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Research Article

Total outer-convex domination number of graphs

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Abstract: In this paper, we initiate the study of total outer-convex domination as a new variant of graph domination and we show the close relationship that exists between this novel parameter and other domination parameters of a graph such as total domination, convex domination, and outer-convex domination. Furthermore, we obtain general bounds of total outer-convex domination number and, for some particular families of graphs, we obtain closed formulas.

Keywords: Domination number, total domination number, convex domination, total outer-convex domination number, grid graphs

AMS Subject classification: 05C69

1. Introduction

Graph theory is one of the most developed branches of modern mathematics and computer applications and dominations in graphs is its most researched sub branch [8]. Its interrelated general concepts allow different domination types to exist [10]. Dominating set and its variants have a wide range of applications and model various real-life problems. In this paper, we introduce total outer-convex domination as a new variant in graph domination and show the close relationship that exists between this novel parameter and other domination parameters of a graph. Further, general

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bounds on total outer-convex domination and closed formulas for some families of graphs were obtained.

Let $G = (V(G), E(G))$ be a simple graph. A graph G is *connected* if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is *disconnected*. For any two vertices u and v in a connected graph, the distance $d_G(u, v)$ between u and v is the length of the shortest path in G. A u-v path of length $d_G(u, v)$ is also referred to as u-v *geodesic*. The *closed interval* $I_G[u, v]$ consists of all those vertices lying on a u-v geodesic in G. For a subset S of vertices of G, the union of all sets $I_G[u, v]$ for $u, v \in S$ is denoted by $I_G[S]$. Hence $x \in I_G[S]$ if and only if x lies on some u-v geodesic, where $u, v \in S$. A set $S \subseteq V(G)$ is *convex* if $I_G[S] = S$. In other words, a set S is *convex* in G if, for every two vertices $u, v \in S$, the vertex set of every u - v *geodesic* is contained in S. Certainly, if G is connected graph, then $V(G)$ is convex. Convexity and geodetic in graphs was studied in [1–5, 9]. Let $K_n, P_n, C_n, W_n, F_n (F_{r,s}$ with $n = r + s)$ and S_n denote a complete graph, the path, the cycle, the wheel, the fan and the star graph of order n, respectively.

A subset S of a vertex set $V(G)$ is a *dominating set* of G if for every vertex $v \in V(G) \backslash S$, there exists a vertex $x \in S$ such that xv is an edge of G. A dominating set S is an outer-convex dominating set if the subgraph induced by $V(G) \setminus S$, denoted $\langle V(G) \backslash S \rangle$, is convex. The set $S \subseteq V(G)$ is a total dominating set if every vertec $v \in V(G)$ is adjacent to an element of S. The minimum cardinality of a dominating set, a total dominating set, an outer-convex dominating set are the domination number $\gamma(G)$, the total domination number $\gamma_t(G)$, and the outer-convex domination number $\widetilde{\gamma}_{con}(G)$, respectively. The outer-convex domination was introduced by Dayap and Enriquez in 2020 [7] and further studied in [6] by using the said parameter as a tool in encrypting messages and as a new variation of domination parameter in [11].

Motivated by the definition of total domination and outer-convex domination in graphs, we define a new domination parameter in graphs called *total outer-convex domination*. A total dominating set S of vertices of a graph G is a total outer-convex dominating set if the subgraph induced by $V(G) \setminus S$ is convex. The *total outer-convex domination number* of G, denoted by $\widetilde{\gamma}_{tcon}(G)$, is the minimum cardinality of a total outer-convex dominating set of G. A total outer-convex dominating set of cardinality $\widetilde{\gamma}_{tcon}(G)$ will be called a $\widetilde{\gamma}_{tcon}$-*set*.

Since every total outer-convex dominating set of G is a total dominating set of G and an outer-convex dominating set of G, we have

$$\gamma_t(G) \leq \widetilde{\gamma}_{tcon}(G), \tag{1}$$

and

$$\widetilde{\gamma}_{con}(G) \leq \widetilde{\gamma}_{tcon}(G). \tag{2}$$

The next result is a direct consequence of inequalities (1) and (2).

