Austrian Journal of Statistics
July 2022, Volume 51, 10–34.
http://www.ajs.or.at/
doi:10.17713/ajs.v51i4.1293

Effect of Dependency on the Estimation of $P[Y < X]$ in Exponential Stress-strength Models

Dipak D. Patil
Department of Statistics,
H. V. Desai College,
India

U. V. Naik-Nimbalkar
Savitribai Phule Pune
University,
India

M. M. Kale
Savitribai Phule Pune
University,
India

Abstract

We consider an expression for the probability $R = P(Y < X)$ where the random variables X and Y denote strength and stress, respectively. Our aim is to study the effect of the dependency between X and Y on R. We assume that X and Y follow exponential distributions and their dependency is modeled by a copula with the dependency parameter θ. We obtain a closed-form expression for R for Farlie-Gumbel-Morgenstern (FGM), Ali-Mikhail-Haq (AMH), Gumbel's bivariate exponential copulas and compute R for Gumbel-Hougaard (GH) copula using a Monte-Carlo integration technique. We plot a graph of R versus θ to study the effect of dependency on R. We estimate R by plugging in the estimates of the marginal parameters and θ in its expression. The estimates of the marginal parameters are based on the marginal likelihood. The estimates of θ are obtained from two different methods; one is based on the conditional likelihood and the other on the method of moments using Blomqvist's beta. Asymptotic distribution of both the estimators of R is obtained. For illustration purpose, we apply our results to a real data set.

Keywords: Blomqvist's beta, maximum likelihood estimation, Monte-Carlo method, reliability, two-stage estimation procedure.

1. Introduction

Estimation of the probability $R = P(Y < X)$, a measure of component reliability, has been considered by many authors when X and Y are independent variables belonging to the same univariate family of distributions, for example, Nadarajah (2003), Kundu and Gupta (2005), Kundu and Gupta (2006), Kotz, Lumelskii, and Pensky (2003), among others and the references therein. Jana (1994) discussed the estimation of R when (X, Y) follows the bivariate exponential (BVE) Marshall-Olkin model. Kotz *et al.* (2003) (pg. 100) discussed R for Gumbel's BVE distributions. Hanagal (1994) considered the studentized test for testing reliability in BVE stress-strength model. Nadarajah and Kotz (2006) considered six BVE distributions to find R and applied these results to the receiver operating characteristic (ROC) curves. Among these, in four of their BVE distributions, the marginal distributions are exponential. Domma and Giordano (2013) obtained a closed-form expression for R by modeling the dependence through FGM copula and generalized FGM copula with the margins belonging to the Burr system. It is of interest to study the effect of the dependency between X and Y on R. Due to Sklar's theorem (Sklar 1959) stated below a convenient approach to model the dependency between the variables is through the copulas. A copula is a multivariate distribution function with all univariate marginal distribution functions uniform on [0, 1].

Sklar's Theorem: Let H be a joint distribution function of random variables X and Y with marginal distribution functions $F(x)$ and $G(y)$ respectively. Then there exists a copula C such that for all x, y in $(-\infty, \infty)$,

$$H(x,\ y){=}C(F(x),\ G(y)).$$

Further, if F and G are continuous then C is unique; otherwise, C is uniquely determined on $RangeF \times RangeG$.

For details on copulas, we refer to Nelsen (1999). Thus using copula, together with any marginal distribution functions, we can construct a joint distribution function. However, it is generally not known which copula function should be used. The best fitting model can be obtained using Akaike information criteria (AIC). In this article we study the expression for R, together with its estimation in case of four important copula models with exponential margins. The reliability R can be expressed as

$$R = P\left[Y < X\right] = \int P\left[Y < x | X = x\right] f(x) dx, \tag{1}$$

where $P[Y < x|X = x]$ denotes the conditional probability and f, the probability density function (p.d.f.) of X.

Exponential distributions have a variety of applications. The most prominent application is in the field of life testing. In the context of reliability, the stress-strength model describes the life of a component that has random strength X subject to random stress Y. We, therefore, consider the exponential distribution as a marginal distribution of random variables X and Y with distribution functions

$$F_1(x; \alpha_1) = 1 - e^{-\alpha_1 x}, \quad x > 0;\ \alpha_1 > 0 \tag{2}$$

and

$$F_2(y; \alpha_2) = 1 - e^{-\alpha_2 y}, \quad y > 0;\ \alpha_2 > 0, \tag{3}$$

respectively.

The copulas involve a parameter θ called the dependency parameter. We consider FGM copula from the non-Archimedean family. Its dependency parameter $\theta \in [-1, 1]$. Next, we consider three copulas from Archimedean family with different ranges for the dependence parameter θ, viz. AMH copula, Gumbel's bivariate exponential copula and GH copula. For the AMH copula $\theta \in [-1, 1]$, for the Gumbel's bivariate exponential copula $\theta \in [0, 1]$ and for the GH copula $\theta \in [1, \infty)$. The range of Kendall's tau (τ) for FGM, AMH, Gumbel's bivariate exponential, and GH copulas are $[-0.22, 0.22]$, $[-0.1817, 0.3333]$, $[-0.4, 0]$ and $[0, 1]$ respectively. The range of Blomqvist's beta (β) for FGM, AMH, Gumbel's bivariate exponential, and GH copulas are $[-0.25, 0.25]$, $[-0.20, 0.3333]$, $[-0.38149, 0]$ and $[0, 1)$ respectively. The FGM and AMH models both positive and negative dependency, Gumbel's bivariate exponential models only negative dependency and GH models only positive dependency.

