Communications in Combinatorics and Optimization
Vol. 6, No. 2, 2021 pp.315-324
DOI: 10.22049/CCO.2021.27067.1188

CCO
Commun. Comb. Optim.

Research Article

On the outer independent 2-rainbow domination number of Cartesian products of paths and cycles

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Received: 29 December 2020; Accepted: 29 January 2021
Published Online: 31 January 2021

Abstract: Let G be a graph. A 2-*rainbow dominating function* (or *2-RDF*) of G is a function f from $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for a vertex $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u \in N_G(v)} f(u) = \{1,2\}$ is fulfilled, where $N_G(v)$ is the open neighborhood of v. The weight of 2-RDF f of G is the value $\omega(f) := \sum_{v \in V(G)} |f(v)|$. The 2-*rainbow domination number* of G, denoted by $\gamma_{r2}(G)$, is the minimum weight of a 2-RDF of G. A 2-RDF f is called an *outer independent* 2-*rainbow dominating function* (or *OI2-RDF*) of G if the set of all $v \in V(G)$ with $f(v) = \emptyset$ is an independent set. The outer independent 2-rainbow domination number $\gamma_{oir2}(G)$ is the minimum weight of an OI2-RDF of G. In this paper, we obtain the outer independent 2-rainbow domination number of $P_m \square P_n$ and $P_m \square C_n$. Also we determine the value of $\gamma_{oir2}(C_m \square C_n)$ when m or n is even.

Keywords: 2-rainbow dominating function, 2-rainbow domination number, outer independent 2-rainbow dominating function, outer independent 2-rainbow domination number, Cartesian product

AMS Subject classification: 05C69

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* $\deg_G(v)$ of a vertex v is the cardinality of its open neighborhood. Consult [11] for notation and terminology which are not defined here.
Let k be a positive integer, and set $[k] := \{1, 2, \ldots, k\}$. A function $f : V(G) \to 2^{[k]}$ is a k-*rainbow dominating function* (or k-*RDF*) of G if for a vertex $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u \in N_G(v)} f(u) = [k]$ is fulfilled. The *weight* of a k-RDF f of G is the value $\omega(f) := \sum_{v \in V(G)} |f(v)|$. The k-*rainbow domination number* of G,

denoted by $\gamma_{rk}(G)$, is the minimum weight of a k-RDF of G. A k-RDF f of G is a γ_{rk}-*function* if $\omega(f) = \gamma_{rk}(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [5]. The k-rainbow domination and its variants have been studied by several authors (see for example [1–4, 6–10, 13–19]).
An *outer independent k-rainbow dominating function* (or *OIk-RDF*) on a graph G is a k-rainbow dominating function f with the additional property that the set of all $v \in V(G)$ with $f(v) = \emptyset$ is an independent set. The outer independent k-rainbow domination number $\gamma_{oirk}(G)$ is the minimum weight of an OIk-RDF of G. Outer independent k-rainbow domination was introduced by Kang et al. in [12] in 2019.
For two graphs G and H, we let $G\Box H$ denote the Cartesian product of G and H.
In this paper we focus on the outer independent 2-rainbow domination number and we obtain the outer independent 2-rainbow domination number of $P_m\Box P_n$ for $m, n \geq 2$, where P_m is the path of order m. Also we determine the outer independent 2-rainbow domination number of $P_m\Box C_n$ when $m \geq 2, n \geq 3$ and $C_m\Box C_n$ when m or n is even, where C_n is the cycle of order n.

2. Outer independent 2-rainbow domination number of $P_m\Box P_n$

Let $V(P_m\Box P_n) = \{v_i^1, v_i^2, \ldots, v_i^m \mid 1 \leq i \leq n\}$ and

$$E(P_m\Box P_n) = \{v_i^j v_i^{j+1} \mid 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{v_i^j v_{i+1}^j \mid 1 \leq i \leq n-1, 1 \leq j \leq m\}.$$

Then for every $1 \leq i \leq m$, the i-th copy of P_n in the grid $P_m\Box P_n$ is denoted by $v_1^i v_2^i \ldots v_n^i$.
Also we use $0, 1, 2, 3$ to encode the sets $\emptyset, \{1\}, \{2\}, \{1, 2\}$.

Theorem 1. For $m, n \geq 2$, $\gamma_{oir2}(P_m\Box P_n) = \lfloor \frac{mn}{2} \rfloor$.

Proof. First we will present constructions of a OI2-RDF of $P_m\Box P_n$ of the desired weight. We consider the following m lines of length n.

