]_{p,q}}. \end{cases}$$ □

Definition 8 The function class $MF-S_{\Sigma,m}^{p,q}(s), (0<q<p\le 1, 0\le s<1, m\in\mathbb{N}, (z,w)\in U)$, contains all the functions f given by relation (3) that satisfy the following conditions:

$$\begin{cases} f\in\Sigma_m \\ \operatorname{Re}\{D_{p,q}f(z)+z(D_{p,q}f(z))'\}>s, \quad z\in U \end{cases} \tag{25}$$

$$\operatorname{Re}\left\{D_{p,q}g(w)+w\left(D_{p,q}g(w)\right)'\right\}>s, \quad w\in U \tag{26}$$

where the function g is of the form (4).

Coefficient bounds for the functions class $MF-S_{\Sigma,m}^{p,q}(s)$ are obtained in the next theorem.

Theorem 9 *Let f be a function in the class $MF-S_{\Sigma,m}^{p,q}(s), (m\in\mathbb{N}, 0<q<p\le 1, 0\le s<1, (z,w)\in U)$, which has the form (3). Then*

$$|a_{m+1}| \le \min\left\{\frac{2(1-s)^2}{(1+m)^2[m+1]_{p,q}^2}, 2\sqrt{\frac{(1-s)}{(m+1)(1+2m)[2m+1]_{p,q}}}\right\} \tag{27}$$

and

$$|a_{2m+1}| \le \frac{2(1-s)}{(1+2m)[2m+1]_{p,q}}. \tag{28}$$

Proof We can see that from (24) and (25) we obtain

$$D_{p,q}f(z)+z\left(D_{p,q}f(z)\right)'=s+(1-s)\alpha(z), \quad z\in U \tag{29}$$

and

$$D_{p,q}g(w)+w\left(D_{p,q}g(w)\right)'=s+(1-s)\beta(w), \quad w\in U, \tag{30}$$

where $\alpha(z)$ and $\beta(w)$ in $\mathcal{P}$ are given by (14) and (15).

Now we compare the coefficients from (28) and (29), and we obtain

$$(1+m)[m+1]_{p,q}a_{m+1}=(1-s)\alpha_m, \tag{31}$$

$$(1+2m)[2m+1]_{p,q}a_{2m+1}=(1-s)\alpha_{2m}, \tag{32}$$

$$-(1+m)[m+1]_{p,q}a_{m+1}=(1-s)\beta_m, \tag{33}$$

$$(1+2m)[2m+1]_{p,q}\left[(m+1)a_{m+1}^2-a_{2m+1}\right]=(1-s)\beta_{2m}. \tag{34}$$

We obtain from (30) and (32) that

$$\alpha_m=-\beta_m \tag{35}$$

and

$$2(1+m)^2[m+1]_{p,q}^2 a_{m+1}^2 = (1-s)^2\left(\alpha_m^2+\beta_m^2\right). \tag{36}$$

From (33) and (31) we obtain

$$(1+2m)[2m+1]_{p,q}(m+1)a_{m+1}^2 = (1-s)(\alpha_{2m}+\beta_{2m}). \tag{37}$$

If we apply Lemma 4 for the coefficients α_m, α_{2m}, β_m, β_{2m}, then we obtain

$$|a_{m+1}| \le 2\sqrt{\frac{1-s}{[2m+1]_{p,q}(m+1)(1+2m)}}.$$

Using (33) and (31) to find the bound on $|a_{2m+1}|$, we obtain

$$\begin{aligned} &-(m+1)(1+2m)[2m+1]_{p,q}a_{m+1}^2 + 2(1+2m)[2m+1]_{p,q}a_{2m+1} \\ &\quad = (1-s)(\alpha_{2m}-\beta_{2m}), \end{aligned} \tag{38}$$

or equivalently

$$a_{2m+1} = \frac{(1-s)(\alpha_{2m}-\beta_{2m})}{2(1+2m)[2m+1]_{p,q}} + \frac{(m+1)}{2}a_{m+1}^2. \tag{39}$$

From (35) we substitute the value of a_{m+1}^2 and obtain

$$a_{2m+1} = \frac{(1-s)(\alpha_{2m}-\beta_{2m})}{2(1+2m)[2m+1]_{p,q}} + \frac{(1-s)^2(\alpha_m^2+\beta_m^2)}{4(1+m)[m+1]_{p,q}^2}. \tag{40}$$

Now, we will apply Lemma 4 for the coefficients α_m, α_{2m}, β_m, β_{2m}, and we obtain

$$|a_{2m+1}| \le \frac{2(1-s)}{(1+2m)[2m+1]_{p,q}} + \frac{2(1-s)^2}{(1+m)[m+1]_{p,q}^2}.$$

From (36) and (38), if we apply again Lemma 4, then we obtain

$$|a_{2m+1}| \le \frac{2(1-s)}{(1+2m)[2m+1]_{p,q}}. \qquad \square$$

In the next theorem we compute the Fekete–Szegö functional for the class $MF-S_{\Sigma,m}^{p,q}(s)$.