In Section 2, we consider the expression for R in terms of θ and the parameters (α_1, α_2) of the margins for the four copulas. To study the effect of dependency between X and Y on R, we plot a graph of R versus θ for different pairs of (α_1, α_2). We get closed-form expressions for R for FGM copula, AMH copula, and Gumbel's bivariate exponential copula. It was not straightforward to solve integrals involved in the expressions for R in case of AMH and Gumbel's bivariate exponential copula thus we solve these integrals using Wolfram Mathematica 6.0. It was not possible to obtain a closed-form expression for R in the case of GH copula. Therefore, in this case, we use a Monte Carlo (MC) integration technique for obtaining R for known values of the parameters of margins and of θ. We plot a graph of R versus θ to study the effect of θ on the R. Based on the observations on n independent pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$, we consider two procedures for the estimation of θ and hence of R. Both the procedures are based on what is known as the 'two-stage estimation procedure' in the literature (Joe 1997); that is, we first obtain the maximum likelihood estimates (mles) of the parameters of the margins based on the respective probability distributions. In the first method, we obtain an estimate of θ by solving $\frac{dlnL}{d\theta} = 0$, using the Newton Raphson (NR) method, where L is the conditional likelihood based on the conditional density of Y given X and by plugging in the estimates of the parameters of the margins. This method is reported in Section 3. For the second method, we estimate θ using a nonparametric method. For the copulas, the nonparametric methods are often based on the inversion of Spearman's rho (ρ), Kendall's tau (τ), Blomqvist's beta (β) etc. (Nelsen 1999). For the models considered in this article, explicit expressions for the population version of Spearman's rho and Kendall's tau are either not available or the inversion for θ is not simple. We, therefore, consider the estimator based on Blomqvist's beta, (see, Nelsen (1999), Blomqvist (1950)). This method is reported in Section 4. We obtain the asymptotic distribution of both the estimators of R in the respective sections. The results of simulations carried out to study the performance of the estimators are reported in Section 5. We plot graphs of the estimates of R versus θ to see the effect of dependency on the estimates. In Section 6, we apply our results to real data. Finally conclusions appear in Section 7.

2. Expressions for R

The derivations of the expressions for R for the four copula families with exponential marginals are given in the following subsections. The reliability R given in Equation (1) with exponential marginals is given by

$$R = P\left[Y < X\right] = \int_0^\infty P\left[Y \leq x | X = x\right] f_1(x) dx, \tag{4}$$

where $f_1(x) = \alpha_1 e^{-\alpha_1 x}, \quad x > 0; \quad \alpha_1 > 0$ is p.d.f.s of X.
We note that if the random variables X and Y are independent then

$$R = P\left[Y < X\right] = \frac{\alpha_2}{\alpha_1 + \alpha_2}.$$

If the joint distribution function of (X, Y) is $C_\theta(F_1(x), F_2(y))$ with the copula function C_θ then

$$P[Y \leq y | X = x] = \frac{\partial C_\theta(u, v)}{\partial u}|_{u=F_1(x), v=F_2(y)} \quad , \tag{5}$$

see Nelsen (1999) (pp. 36).

2.1. Farlie-Gumbel-Morgenstern copula

The FGM copula (Nelsen 1999) is given by

$$C_\theta(u, v) = uv\left[1 + \theta(1-u)(1-v)\right], \quad 0 \leq u, v \leq 1; \quad -1 \leq \theta \leq 1. \tag{6}$$

The bivariate copula is the joint distribution function of two random variables (r.v.s) with uniform margins. We denote these r.v.s by U and V throughout the paper. Hence,

$$\frac{\partial C_\theta(u, v)}{\partial u} = v\left[1 + \theta(1-v)(1-2u)\right].$$

Therefore, from Equation (5),

$$\begin{aligned} P\left(Y \leq y|X=x\right) &= \left(1-e^{-\alpha_2 y}\right)\left[1+\theta e^{-\alpha_2 y}\left(1-2\left(1-e^{-\alpha_1 x}\right)\right)\right] \\ &= \left(1-e^{-\alpha_2 y}\right)\left[1+\theta e^{-\alpha_2 y}\left(-1+2e^{-\alpha_1 x}\right)\right], \quad x>0;\ y>0. \end{aligned} \tag{7}$$

Thus, the reliability R given in Equation (4) is

$$\begin{aligned} R &= \int_0^\infty \left(1-e^{-\alpha_2 x}\right)\left[1+\theta e^{-\alpha_2 x}\left(-1+2e^{-\alpha_1 x}\right)\right]\alpha_1 e^{-\alpha_1 x}dx \\ &= \frac{\alpha_2}{\alpha_1+\alpha_2}\left[\frac{\left(2\alpha_1+\alpha_2\right)\left(\alpha_1+2\alpha_2\right)+\alpha_1\left(-\alpha_1+\alpha_2\right)\theta}{\left(2\alpha_1+\alpha_2\right)\left(\alpha_1+2\alpha_2\right)}\right]. \end{aligned} \tag{8}$$

It is straightforward to solve the above integral. The reliability R is a linear function of the dependence parameter θ for this copula.