Corollary 1. *Let G be a non-trivial connected graph. Then, we have the following:*

(i) $\gamma(G) \leq \widetilde{\gamma}_{con}(G) \leq \widetilde{\gamma}_{tcon}(G)$

(ii) $\gamma(G) \leq \gamma_t(G) \leq \widetilde{\gamma}_{tcon}(G)$.

2. Preliminary Results

In this section, we study basic properties of the total outer-convex domination number of graphs.

Proposition 1. *Let G be a connected graph of order $n \geq 3$ and minimum degree $\delta(G) = 1$. Then any $\widetilde{\gamma}_{tcon}$-set of G contains all support vertices of G.*

Proof. Let S be a $\widetilde{\gamma}_{tcon}(G) - set$. Let p be a support vertex and q a leaf adjacent to p. Since S is a total dominating set in G, to total dominate q we must have $p \in S$. Thus, S contains all support vertices of G. □

Proposition 2. *Let G be a connected graph of order $n \geq 3$ with $\widetilde{\gamma}_{tcon}(G) \leq n-2$. Then any $\widetilde{\gamma}_{tcon}$-set of G contains all leaves of G.*

Proof. The result is trivial if $\delta(G) \geq 2$. Let $\delta(G) = 1$ and S be a $\widetilde{\gamma}_{tcon}(G) - set$ of G. Let q be a leaf of G and let p be its support vertex. By Proposition 1, $p \in S$. If $q \notin S$, then since $V(G)\backslash S$ is convex we have $V(G)\backslash\{q\} \subseteq S$, a contrary to our assumption that $\widetilde{\gamma}_{tcon}(G) \leq n-2$. Thus S contains all leaves of G. □

Proposition 3. *For any connected graph G of order $n \geq 4$ and any edge uv, where* $\min\{\deg(u), \deg(v)\} \geq 2$, *$v$ is not a support vertex and $N(u) \subseteq N(v)$, $\widetilde{\gamma}_{tcon}(G) \leq n-2$.*

Proof. If $\deg(v) = 2$, then $\deg(u) = 2$ and clearly $V(G) - \{u, v\}$ is a total outer-convex dominating set of G. Assume that $\deg(v) \geq 3$. If u has a neighbor w different from v which is not a support vertex, then $V(G) - \{u, w\}$ is a total outer-convex dominating set of G. Let any neighbor of u different from v be a support vertex. Then $V(G) - \{u, v\}$ is a total outer-convex dominating set of G and thus $\widetilde{\gamma}_{tcon}(G) \leq n-2$. □

Theorem 1. *Let G be a connected graph of order $n \geq 3$. Then*

$$\widetilde{\gamma}_{tcon}(G) \leq n-1.$$

The equality holds if and only if for any vertex v of G with degree at least two either v is a support vertex or all neighbors of v are support vertices.

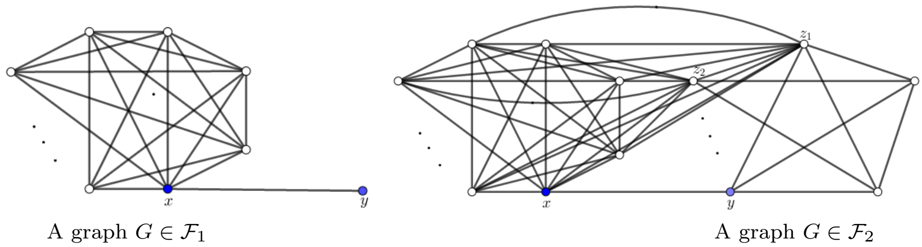

Figure 1. **Families $\mathcal{F}_1$ and $\mathcal{F}_2$**

Proof. Let T be a spanning tree of G and u be a leaf of T. Clearly $V(G) - \{u\}$ is a total outer-convex dominating set of G implying that $\widetilde{\gamma}_{tcon}(G) \leq n - 1$.
Assume that $\widetilde{\gamma}_{tcon}(G) = n - 1$. Let v be a vertex of G with degree at least 2. If v is a support vertex, then we are done. Suppose v is not a support vertex. By Proposition 3, we have $N(u) \setminus N[v] \neq \emptyset$ for any vertex $u \in N(v)$. If v has a neighbor w which is not a support vertex, then $V(G) \setminus \{v, w\}$ is a total outer-convex dominating set of G, a contradiction. Thus each neighbor of v is a support vertex.
Conversely, let G be a connected graph of order $n \geq 3$ such that for any vertex v of G with degree at least two either v is a support vertex or all neighbors of v are support vertices. Suppose S is a $\widetilde{\gamma}_{tcon}(G)$-set. If G has a vertex v with degree at least two such that $v \notin S$, then by assumption all neighbors of v are in S and since $V(G) \setminus S$ is convex, we must have $V(G) \setminus \{v\} \subseteq S$ implying that $\widetilde{\gamma}_{tcon}(G) = n - 1$. Thus we may assume that S contains all non-leaf vertices of G. It follows from Proposition 2 that $\widetilde{\gamma}_{tcon}(G) = n - 1$ and the proof is complete. □