Theorem 10 *Let f be a function of the form* (3) *in the class $MF-S_{\Sigma,m}^{p,q}(s)$. Then*

$$\left|a_{2m+1}-\rho a_{m+1}^2\right| \le \begin{cases} \frac{2(1-s)}{(1+2m)[2m+1]_{p,q}}, & |l(\rho)| \le \frac{1}{2(1+2m)[2m+1]_{p,q}}, \\ 4(1+2m)(1-s)[2m+1]_{p,q}^2|l(\rho)|, & |l(\rho)| \ge \frac{1}{2(1+2m)[2m+1]_{p,q}}, \end{cases} \tag{41}$$

where $l(\rho)$ is given by

$$l(\rho) = \frac{(1-s)(1+2m)[2m+1]_{p,q} - 4\rho[m+1]_{p,q}^2}{4(1+m)[m+1]_{p,q}^2(1+2m)[2m+1]_{p,q}}.$$

Proof Using the values of a_{m+1}^2 and a_{2m+1} from the proof of Theorem 9, we can compute $a_{2m+1} - \rho a_{m+1}^2$.

$$a_{2m+1} = \frac{(1-s)(\alpha_{2m} - \beta_{2m})}{2[1+2m]_{p,q}(1+2m)} + \frac{(1-s)^2(\alpha_{2m} + \beta_{2m})}{4(1+m)[m+1]_{p,q}^2},$$
$$a_{m+1}^2 = \frac{(1-s)(\alpha_{2m} + \beta_{2m})}{(1+2m)[2m+1]_{p,q}(m+1)}.$$

We obtain

$$\begin{aligned}
&a_{2m+1} - \rho a_{m+1}^2 \\
&\quad = (1-s)\Bigg\{\alpha_{2m}\Bigg[\frac{1}{2[1+2m]_{p,q}(1+2m)} \\
&\qquad + \frac{(1-s)(2m+1)[1+2m]_{p,q} - 4\rho[m+1]_{p,q}^2}{4(1+m)[m+1]_{p,q}^2(1+2m)[2m+1]_{p,q}}\Bigg] \\
&\qquad + \beta_{2m}\Bigg[\frac{(1-s)(1+2m)[2m+1]_{p,q} - 4\rho[m+1]_{p,q}^2}{4(1+m)[m+1]_{p,q}^2(1+2m)[2m+1]_{p,q}} \\
&\qquad - \frac{1}{2[1+2m]_{p,q}(1+2m)}\Bigg]\Bigg\}.
\end{aligned}$$

The next inequality is obtained after some computations and according to Lemma 4:

$$\left|a_{2m+1} - \rho a_{m+1}^2\right| \le \begin{cases} \frac{2(1-s)}{(1+2m)[2m+1]_{p,q}}, & |l(\rho)| \le \frac{1}{2(1+2m)[2m+1]_{p,q}}, \\ 4(1+2m)(1-s)[2m+1]_{p,q}^2|l(\rho)|, & |l(\rho)| \ge \frac{1}{2(1+2m)[2m+1]_{p,q}}. \end{cases} \qquad \square$$

Definition 11 Let $b, d : U \to \mathbb{C}$ be analytic functions with the property $\min\{\mathrm{Re}(b(z)), \mathrm{Re}(d(z))\} > 0$, where $z \in U$, $b(0) = d(0) = 1$.

The class $MF - S_{\Sigma,m}^{b,d}$ contains all the functions f given by (3) if the following conditions are satisfied:

$$\left(D_{p,q}f(z) + z\left(D_{p,q}f(z)\right)'\right) \in b(U), \quad z \in U \tag{42}$$

and

$$\left(D_{p,q}g(w) + w\left(D_{p,q}g(w)\right)'\right) \in d(U), \quad w \in U \tag{43}$$

where the function g is given by (4).

In the next theorem we obtain the coefficient bounds for the function class $MF - S_{\Sigma,m}^{b,d}$.