2.2. Ali-Mikhail-Haq copula

The AMH copula (Nelsen 1999) is given by

$$C_\theta(u,v) = \frac{uv}{1-\theta\left(1-u\right)\left(1-v\right)}, \quad 0 \leq u,v \leq 1; \ -1 \leq \theta \leq 1. \tag{9}$$

Then,

$$\frac{\partial C_\theta(u,v)}{\partial u} = \frac{v\left[1-\theta\left(1-v\right)\right]}{\left[1-\theta\left(1-u\right)\left(1-v\right)\right]^2}.$$

Hence, from Equation (5),

$$P\left(Y \leq y|X=x\right) = \frac{\left(1-e^{-\alpha_2 y}\right)\left(1-\theta e^{-\alpha_2 y}\right)}{\left(1-\theta e^{-\alpha_1 x}e^{-\alpha_2 y}\right)^2}, \quad x>0;\ y>0. \tag{10}$$

Thus, the reliability R given in Equation (4) is

$$\begin{aligned} R &= \int_0^\infty \frac{\left(1-e^{-\alpha_2 x}\right)\left(1-\theta e^{-\alpha_2 x}\right)}{\left(1-\theta e^{-(\alpha_1+\alpha_2)x}\right)^2}\alpha_1 e^{-\alpha_1 x}dx \\ &= Hypergeometric2F1\left[2, \frac{\alpha_1}{\alpha_1+\alpha_2}, 1+\frac{\alpha_1}{\alpha_1+\alpha_2}, \theta\right] + \frac{\alpha_1}{\alpha_1+\alpha_2} \\ &\quad \times \left[\frac{1+\theta}{-1+\theta} + \theta\frac{Gamma\left(\frac{\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}\right) Hypergeometric2F1\left[2, \frac{\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}, 1+\frac{\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}, \theta\right]}{Gamma\left(\frac{2\alpha_1+3\alpha_2}{\alpha_1+\alpha_2}\right)}\right], \end{aligned} \tag{11}$$

where $Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ and the hypergeometric function $Hypergeometric2F1\left[a,b,c,z\right]$ is defined for $|z|<1$ by the power series $Hypergeometric2F1\left[a,b,c,z\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!}$, where

$$(a)_n = \begin{cases} 1 & \text{if n=0} \\ a(a+1)\ldots(a+n+1) & \text{if n>0} \end{cases}$$

The above power series diverges to infinity for $c<0$. For given values of parameters a, b, c, and z, Wolfram Mathematica 6.0 calculates the $Hypergeometric2F1\left[a,b,c,z\right]$ function.

2.3. Gumbel's bivariate exponential copula

Gumbel's bivariate exponential copula (Nelsen 1999) is given by

$$C_\theta(u,v) = u + v - 1 + (1-u)(1-v)\, e^{-\theta \ln(1-u)\ln(1-v)}, \quad 0 \le u, v \le 1;\ 0 \le \theta \le 1. \tag{12}$$

Then,

$$\frac{\partial C_\theta(u,v)}{\partial u} = 1 - e^{-\theta \ln(1-u)\ln(1-v)}\,(1-v)\,[1 - \theta \ln(1-v)]\,.$$

Hence, from Equation (5),

$$\begin{aligned} P(Y \le y | X = x) &= 1 - e^{-\theta \ln\left(e^{-\alpha_1 x}\right)\ln\left(e^{-\alpha_2 y}\right)} e^{-\alpha_2 y}\left[1 - \theta \ln\left(e^{-\alpha_2 y}\right)\right] \\ &= 1 - e^{-\theta\alpha_1\alpha_2 x y - \alpha_2 y}\,(1 + \theta\alpha_2 y)\,, \qquad x > 0;\ y > 0. \end{aligned} \tag{13}$$

Thus, the reliability R given in Equation (4) is given by

$$\begin{aligned} R &= \int_0^\infty \left[1 - e^{-\theta\alpha_1\alpha_2 x^2 - \alpha_2 x}\,(1 + \theta\alpha_2 x)\right] \alpha_1 e^{-\alpha_1 x} dx \\ &= \frac{1}{2} - \frac{\sqrt{\pi}\,(\alpha_1 - \alpha_2)}{4\sqrt{\alpha_1\alpha_2\theta}} e^{\frac{(\alpha_1+\alpha_2)^2}{4\alpha_1\alpha_2\theta}}\, Erfc\left[\frac{\alpha_1 + \alpha_2}{2\sqrt{\alpha_1\alpha_2\theta}}\right], \end{aligned} \tag{14}$$

where $Erfc[z]$ is the integration of Gaussian distribution defined as $Erfc[z] = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$. It is also discussed by Kotz *et al.* (2003) (pg. 100).