Next we characterize all graphs G with $\widetilde{\gamma}_{tcon}(G) = 2$.
Let $\mathcal{F}_1$ be the family of all graphs G obtained from some complete graph K_p ($p \geq 1$) by adding a new vertex y and joining it to at least one vertex of K_p (see Figure 1).
Let $H_{p,q}$ ($p, q \geq 1$) be a graph obtained from two complete graphs K_p and K_q by adding some edges between $V(K_p)$ and $V(K_q)$ such that the resulting graph $H_{p,q}$ has diameter at most two, and let $G_{p,q}$ be a graph obtained from some $H_{p,q}$ by adding two new vertices x, y and joining x to all vertices of $V(K_p)$, y to x and all vertices of $V(K_q)$ and some vertices of $H_{p,q}$ with degree $p + q - 1$ (see Figure 1). Let $\mathcal{F}_2$ be the family of all graphs $G_{p,q}$.

Theorem 2. *Let G be a connected graph of order $n \geq 2$. Then $\widetilde{\gamma}_{tcon}(G) = 2$ if and only if $G \in \mathcal{F}_1 \cup \mathcal{F}_2$.*

Proof. If $G \in \mathcal{F}_1$ and x is a vertex of K_p adjacent to y, then clearly $\{x, y\}$ is a total outer-convex dominating set of G and so $\widetilde{\gamma}_{tcon}(G) = 2$. If $G \in \mathcal{F}_2$, then obviously $\{x, y\}$ is a total outer-convex dominating set of G implying that $\widetilde{\gamma}_{tcon}(G) = 2$.
Conversely, let $\widetilde{\gamma}_{tcon}(G) = 2$ and let $S = \{x, y\}$ be a $\widetilde{\gamma}_{tcon}(G)$. This implies that S is a total dominating set and $V(G) \setminus S$ is a convex set by definition. Assume without

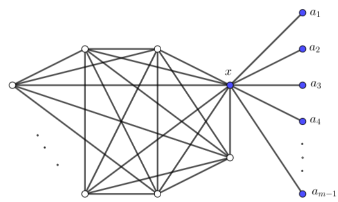

Figure 2. **A graph G of order n with $\widetilde{\gamma}_{tcon}(G) = m$ where $2 \leq m \leq n-1$**

loss of generality that $\deg(x) \geq \deg(y)$. Since $V(G) \setminus S$ is a convex set, the subgraph induced by $N(x)$ is a complete graph. If $N(y) \subseteq N[x]$, then clearly $G \in \mathcal{F}_1$ and we are done. Assume that $N(y) \not\subseteq N[x]$. As before, the subgraph induced by $N(y)$ is a complete graph. Considering the set $N(x)$ as the set of vertices of a complete graph K_p and the set of $N(y) \setminus N[x]$ as the set of vertices of a complete graph K_q, we can see that $G \in \mathcal{F}_2$ and this completes the proof. $\square$

The proof of the next result is straightforward and therefore omitted.

Proposition 4. *Let n be a positive integer.*

(i) For $n \geq 2$, $\widetilde{\gamma}_{tcon}(K_n) = 2$.

(ii) For $n \geq 3$,

$$\widetilde{\gamma}_{tcon}(P_n) = \begin{cases} n-1 & \text{if } n \leq 5 \\ n-2 & \text{if } n > 5. \end{cases}$$

(iii) For $n \geq 3$,

$$\widetilde{\gamma}_{tcon}(C_n) = \begin{cases} 2 & \text{if } n = 3 \\ n-2 & \text{if } n > 3. \end{cases}$$

(iv) For $n \geq 3$,

$$\widetilde{\gamma}_{tcon}(W_n) = \begin{cases} \frac{n}{2} & if n \equiv 2 (mod 4) \\ 2\lfloor \frac{n+1}{4} \rfloor & if n \not\equiv 2 (mod 4). \end{cases}$$

(v) For $n \geq 2$, $\widetilde{\gamma}_{tcon}(S_n) = n-1$.