Theorem 12 *If the function f of the form (3) is in the class $MF - S_{\Sigma,m}^{b,d}$, then the following inequalities are satisfied*:

$$|a_{m+1}| \le \min\left\{\sqrt{\frac{|b_1'(0)|^2 + |d_1'(0)|^2}{2(1+m)^2[m+1]_{p,q}^2}}, \sqrt{\frac{|b_2''(0)| + |d_2''(0)|}{(1+2m)(m+1)[2m+1]_{p,q}}}\right\} \tag{44}$$

and (45)

$$|a_{2m+1}| \le \min\left\{ \frac{(|b'(0)|^2 + |d'(0)|^2)}{4(1+m)[2m+1]_{p,q}^2} + \frac{|b''(0)|^2 + |d''(0)|^2}{2(1+2m)[2m+1]_{p,q}}, \frac{|b''(0)| + |d''(0)|}{2(1+2m)[2m+1]_{p,q}} + \frac{(|b''(0)| + |d''(0)|)}{2(1+2m)[2m+1]_{p,q}} \right\}. \tag{46}$$

Proof We can write relations (42) and (43) as follows:

$$D_{p,q}f(z) + z\big(D_{p,q}f(z)\big)' = b(z) \tag{47}$$

and

$$D_{p,q}g(w) + w\big(D_{p,q}g(w)\big)' = d(w), \tag{48}$$

where the functions b and d have the following forms and satisfy the conditions from Definition 11:

$$b(z) = 1 + b_1 z + b_2 z^2 + \cdots, \tag{49}$$

$$d(w) = 1 + d_1 w + d_2 w^2 + \cdots. \tag{50}$$

Substituting relations (49) and (50) into (47) and (48), respectively, and equating the coefficients, we obtain

$$(1+m)[m+1]_{p,q} a_{m+1} = b_1; \tag{51}$$

$$(1+2m)[2m+1]_{p,q} a_{2m+1} = b_2; \tag{52}$$

$$-(1+m)[m+1]_{p,q} a_{m+1} = d_1; \tag{53}$$

$$(1+2m)[2m+1]_{p,q}\big((m+1)a_{m+1}^2 - a_{2m+1}\big) = d_2. \tag{54}$$

We obtain from (51) and (53) that

$$b_1 = -d_1 \tag{55}$$

and

$$b_1^2 + d_1^2 = 2(1+m)^2[m+1]_{p,q}^2 a_{m+1}^2. \tag{56}$$

Adding relations (52) and (54), we obtain

$$\big\{(1+2m)(m+1)[2m+1]_{p,q}\big\} a_{m+1}^2 = b_2 + d_2. \tag{57}$$

From (56) and (57), we obtain

$$a_{m+1}^2 = \frac{b_1^2 + d_1^2}{2(1+m)^2[m+1]_{p,q}^2} \tag{58}$$

and

$$a_{m+1}^2 = \frac{b_2 + d_2}{(1+2m)(m+1)[2m+1]_{p,q}}. \tag{59}$$

We find from (58) and (59) that

$$|a_{m+1}|^2 \le \frac{|b_1'(0)|^2 + |d_1'(0)|^2}{2(1+m)^2[m+1]_{p,q}^2}$$

and

$$|a_{m+1}|^2 \le \frac{|b_2''(0)| + |d_2''(0)|}{(1+2m)(m+1)[2m+1]_{p,q}}.$$

We get in this way the desired estimate on the coefficient $|a_{m+1}|$ as asserted in (44).

By subtracting (54) from (52), we obtain

$$\begin{aligned}&2(1+2m)[2m+1]_{p,q}a_{2m+1} - (1+2m)[2m+1]_{p,q}(m+1)a_{m+1}^2\\&\quad= b_2 - d_2.\end{aligned} \tag{60}$$

It follows that

$$a_{2m+1} = \frac{b_2 - d_2}{2(1+2m)[2m+1]_{p,q}} + \frac{b_1^2 + d_1^2}{4(1+m)[2m+1]_{p,q}^2},$$

using the value of a_{m+1}^2 from (58) into (60).
Hence,

$$|a_{2m+1}| \le \frac{(|b'(0)|^2 + |d'(0)|^2)}{4(1+m)[2m+1]_{p,q}^2} + \frac{|b''(0)|^2 + |d''(0)|^2}{2(1+2m)[2m+1]_{p,q}}.$$

Using in (60) a_{m+1}^2 given by (59), we have

$$a_{2m+1} = \frac{b_2 - d_2}{2(1+2m)[2m+1]_{p,q}} + \frac{b_2 + d_2}{2(1+2m)[2m+1]_{p,q}}.$$

It follows that

$$|a_{2m+1}| \le \frac{|b''(0)| + |d''(0)|}{2(1+2m)[2m+1]_{p,q}} + \frac{|b''(0)| + |d''(0)|)}{2(1+2m)[2m+1]_{p,q}}. \qquad \square$$

3 Conclusion

These classes of functions introduced in this paper can be extended and similar properties to those presented can be studied. Using the same research method as in the paper [36], we introduce in Definitions 5, 8, and 11 three new classes of m-fold symmetric bi-univalent functions. As future research, using other operators or the (p,q)-derivative operator, properties of starlikeness, convexity, and close-to-convexity of the new classes of functions could be investigated, and we can study the properties of symmetry of (p,q)-derivative operator. We believe that this study will motivate a number of researchers to extend this idea for other functions and classes of functions.

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