2.4. Gumbel-Hougaard copula

This family of copula was first discussed by Gumbel (1960b) and also appears in Hougaard (1986). Hutchinson and Lai (1990) referred to it as the Gumbel-Hougaard (GH) copula. Also, Nelsen (1999) (pp. 96) has listed this copula in his book as the fourth family. It is given by

$$C_\theta(u,v) = Exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right), \quad 0 \le u, v \le 1;\ 1 \le \theta < \infty. \tag{15}$$

Then,

$$\frac{\partial C_\theta(u,v)}{\partial u} = \frac{(-\ln u)^{\theta-1}}{u}\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{\frac{1}{\theta}-1} Exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{\frac{1}{\theta}}\right).$$

Hence, from Equation (5),

$$\begin{aligned} P(Y \le y | X = x) &= \frac{\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^{\theta-1}}{(1 - e^{-\alpha_1 x})}\left[\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^\theta + \left(-\ln\left[1 - e^{-\alpha_2 y}\right]\right)^\theta\right]^{\frac{1}{\theta}-1} \\ &\times\ Exp\left(-\left[\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^\theta + \left(-\ln\left[1 - e^{-\alpha_2 y}\right]\right)^\theta\right]^{\frac{1}{\theta}}\right), \ x > 0;\ y > 0. \end{aligned} \tag{16}$$

It was not possible to obtain an explicit expression for R, for this copula. We, therefore, compute the integration involved in R using a MC integration technique. The details about a MC integration technique are given below. For known values of α_1, α_2 and θ, we generate N values of X_i randomly from the exponential distribution with parameter α_1 having p.d.f. $f_1(x)$ and approximate the integral $\int_0^\infty g(x) f_1(x) dx$ by $\frac{1}{N}\sum_{i=1}^N g(X_i)$, where

$$\begin{aligned} g(x) &= \frac{\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^{\theta-1}}{(1 - e^{-\alpha_1 x})}\left[\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^\theta + \left(-\ln\left[1 - e^{-\alpha_2 x}\right]\right)^\theta\right]^{\frac{1}{\theta}-1} \\ &\times Exp\left(-\left[\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^\theta + \left(-\ln\left[1 - e^{-\alpha_2 x}\right]\right)^\theta\right]^{\frac{1}{\theta}}\right), \qquad x > 0. \end{aligned}$$

The estimated answer converges to the correct result as $N \to \infty$ (from the law of large numbers). We took N = 100000.

2.5. Variation in R with respect to θ

Next, to study the effect of the dependency between X and Y on R, we plot a graph of R versus the dependency parameter θ for four different pairs of (E(X), E(Y)) for the four copulas where E(Z) denotes the expected value of Z. We consider graphs for two cases $E(X) > E(Y)$ and $E(X) < E(Y)$. We note that $E(X) = 1/\alpha_1$ and $E(Y) = 1/\alpha_2$. It is easily verified that: i) if $\alpha_1 < \alpha_2$ then $E(X) > E(Y)$ and ii) if $\alpha_1 > \alpha_2$ then $E(X) < E(Y)$. For the parameter values

$$(\alpha_1, \alpha_2) = \{(2,3),(2,5)\},$$

$E(X) > E(Y)$ and for

$$(\alpha_1, \alpha_2) = \{(3,2),(5,2)\},$$

$E(X) < E(Y)$.

Figure 1 illustrates the variation in R with respect to (w. r. to) θ, for the four copulas for

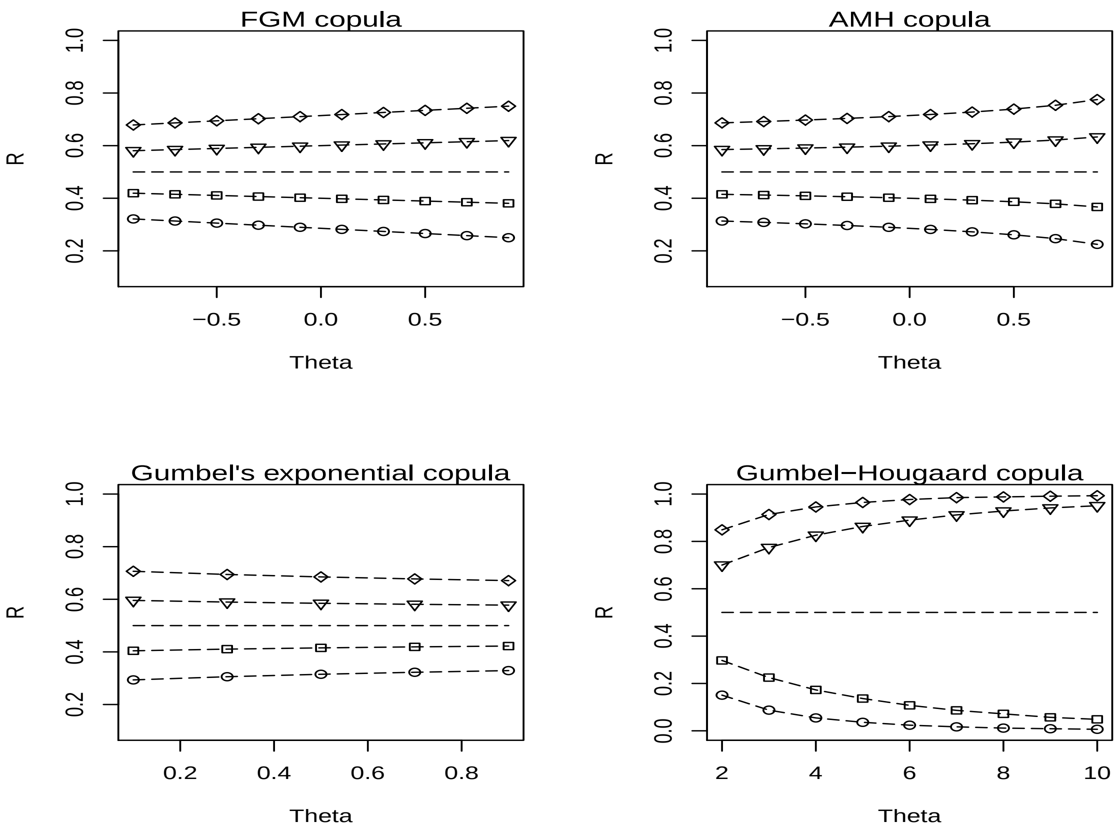

Figure 1: Variation in R against dependence parameter θ (Theta) for the four pairs of $\{(E(X), E(Y)) = (0.5, 0.33)\triangledown,\ (0.5, 0.2)\Diamond,\ (0.33, 0.5)\Box,\ (0.2, 0.5)\bigcirc\}$