(vi) For $n \geq 3$,

$$\widetilde{\gamma}_{tcon}(F_{r,s}) = \begin{cases} \frac{s+1}{2} & if r = 1\, and\, s \equiv 1 (mod 4) \\ 2\lceil \frac{s}{4} \rceil & if r = 1\, and\, s \not\equiv 1 (mod 4) \\ r & if r \geq 2\, and\, s \leq 3 \\ r+s-3 & if r \geq 2\, and\, s > 3. \end{cases}$$

Next we present a realization result.

Theorem 3. *Given positive integers m and n where $n \geq 3$ and $2 \leq m \leq n-1$, there exists a connected graph G of order n with $\widetilde{\gamma}_{tcon}(G) = m$.*

Proof. Let G_m be the graph obtained from a complete graph K_{n-m+1} by adding $m-1$ pendant edges $xa_1, \ldots, xa_{m-1}$ at a vertex x of K_{n-m+1} (see Figure 2). If $m = 2$, then clearly $\widetilde{\gamma}_{tcon}(G_m) = m$. If $m = n-1$, then by Theorem 1 we have $\widetilde{\gamma}_{tcon}(G_m) = m$. Let $3 \leq m \leq n-2$. Clearly, the set $\{x, a_1, a_2, \ldots, a_{m-1}\}$ is a total outer-convex dominating set since every complete graph is convex. This implies that $\widetilde{\gamma}_{tcon}(G_m) \leq m$. To show the inverse inequality, let S be a $\widetilde{\gamma}_{tcon}(G_m) - set$. Since $\widetilde{\gamma}_{tcon}(G_m) \leq n-2$, it follows from Proposition 2 that $\{a_1, a_2, \ldots, a_{m-1}\} \subseteq S$. On the other hand, to dominate the vertices of K_{n-m+1}, we must have $|S \cap V(G_m)| \geq 1$ implying that $\widetilde{\gamma}_{tcon}(G_m) \geq m$. Thus, $\widetilde{\gamma}_{tcon}(G_m) = m$. □

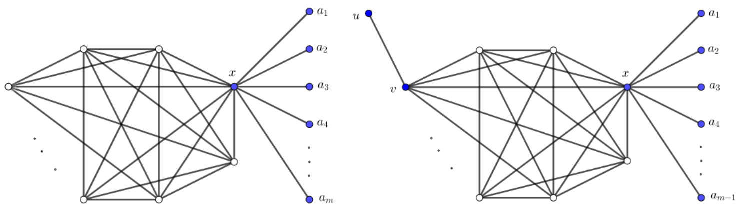

Figure 3. **A graph G with $\widetilde{\gamma}_{tcon}(G) - \gamma(G) = m$ and $\widetilde{\gamma}_{tcon}(G) - \gamma_t(G) = m$**

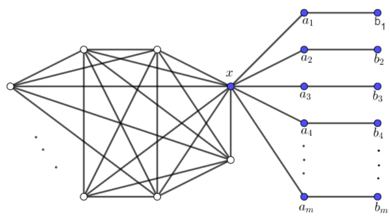

Figure 4. **A graph G with $\widetilde{\gamma}_{tcon}(G) - \widetilde{\gamma}_{con}(G) = m$**

Proposition 5. *The differences $\widetilde{\gamma}_{tcon}(G) - \gamma(G)$, $\widetilde{\gamma}_{tcon}(G) - \gamma_t(G)$, and $\widetilde{\gamma}_{tcon}(G) - \widetilde{\gamma}_{con}(G)$ can be made arbitrarily large.*