1. m and n are even:

   $1010\ldots10$

   $0202\ldots02$

   . . . .

   . . . .

   . . . .

   $1010\ldots10$

   $0202\ldots02$

2. m is even and n is odd:

   $1010\ldots10\ 1$

   $0202\ldots02\ 0$

   . . . .

   . . . .

   . . . .

   $1010\ldots10\ 1$

   $0202\ldots02\ 0$

3. m is odd and n is even:

   $0202\ldots02$

   $1010\ldots10$

   . . . .

   . . . .

   . . . .

   $0202\ldots02$

   $1010\ldots10$

   $0202\ldots02$

4. m and n are odd:

   $0202\ldots02\ 0$

   $1010\ldots10\ 1$

   . . . .

   . . . .

   . . . .

   $0202\ldots02\ 0$

   $1010\ldots10\ 1$

   $0202\ldots02\ 0$

Now we show that $\gamma_{oir2}(P_m \square P_n) \geq \lfloor \frac{mn}{2} \rfloor$. Let f be an OI2-RDF with $\omega(f) = \gamma_{oir2}(P_m \square P_n)$ and for every $1 \leq i \leq n$, $\omega(f_i) = |f(v_i^1)| + |f(v_i^2)| + \cdots + |f(v_i^m)|$. First let m is even. By defination of function f, for every $1 \leq i \leq n$, $\omega(f_i) \geq \frac{m}{2}$. Hence

$$\omega(f) = \sum_{1 \leq i \leq n} \omega(f_i) \geq \frac{mn}{2} = \left\lfloor \frac{mn}{2} \right\rfloor.$$

Now let m is odd. Then $\omega(f_i) \geq \frac{m-1}{2}$ for every $1 \leq i \leq n$. It is easy to see that, if $\omega(f_i) = \frac{m-1}{2}$, then $\omega(f_{i-1}) \geq \frac{m+1}{2}$ and $\omega(f_{i+1}) \geq \frac{m+1}{2}$. First let n is odd. Then

$$\begin{aligned}
2\omega(f) &= 2\sum_{1\leq i\leq n} \omega(f_i) \\
&= \omega(f_1) + \omega(f_n) + \sum_{1\leq i\leq n-1} (\omega(f_i) + \omega(f_{i+1})) \\
&\geq \frac{m-1}{2} + \frac{m-1}{2} + (n-1)(\frac{m-1}{2} + \frac{m+1}{2}) \\
&= mn - 1.
\end{aligned}$$

Therefore $\omega(f) \geq \frac{mn-1}{2} = \lfloor \frac{mn}{2} \rfloor$, when m and n are odd. Now let n is even. If $\omega(f_1) + \omega(f_n) \geq m$, then

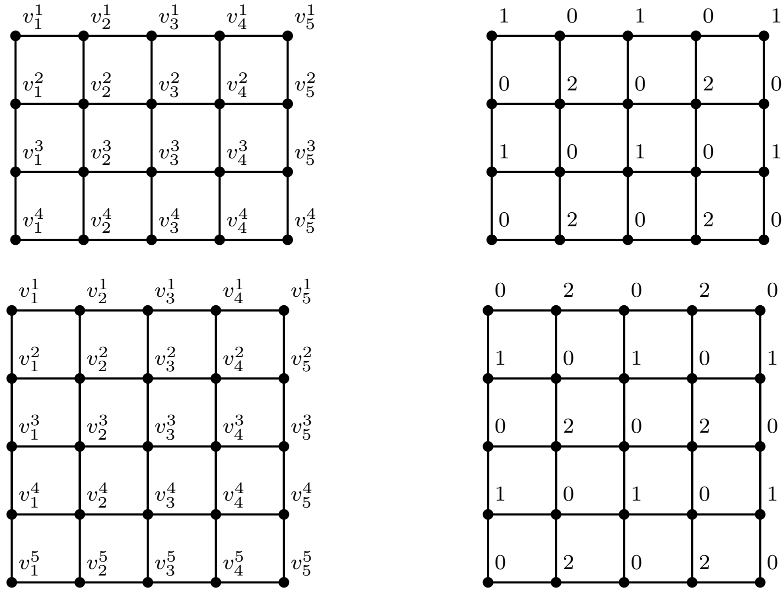
$$\begin{aligned}
2\omega(f) &= 2\sum_{1\leq i\leq n} \omega(f_i) \\
&= \omega(f_1) + \omega(f_n) + \sum_{1\leq i\leq n-1} (\omega(f_i) + \omega(f_{i+1})) \\
&\geq m + (n-1)(\frac{m-1}{2} + \frac{m+1}{2}) \\
&= mn.
\end{aligned}$$