four different pairs of $(E(X), E(Y))$. From the Figure, the pattern of variation in R w. r. to θ is found to be:

1. If $E(X) > E(Y)$ then $R > 1/2$ and R increases with θ for FGM, AMH, and GH copulas and decreases for Gumbel's bivariate exponential copula. The graph of R appears to be

almost linear in θ except for the GH copula. The increase in R w. r. to θ is faster for the GH copula than for the FGM and AMH copulas.

2. If $E(X) < E(Y)$ then $R < 1/2$ and R decreases with θ for FGM, AMH, and GH copulas and increases for Gumbel's bivariate exponential copula. The graph of R appears to be almost linear in θ except for the GH copula. The decrease in R w. r. to θ is faster for the GH copula than for the FGM and AMH copulas.

Also, for a given E(X), smaller the value of E(Y) larger is the value of R for all θ and, for a given E(Y), smaller the value of E(X) smaller is the value of R for all θ for the four copulas considered. We observed that the pattern of variation in R discussed above holds true for other combinations of the parameters of the margins that lead to the same inequality between E(X) and E(Y).
The graphs show that GH copula has maximum variation in R with respect to θ and gives a maximum range for R. The variation in R is the least for FGM copula. We see that the difference between the value of R with independent margins and its value with dependent margins is small in case of FGM copula, AMH copula, and Gumbel's bivariate exponential copula. This could be because the GH copula captures a larger range of the strength of dependence as seen from the values of Kendall's tau (τ) and Blomqvist's beta (β) reported in the introduction. Thus, the value of R differes with the strength of dependence.

3. Likelihood-based estimation

Let $\{(x_i, y_i), i = 1, 2, \cdots, n\}$ be a realization of a random sample from the joint distribution function $H(x, y)$. We follow a two-stage estimation procedure (Joe 1997) and estimate α_1 and α_2 using the marginal densities. Thus the likelihood function based on $(x_1, x_2, \cdots, x_n)$ is given by

$$L(x_1, x_2, \cdots, x_n; \alpha_1) = \alpha_1^n e^{-\alpha_1 \sum_{i=1}^{n} x_i}$$

Based on the above likelihood, the mle $\hat{\alpha}_1$ of α_1 is given by

$$\hat{\alpha}_1 = n/\sum_{i=1}^{n} x_i = 1/\bar{x}. \tag{17}$$

Similarly, the mle $\hat{\alpha}_2$ of α_2 is given by

$$\hat{\alpha}_2 = n/\sum_{i=1}^{n} y_i = 1/\bar{y}. \tag{18}$$

Since the joint density $h(x_i, y_i) = f(y_i|x_i)f_X(x_i)$ and since $f_X(x_i)$ is free from θ we maximize $\prod_{i=1}^{n} f(y_i|x_i)$ to obtain the estimate of θ. We substitute the parameters of the margins by their estimates given in Equation (17) and Equation (18). Let $L(\underline{y}|\underline{x}) = \prod_{i=1}^{n} f(y_i|x_i)$. The expression for $\frac{d \ln L(\underline{y}|\underline{x})}{d\theta}$, using the conditional distribution of $y_1, y_2, \cdots, y_n$ given $x_1, x_2, \cdots, x_n$ for the four copulas considered, are given in the following subsections.

3.1. FGM copula

The conditional density function of Y given $X = x$, using Equation (7), is given by

$$f(y|X = x) = \alpha_2 e^{-\alpha_1 x - 2\alpha_2 y} \left[4\theta - 2\theta e^{\alpha_1 x} - 2\theta e^{\alpha_2 y} + (1 + \theta)e^{\alpha_1 x + \alpha_2 y}\right]. \tag{19}$$

Hence,

$$\frac{d\ln L}{d\theta}=\sum_{i=1}^{n}\frac{4-2e^{\alpha_1x_i}-2e^{\alpha_2y_i}+e^{\alpha_1x_i+\alpha_2y_i}}{4\theta-2\theta e^{\alpha_1x}-2\theta e^{\alpha_2y}+(1+\theta)e^{\alpha_1x+\alpha_2y}}. \tag{20}$$

We substitute (α_1,α_2) by their estimates and solve $\frac{d\ln L}{d\theta}=0$ by using a NR method to estimate θ. This reduces solving polynomial in θ of degree n-1 for n >1.