Proof. Let m be a positive integer. To show that $\widetilde{\gamma}_{tcon}(G) - \gamma(G)$, $\widetilde{\gamma}_{tcon}(G) - \gamma_t(G)$, and $\widetilde{\gamma}_{tcon}(G) - \widetilde{\gamma}_{con}(G)$ can be made arbitrarily large, it is enough to show that there exists a graph such that $\widetilde{\gamma}_{tcon}(G) - \gamma(G) = m$, $\widetilde{\gamma}_{tcon}(G) - \gamma_t(G) = m$, and $\widetilde{\gamma}_{tcon}(G) - \widetilde{\gamma}_{con}(G) = m-1$. If G_{m+1} is the graph defined as before (see the first graph illustrated in Figure 3), then we have $\gamma(G) = 1$ and $\widetilde{\gamma}_{tcon}(G) = m+1$ by Theorem 3. Thus, $\widetilde{\gamma}_{tcon}(G) - \gamma(G) = (m+1) - 1 = m$. Now, let G be a graph obtained from G_m by adding a pendant edge uv (see the second graph in Figure 3). Clearly, the $\widetilde{\gamma}_{tcon}(G) = m+2$ and $\gamma_t(G) = 2$. Thus, $\widetilde{\gamma}_{tcon}(G) - \gamma_t(G) = (m+2) - 2 = m$. Now, consider the graph G obtained from G_{m+1} by adding the pendant edges, $a_1b_1, a_2b_2, \ldots, a_mb_m$ (see Figure 4). Clearly, the $\widetilde{\gamma}_{tcon}(G) = 2m+1$ and $\widetilde{\gamma}_{con}(G) = m+1$. Hence, $\widetilde{\gamma}_{tcon}(G) - \widetilde{\gamma}_{con}(G) = (2m+1) - (m+1) = m$. This proves the assertion. □

3. Total outer-convex domination number of two-dimensional grid graphs

In this section we determine the total outer-convex domination number of two-dimensional grid graphs. A two-dimensional grid graph, also known as a rectangular grid graph or two-dimensional lattice graph is the Cartesian product $P_m \Box P_n$ of path graphs on m and n vertices. Let $V(P_m \Box P_n) = \{(i,j) \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and $E(P_m \Box P_n) = \{(i,j)(i,j+1) \mid 1 \leq i \leq m, 1 \leq j \leq n-1\} \cup \{(i,j)(i+1,j) \mid 1 \leq i \leq m-1, 1 \leq j \leq n\}$.

Proposition 6. *For any $n \geq 2$, we have*

$$\widetilde{\gamma}_{tcon}(P_1 \Box P_n) = \begin{cases} n-1 & \text{if } n \leq 5 \\ n-2 & \text{if } n > 5. \end{cases}$$

Proof. Since $P_1 \Box P_n \cong P_n$, the result follows directly. □

Proposition 7. *For $n \geq 2$, $\widetilde{\gamma}_{tcon}(P_2 \Box P_n) = n$.*

Proof. Clearly, the set $S = \{(1,j) \mid j = 1,2,\ldots,n\}$ is a total outer-convex dominating set of $P_2 \Box P_n$ and so $\widetilde{\gamma}_{tcon}(P_2 \Box P_n) \leq n$. To prove $\widetilde{\gamma}_{tcon}(P_2 \Box P_n) \geq n$, let S be a $\widetilde{\gamma}_{tcon}(P_2 \Box P_n)$-set. Suppose, to the contrary, that $\widetilde{\gamma}_{tcon}(P_2 \Box P_n) < n$. Then for some $1 \leq j \leq n$ we must have $(1,j),(2,j) \in V(P_2 \Box P_n) \setminus S$. If $j = 1$ (the case $j = n$ is similar), then to dominate the vertices $(1,1),(2,1)$, we must have $(1,2),(2,2) \in S$. Since $V(P_2 \Box P_n) \setminus S$ is convex, we have $V(P_2 \Box P_n) \setminus \{(1,1),(2,1)\} \subseteq S$ implying that $|S| \geq 2n-2 \geq n$, a contradiction. Let $1 < j < n$. If $(1,j-1)$, $(2,j-1)$, $(1,j+1)$, $(2,j+1) \in S$, then as above we obtain a contradiction. Therefore, $\{(1,j-1),(2,j-1),(1,j+1),(2,j+1)\} \nsubseteq S$. Assume without loss of generality that $(1,j+1) \in V(P_2 \Box P_n) \setminus S$. Since $V(P_2 \Box P_n) \setminus S$ is convex, we must have $(2,j+1) \in V(P_2 \Box P_n) \setminus S$. To dominate the vertices $(1,j),(2,j),(1,j+1),(2,j+1)$, we have $(1,j-1)$, $(2,j-1)$, $(1,j+2)$, $(2,j+2) \in S$ and since $V(P_2 \Box P_n) \setminus S$ is convex, we must have $V(P_2 \Box P_n) \setminus \{(1,j),(2,j),(1,j+1),(2,j+1)\} \subseteq S$. It follows that $|S| \geq 2n-4 \geq n$ (note that $n \geq 4$) which is a contradiction. Thus $|S| \geq n$ and so $\widetilde{\gamma}_{tcon}(P_2 \Box P_n) = n$. □