Now if $\omega(f_1) = \omega(f_n) = \frac{m-1}{2}$, we can see that, there is $2 \leq k \leq n-2$ such that $\omega(f_k), \omega(f_{k+1}) \geq \frac{m+1}{2}$. Hence

$$\begin{aligned}
2\omega(f) &= 2\sum_{1\leq i\leq n} \omega(f_i) \\
&= \omega(f_k) + \omega(f_{k+1}) + \omega(f_0) + \omega(f_{n-1}) + \sum_{1\leq i\leq n-1, i\neq k} (\omega(f_i) + \omega(f_{i+1})) \\
&\geq \frac{m+1}{2} + \frac{m+1}{2} + \frac{m-1}{2} + \frac{m-1}{2} + (n-2)(\frac{m-1}{2} + \frac{m+1}{2}) \\
&= mn,
\end{aligned}$$

and $\omega(f) \geq \lfloor \frac{mn}{2} \rfloor$ when m is odd and n is even. This complete the proof. $\square$

The following figures are the values of the vertices of $P_4 \square P_5$ and $P_5 \square P_5$.



3. Outer independent 2-rainbow domination number of $P_m \square C_n$

We write $V(P_m \square C_n) = \{v_i^1, v_i^2, \ldots, v_i^m \mid 1 \le i \le n\}$ and let

$$E(P_m \square C_n) = \{v_1^j v_n^j \mid 1 \le j \le m\} \cup \{v_i^j v_i^{j+1} \mid 1 \le i \le n, 1 \le j \le m-1\}$$

$$\cup \{v_i^j v_{i+1}^j \mid 1 \le i \le n-1, 1 \le j \le m\}.$$

Also we use $0, 1, 2, 3$ to encode the sets $\emptyset, \{1\}, \{2\}, \{1,2\}$.

Theorem 2. For $m \ge 2$ and $n \ge 3$,

$$\gamma_{oir2}(P_m \square C_n) = \begin{cases} mn/2 & \text{if } n \text{ is even,} \\ m(n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First we will present constructions of a OI2-RDF of $P_m \square C_n$ of the desired weight. We consider the following m lines of length n.

1. m and n are even:

   $1010\ldots10$

   $0202\ldots02$

   $\ldots\ldots\ldots$

   $\ldots\ldots\ldots$

   $\ldots\ldots\ldots$

   $1010\ldots10$

   $0202\ldots02$

2. m is even and n is odd:

   $1010\dots10\ 1$

   $0202\dots02\ 1$

   $.\ .\ .\ .\ .\ .\ .\ .\ .\ .$

   $.\ .\ .\ .\ .\ .\ .\ .\ .\ .$

   $.\ .\ .\ .\ .\ .\ .\ .\ .\ .$

   $1010\dots10\ 1$

   $0202\dots02\ 1$

3. m is odd and n is even:

   $1010\dots10$

   $0202\dots02$

   $.\ .\ .\ .\ .\ .\ .\ .\ .$

   $.\ .\ .\ .\ .\ .\ .\ .\ .$

   $.\ .\ .\ .\ .\ .\ .\ .\ .$

   $1010\dots10$

   $0202\dots02$

   $1010\dots10$

4. m and n are odd:

   $1010\dots10\ 1$

   $0202\dots02\ 1$

   $.\ .\ .\ .\ .\ .\ .\ .\ .\ .$

   $.\ .\ .\ .\ .\ .\ .\ .\ .\ .$

   $.\ .\ .\ .\ .\ .\ .\ .\ .\ .$

   $1010\dots10\ 1$

   $0202\dots02\ 1$

   $1010\dots10\ 1$

Now let f be an OI2-RDF of $P_m\Box C_n$ with $\omega(f) = \gamma_{oir2}(P_m\Box C_n)$.
First let n is even and for every $1 \leq i \leq n$, $\omega(f_i) = |f(v_i^1)| + |f(v_i^2)| + \cdots + |f(v_i^m)|$.
If m is even then for every $1 \leq i \leq n$, $\omega(f_i) \geq \frac{m}{2}$. Hence

$$\omega(f) = \sum_{1\leq i\leq n} \omega(f_i) \geq \frac{mn}{2}.$$

Now let m is odd. Then $\omega(f_i) \geq \frac{m-1}{2}$ for every $1 \leq i \leq n$. It is easy to see that, if $\omega(f_i) = \frac{m-1}{2}$, then $\omega(f_{i-1}) \geq \frac{m+1}{2}$ and $\omega(f_{i+1}) \geq \frac{m+1}{2}$. Hence

$$\begin{aligned} 2\omega(f) &= 2\sum_{1\leq i\leq n} \omega(f_i) \\ &\geq n(\frac{m-1}{2} + \frac{m+1}{2}) \\ &= mn. \end{aligned}$$