3.2. AMH copula

The conditional density function of Y given $X=x$, using Equation (10), is given by

$$f\left(y|X=x\right)=\frac{e^{2\alpha_1x+2\alpha_2y}}{\left(e^{\alpha_1x+\alpha_2y}-\theta\right)^3}\left[-2\theta e^{\alpha_1x}-2\theta e^{\alpha_2y}+(1+\theta)e^{\alpha_1x+\alpha_2y}+\theta(1+\theta)\right], \tag{21}$$

and with the parameters of the margins substituted by their estimates

$$\frac{d\ln L}{d\theta}=\sum_{i=1}^{n}\frac{\left[2e^{\hat\alpha_1x_i}+2e^{\hat\alpha_2y_i}-e^{\hat\alpha_1x_i+\hat\alpha_2y_i}-1-2\theta\right]}{\left[2\theta e^{\hat\alpha_1x_i}+2\theta e^{\hat\alpha_2y_i}-(1+\theta)e^{\hat\alpha_1x_i+\hat\alpha_2y_i}-\theta(1+\theta)\right]}+\sum_{i=1}^{n}\frac{3}{e^{\hat\alpha_1x_i+\hat\alpha_2y_i}-\theta}. \tag{22}$$

We solve $\frac{d\ln L}{d\theta}=0$ by using a NR method to estimate θ.

3.3. Gumbel's bivariate exponential copula

The conditional density function of Y given $X=x$, using Equation (13), is given by

$$f(y|X=x)=\alpha_2e^{-\alpha_2(1+\theta\alpha_1x)y}\left[(1+\theta\alpha_1x)(1+\theta\alpha_2y)-\theta\right], \tag{23}$$

and with the parameters of the margins substituted by their estimates

$$\frac{d\ln L}{d\theta}=-\hat\alpha_1\hat\alpha_2\sum_{i=1}^{n}x_iy_i+\sum_{i=1}^{n}\frac{\hat\alpha_1x_i(1+\theta\hat\alpha_2y_i)+\hat\alpha_2y_i(1+\theta\hat\alpha_1x_i)-1}{(1+\theta\hat\alpha_1x_i)(1+\theta\hat\alpha_2y_i)-\theta}. \tag{24}$$

We solve $\frac{d\ln L}{d\theta}=0$ by using a NR method to estimate θ.

3.4. GH copula

The conditional density function of Y given $X=x$, using Equation (16), is given by

$$\begin{aligned} f(y|X=x) &= \frac{\left(-\ln\left[1-e^{-\alpha_1x}\right]\right)^{\theta-1}\left(-\ln\left[1-e^{-\alpha_2y}\right]\right)^{\theta-1}}{\left(1-e^{-\alpha_1x}\right)\left(1-e^{-\alpha_2y}\right)}\\ &\quad\times\left[\left(-\ln\left[1-e^{-\alpha_1x}\right]\right)^{\theta}+\left(-\ln\left[1-e^{-\alpha_2y}\right]\right)^{\theta}\right]^{-2+\frac{1}{\theta}}\\ &\quad\times\left\{(\theta-1)+\left[\left(-\ln\left[1-e^{-\alpha_1x}\right]\right)^{\theta}+\left(-\ln\left[1-e^{-\alpha_2y}\right]\right)^{\theta}\right]^{\frac{1}{\theta}}\right\}\\ &\quad\times Exp\left(-\alpha_2y-\left[\left(-\ln\left[1-e^{-\alpha_1x}\right]\right)^{\theta}+\left(-\ln\left[1-e^{-\alpha_2y}\right]\right)^{\theta}\right]^{\frac{1}{\theta}}\right), \end{aligned} \tag{25}$$

and with the parameters of the margins substituted by their estimates

$$\begin{aligned}
&\frac{d\ln L}{d\theta} \\
=\;& \sum_{i=1}^{n}\ln(\hat{E}_{1(i)})+\sum_{i=1}^{n}\ln(\hat{E}_{2(i)})+\sum_{i=1}^{n}\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]^{\frac{1}{\theta}} \\
&\times\left[-\frac{\ln\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]}{\theta^{2}}+\frac{(\hat{E}_{1(i)})^{\theta}\ln\hat{E}_{1(i)}+(\hat{E}_{2(i)})^{\theta}\ln(\hat{E}_{2(i)})}{\theta\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]}\right] \\
&-\sum_{i=1}^{n}\frac{\ln\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]}{\theta^{2}}+\sum_{i=1}^{n}\frac{\left(-2+\frac{1}{\theta}\right)\left[(\hat{E}_{1(i)})^{\theta}\ln\hat{E}_{1(i)}+(\hat{E}_{2(i)})^{\theta}\ln(\hat{E}_{2(i)})\right]}{(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}} \\
&+\sum_{i=1}^{n}\frac{1+\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]^{\frac{1}{\theta}}\left[-\frac{\ln\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]}{\theta^{2}}+\frac{(\hat{E}_{1(i)})^{\theta}\ln\hat{E}_{1(i)}+(\hat{E}_{2(i)})^{\theta}\ln(\hat{E}_{2(i)})}{\theta\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]}\right]}{-1+\theta+\left[(\hat{E}_{1(i)})^{\theta}+(\hat{E}_{2(i)})^{\theta}\right]^{\frac{1}{\theta}}}
\end{aligned} \tag{26}$$

where $\hat{E}_{1(i)}=-\ln\left[1-e^{-\hat{\alpha}_1 x_i}\right]$ and $\hat{E}_{2(i)}=-\ln\left[1-e^{-\hat{\alpha}_2 y_i}\right]$.
In case of GH copula, we use r-package 'Gumbel' (version 1.10-2, Caillat, Dutang, Larrieu, and Nguyen (2018)) to estimate θ.