Proposition 8. *For $n \geq 3$,*

$$\widetilde{\gamma}_{tcon}(P_3 \Box P_n) = \begin{cases} 3\lfloor \frac{2n}{3} \rfloor & \text{if } n < 6 \\ 2n & \text{if } n \geq 6. \end{cases}$$

Proof. By a simple calculation we can verify the result for $n = 3, 4, 5$. Assume that $n \geq 6$. Clearly, the set $S = \{(1,j), (3,j) \mid 1 \leq j \leq n\}$ is a total outer-convex dominating set of $P_3 \Box P_n$ and so $\widetilde{\gamma}_{tcon}(P_3 \Box P_n) \leq 2n$. To prove $\widetilde{\gamma}_{tcon}(P_3 \Box P_n) \geq 2n$, let S be a $\widetilde{\gamma}_{tcon}(P_3 \Box P_n)$-set. Suppose, to the contrary, that $\widetilde{\gamma}_{tcon}(P_3 \Box P_n) < 2n$. Then for some $1 \leq j \leq n$ we must have $|\{(1,j), (2,j), (3,j)\} \cap (V(P_3 \Box P_n) \setminus S)| \geq 2$. We distinguish two cases.

Case 1. $(1,j), (3,j) \in V(P_3 \Box P_n) \setminus S$.
Since $V(P_3 \Box P_n) \setminus S$ is convex, we have $(2,j) \in V(P_3 \Box P_n) \setminus S$. If $j = 1$ (the case $j = n$ is similar), then to dominate the vertices $(1,1), (2,1), (3,1)$, we must have $(1,2), (2,2), (3,2) \in S$. Since $V(P_3 \Box P_n) \setminus S$ is convex, we have $V(P_3 \Box P_n) \setminus \{(1,1), (2,1), (3,1)\} \subseteq S$ yielding $|S| \geq 3n - 3 \geq 2n$, a contradiction. Let $1 < j < n$. If $(1, j-1)$, $(2, j-1)$, $(3, j-1)$, $(1, j+1)$, $(2, j+1), (3, j+1) \in S$, then as before we get a contradiction. Let $\{(1, j-1), (2, j-1), (3, j-1), (1, j+1), (2, j+1), (3, j+1)\} \not\subseteq S$. Assume without loss of generality that $(1, j+1) \in V(P_3 \Box P_n) \setminus S$. Since $V(P_3 \Box P_n) \setminus S$ is convex, we must have $(3, j+1) \in V(P_3 \Box P_n) \setminus S$. By repeating this process we deduce that $(3, j+1) \in V(P_3 \Box P_n) \setminus S$. To dominate the vertices $(1,j), (2,j), (3,j), (1, j+1), (2, j+1), (3, j+1)$, we must have $(1, j-1)$, $(2, j-1)$, $(3, j-1)$, $(1, j+2)$, $(2, j+2), (3, j+2) \in S$ and we conclude from the convexity of $V(P_3 \Box P_n) \setminus S$ that $|S| \geq 3n - 6 \geq 2n$ (note that $n \geq 6$) which is a contradiction.