Where the sum is taken modulo n.
Finally let n is odd. Assume that $\omega(f_i) = |f(v_1^i)| + |f(v_2^i)| + \cdots + |f(v_n^i)|$ when $1 \leq i \leq m$. Then $\omega(f_i) \geq \frac{n+1}{2}$ for every $1 \leq i \leq m$. Hence

$$\omega(f) = \sum_{1\leq i\leq m} \omega(f_i) \geq \frac{m(n+1)}{2}.$$

This complete the proof. □

4. Outer independent 2-rainbow domination number of $C_m \Box C_n$

We write $V(C_m \Box C_n) = \{v_i^1, v_i^2, \ldots, v_i^m \mid 1 \leq i \leq n\}$ and let

$$E(C_m \Box C_n) = \{v_i^j v_i^{j+1} \mid 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{v_i^j v_{i+1}^j \mid 1 \leq i \leq n-1, 1 \leq j \leq m\}$$

$$\cup \{v_i^1 v_i^m \mid 1 \leq i \leq n\} \cup \{v_1^j v_n^j \mid 1 \leq j \leq m\}.$$

Also we use $0, 1, 2, 3$ to encode the sets $\emptyset, \{1\}, \{2\}, \{1, 2\}$.

Theorem 3. For $m, n \geq 3$,

$$\gamma_{oir2}(C_m \Box C_n) = \begin{cases} mn/2 & \text{if} \quad m \text{ and } n \text{ are even,} \\ n(m+1)/2 & \text{if} \quad m \text{ is odd and } n \text{ is even,} \\ m(n+1)/2 & \text{if} \quad m \text{ is even and } n \text{ is odd.} \end{cases}$$

Proof. First we will present constructions of a OI2-RDF of $C_m \Box C_n$ of the desired weight. We consider the following m lines of length n.

1. m and n are even:

   $1010\ldots10$

   $0202\ldots02$

. . . .

. . . .

. . . .

$1010\ldots10$

$0202\ldots02$

2. m is even and n is odd:

$1010\ldots10\ 1$

$0202\ldots02\ 1$

. . . .

. . . .

. . . .

$1010\ldots10\ 1$

$0202\ldots02\ 1$

3. m is odd and n is even:

$0202\ldots02$

$1010\ldots10$

. . . .

. . . .

. . . .

$0202\ldots02$

$1010\ldots10$

$1111\ldots11$

Now let f be an OI2-RDF of $C_m\Box C_n$ with $\omega(f)=\gamma_{oir2}(C_m\Box C_n)$.
First let n is even and for every $1\leq i\leq n$, $\omega(f_i)=|f(v_i^1)|+|f(v_i^2)|+\cdots+|f(v_i^m)|$.
If m is even then for every $1\leq i\leq n$, $\omega(f_i)\geq\frac{m}{2}$. Hence

$$\omega(f)=\sum_{1\leq i\leq n}\omega(f_i)\geq\frac{mn}{2}.$$

Now let m is odd. Then $\omega(f_i)\geq\frac{m+1}{2}$ for every $1\leq i\leq n$. Hence

$$\omega(f)=\sum_{1\leq i\leq n}\omega(f_i)\geq\frac{n(m+1)}{2}.$$

Finally let n is odd and m is even. Assume that $\omega(f_i) = |f(v_1^i)|+|f(v_2^i)|+\cdots+|f(v_n^i)|$ when $1 \leq i \leq m$. Then $\omega(f_i) \geq \frac{n+1}{2}$ for every $1 \leq i \leq m$. Hence

$$\omega(f) = \sum_{1\leq i\leq m} \omega(f_i) \geq \frac{m(n+1)}{2}.$$

This complete the proof. □

If K_n be a complete graph of order n, then similarly we obtain the following results.

Theorem 4. For $m, n \geq 2$, $\gamma_{oir2}(P_m \Box K_n) = m(n-1)$.

Theorem 5. For $m, n \geq 3$, $\gamma_{oir2}(C_m \Box K_n) = m(n-1)$.

Theorem 6. For $m, n \geq 4$, $\gamma_{oir2}(K_m \Box K_n) = m(n-1)$.

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