3.5. Asymptotic properties of the likelihood-based estimators

Let $\{(X_i,Y_i), i=1,2,\cdots,n\}$ be a bivariate random sample from (X,Y). Let $\eta=(\alpha_1,\alpha_2,\theta)$. The estimator $\hat{\eta}=(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta}\ )$ of η is consistent for η (Joe 1997).
To obtain the asymptotic distribution, let

$$g=\left(\frac{\partial\ln L_1}{\partial\alpha_1},\frac{\partial\ln L_2}{\partial\alpha_2},\frac{\partial\ln L}{\partial\theta}\right)=(g_1,g_2,g_3)$$

be a row vector, where $L_1=\alpha_1^n e^{-\alpha_1\sum_{i=1}^{n}X_i}$, $L_2=\alpha_2^n e^{-\alpha_2\sum_{i=1}^{n}Y_i}$ and L is the conditional likelihood of $Y_i'^s$ given $X_i'^s$ and depends on the copula function considered. We get $g_1=\frac{n}{\alpha_1}-\sum_{i=1}^{n}X_i$, $g_2=\frac{n}{\alpha_2}-\sum_{i=1}^{n}Y_i$ and g_3 for FGM, AMH, Gumbel's bivariate exponential, and GH copulas are given by Equations (20), (22), (24) and (26) respectively.
Let $\hat{\eta}=\left(\hat{\alpha}_1,\hat{\alpha_2},\hat{\theta}\right)$ be the estimator obtained by the two-stage estimation procedure and let $\underline{X}_n=(X_1,\cdots,X_n)$ and $\underline{Y}_n=(Y_1,\cdots,Y_n)$. The asymptotic distribution of $\sqrt{n}(\hat{\eta}-\eta)^T$ is equivalent to the asymptotic distribution of $\left\{-E\left(\frac{\partial g^T(\underline{X}_n,\underline{Y}_n,\eta)}{\partial\eta}\right)\right\}^{-1}Z$, where $Z\sim N\left(0,Cov\left(g(\underline{X}_n,\underline{Y}_n,\eta)\right)\right)$ (see Joe (1997), pp. 301). The asymptotic variance-covariance matrix of $\sqrt{n}(\hat{\eta}-\eta)^T$, called the inverse Godambe information matrix (see Joe (1997), pp. 301), is

$$V=D_g^{-1}M_g(D_g^{-1})^T$$

where

$$D_g=E\left[\frac{\partial g^T(\underline{X}_n,\underline{Y}_n,\ \eta)}{\partial\eta}\right]=E\begin{bmatrix}\frac{\partial g_1}{\partial\alpha_1} & \frac{\partial g_1}{\partial\alpha_2} & \frac{\partial g_1}{\partial\theta}\\ \frac{\partial g_2}{\partial\alpha_1} & \frac{\partial g_2}{\partial\alpha_2} & \frac{\partial g_2}{\partial\theta}\\ \frac{\partial g_3}{\partial\alpha_1} & \frac{\partial g_3}{\partial\alpha_2} & \frac{\partial g_3}{\partial\theta}\end{bmatrix}=\begin{bmatrix}-\frac{n}{\alpha_1^2} & 0 & 0\\ 0 & -\frac{n}{\alpha_2^2} & 0\\ E\left(\frac{\partial g_3}{\partial\alpha_1}\right) & E\left(\frac{\partial g_3}{\partial\alpha_2}\right) & E\left(\frac{\partial g_3}{\partial\theta}\right)\end{bmatrix} \tag{27}$$

and

$$\begin{aligned}
M_g = E\left[g^T(\underline{X}_n,\underline{Y}_n,\ \eta)\ g(\underline{X}_n,\underline{Y}_n,\ \eta)\right] &= E\begin{bmatrix} g_1^2 & g_1g_2 & g_1g_3 \\ g_2g_1 & g_2^2 & g_2g_3 \\ g_3g_1 & g_3g_2 & g_3^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{n}{\alpha_1^2} & \frac{n\theta}{4\alpha_1\alpha_2} & E\left(g_1g_3\right) \\ \frac{n\theta}{4\alpha_1\alpha_2} & \frac{n}{\alpha_2^2} & E\left(g_2g_3\right) \\ E\left(g_3g_1\right) & E\left(g_3g_2\right) & E\left(g_3^2\right) \end{bmatrix}. \qquad (28)
\end{aligned}$$

An estimator of the asymptotic variance-covariance matrix is obtained by plugging in the estimates of $\alpha_1, \alpha_2, \theta$ and then the expected values in D_g and M_g are obtained using MC methods. We have given an example of the Godambe information criterion for one combination of the parameters in Section 5.
Further, for the copulas considered in Section 2, R is a continuous function of η. Let $R = h(\eta)$. The function h is a continuous function of η. Hence, $\hat{R} = h(\hat{\eta})$ is a consistent estimator of R. Further, the function $h(.)$ has continuous first order partial derivatives. Thus using the Delta method we get

$$\sqrt{n}\left(\hat{R} - R\right) \xrightarrow{d} N\left(0, \left\{h^{'}(\eta)\right\} V \left\{h^{'}(\eta)\right\}^T\right),$$

where $h^{'}(\eta) = \left(\frac{\partial h}{\partial \alpha_1}, \frac{\partial h}{\partial \alpha_2}, \frac{\partial h}{\partial \theta}\right)$ and $\xrightarrow{d}$ denotes 'convergence in distribution'.