Case 2. $(1,j), (2,j) \in V(P_3 \Box P_n) \setminus S$ (the case $(3,j), (2,j) \in V(P_3 \Box P_n) \setminus S$ is similar). According Case 1, we may assume that $(3,j) \in S$. To dominate $(1,j)$, we may assume without loss of generality that $(1, j+1) \in S$. It follows from the convexity of S that $(2, j+1), (3, j+1) \in S$ and $(3, j-1) \in S$ if $j \geq 2$. If $j = 1$, then by the convexity of S we must have $V(P_3 \Box P_n) \setminus \{(1,j), (2,j)\} \subseteq S$ implying that $|S| \geq 3n - 2 > 2n$ which is a contradiction. Let $j \geq 2$. Using above argument we can see that $|S| \geq 3n - 4 > 2n$, a contradiction again.
Thus $|S| \geq 2n$ and so $\widetilde{\gamma}_{tcon}(P_3 \Box P_n) = 2n$. This completes the proof. □

Proposition 9. *For $n \geq 4$, $\widetilde{\gamma}_{tcon}(P_4 \Box P_n) = 2n$.*

Proof. We can check that $S = \{(1,j), (4,j) \mid 1 \leq j \leq n\}$ is a total outer-convex dominating set of $P_4 \Box P_n$ and so $\widetilde{\gamma}_{tcon}(P_4 \Box P_n) \leq 2n$.
To prove $\widetilde{\gamma}_{tcon}(P_3 \Box P_n) \geq 2n$, let S be a $\widetilde{\gamma}_{tcon}(P_4 \Box P_n)$-set. Suppose, to the contrary, that $\widetilde{\gamma}_{tcon}(P_4 \Box P_n) < 2n$. Then for some $1 \leq j \leq n$ we must have $|\{(1,j), (2,j), (3,j), (4,j)\} \cap S| \leq 1$. We distinguish two cases.

Case 1. $(1,j), (4,j) \in V(P_3 \Box P_n) \setminus S$.
Since $V(P_4 \Box P_n) \setminus S$ is convex, we have $(2,j), (3,j) \in V(P_4 \Box P_n) \setminus S$. If $j = 1$ (the case $j = n$ is similar), then as in the proof of Proposition 8 we can see that

$V(P_4\Box P_n)\setminus\{(1,1),(2,1),(3,1),(4,1)\} \subseteq S$ yielding $|S| \geq 4n-4 \geq 2n$, a contradiction. Let $1 < j < n$. If $\{(i,j-1),(i,j+1) \mid 1 \leq i \leq 4\} \subseteq S$, then we deduce from the convexity of $V(P_4\Box P_n) \setminus S$ that $V(P_4\Box P_n) \setminus \{(1,1),(2,1),(3,1),(4,1)\} \subseteq S$ which leads to a contradiction again. Let $\{(i,j-1),(i,j+1) \mid 1 \leq i \leq 4\} \not\subseteq S$. Assume without loss of generality that $(1,j+1) \in V(P_4\Box P_n) \setminus S$. Since $V(P_4\Box P_n) \setminus S$ is convex, we must have $(2,j+1) \in V(P_4\Box P_n) \setminus S$. Using a similar argument, we have $(3,j+1),(4,j+1) \in V(P_4\Box P_n) \setminus S$. To dominate the vertices $(i,j),(i,j+1)$, we must have $(i,j-1),(i,j+2) \in S$ for each i. We conclude from the convexity of $V(P_4\Box P_n) \setminus S$ that $|S| \geq 4n-8 \geq 2n$ (note that $n \geq 4$) which is a contradiction.

Case 2. $\{(1,j),(4,j)\} \not\subseteq V(P_4\Box P_n) \setminus S$.
Assume without loss of generality that $(4,j) \in S$. By our earlier assumption we have $(1,j),(2,j),(3,j) \in V(P_4\Box P_n) \setminus S$. To dominate $(1,j)$, we may assume without loss of generality that $(1,j+1) \in S$. It follows from the convexity of S that $(2,j+1),(3,j+1),(4,j+1) \in S$ and $(4,j-1) \in S$ if $j \geq 2$. If $j = 1$, then by the convexity of $V(P_4\Box P_n) \setminus S$ we must have $V(P_4\Box P_n) \setminus \{(1,j),(2,j),(3,j)\} \subseteq S$ implying that $|S| \geq 4n-3 > 2n$ which is a contradiction. Let $j \geq 2$. Using above argument we can see that $|S| \geq 4n-6 > 2n$, a contradiction again.
Thus $|S| \geq 2n$ and so $\widetilde{\gamma}_{tcon}(P_4\Box P_n) = 2n$. This completes the proof. □

We close this section with a conjecture.

Conjecture. For positive integer $n \geq m \geq 5$, $\widetilde{\gamma}_{tcon}(P_m\Box P_n) = (m-2)n$.

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