4. Estimation using Blomqvist's beta

Blomqvist's beta is the medial correlation coefficient (Nelsen (1999), Blomqvist (1950)). The closed-form of the population version of Blomqvist's beta,

$$\beta(C_\theta) = -1 + 4C_\theta(1/2, 1/2), \qquad (29)$$

is available for many copulas. In addition, Blomqvist's beta often provides an accurate approximation to Spearman's rho and Kendall's tau (Nelsen (1999), pp. 148). Based on a random sample $(X_i, Y_i), i = 1, 2, \ldots, n$, from the bivariate distribution, the sample version β_n of Blomqvist's beta is:

$$\beta_n = \frac{n_1 - n_2}{n_1 + n_2} \qquad (30)$$

where
n_1: the number of sample points in x-y plane such that either $X_i > \tilde{X}_n$ and $Y_i > \tilde{Y}_n$ or $X_i < \tilde{X}_n$ and $Y_i < \tilde{Y}_n$
n_2: the number of sample points in x-y plane such that either $X_i > \tilde{X}_n$ and $Y_i < \tilde{Y}_n$ or $X_i < \tilde{X}_n$ and $Y_i > \tilde{Y}_n$
and $\tilde{X}_n$ and $\tilde{Y}_n$ are the sample medians of the corresponding random variables.
The asymptotic distribution of β_n is investigated by Schmid and Schmidt (2007) in the multivariate case and for the bivariate case the result is:

$$\sqrt{n}\left(\beta_n - \beta\right) \xrightarrow{d} N\left(0, \sigma^2_{\beta,C}\right) \qquad as \quad n \to \infty,$$

where $\xrightarrow{d}$ denotes 'convergence in distribution'. The asymptotic variance $\sigma^2_{\beta,C}$ is reported in Genest, Aguirre, and Harvey (2013) as

$$\begin{aligned}
\sigma^2_{\beta,C} &= 16C\left(\frac{1}{2},\frac{1}{2}\right)\left\{1 - C\left(\frac{1}{2},\frac{1}{2}\right)\right\} + 4\left\{C_1\left(\frac{1}{2},\frac{1}{2}\right) - C_2\left(\frac{1}{2},\frac{1}{2}\right)\right\}^2 \\
&\quad + 16C\left(\frac{1}{2},\frac{1}{2}\right)\left\{-C_1\left(\frac{1}{2},\frac{1}{2}\right) - C_2\left(\frac{1}{2},\frac{1}{2}\right) + 2C_1\left(\frac{1}{2},\frac{1}{2}\right)C_2\left(\frac{1}{2},\frac{1}{2}\right)\right\}, \qquad (31)
\end{aligned}$$

where $C_1(u,v) = \frac{\partial C(u,v)}{\partial u}$ and $C_2(u,v) = \frac{\partial C(u,v)}{\partial v}$ must exist everywhere and be continuous on $[0,1]^2$. Then solving the equation $\beta(C_\theta) = \beta_n$ for θ we get the sample estimate $\theta_{\beta,n}$ of θ for a given copula. For details see Genest *et al.* (2013) . The estimator of θ can be expressed as $\theta_{\beta,n} = g_\beta(\beta_n)$ where $\theta = g_\beta(\beta)$. We note that an estimate of θ obtained from Blomqvist's beta does not involve the estimates of the parameters of the margins. Further, assuming $g'_\beta(\beta)$ exists and is non zero, the Delta method gives the asymptotic behaviour of $\theta_{\beta,n}$ as

$$\sqrt{n}\,(\theta_{\beta,n} - \theta) \xrightarrow{d} N\left(0, \sigma^2_{\theta,C}\right), \qquad as \quad n \to \infty,$$

where $\sigma^2_{\theta,C} = \left\{g'_\beta(\beta)\right\}^2 \sigma^2_{\beta,C}$.
Using the estimates $(\hat{\alpha}_1, \hat{\alpha}_2, \theta_{\beta,n})$ of the parameters $(\alpha_1, \alpha_2, \theta)$, we get an estimate $\hat{R}_\beta$ of R. Note that, for the copulas considered in Section 2, R is a continuous function of $\eta = (\alpha_1, \alpha_2, \theta)$. Now, the estimator $\hat{\eta}_\beta$ of η is consistent for η (see Genest *et al.* (2013)). Therefore, $\hat{R}_\beta$ is consistent for R.
For obtaining the asymptotic distribution of the estimator of R, we first consider the case when α_1 and α_2 are known. Therefore, for known α_1 and α_2, the estimator of R can be expressed as $R_{\theta,n} = h_\theta(\theta_n)$, a function of θ only. The Delta method gives the asymptotic behaviour of $R_{\theta,n}$ as

$$\sqrt{n}\,(R_{\theta,n} - R) \xrightarrow{d} N\left(0, \sigma^2_{R,C}\right), \qquad as \quad n \to \infty,$$

where $\sigma^2_{R,C} = \left\{h'_\theta(\theta)\right\}^2 \sigma^2_{\theta,C}$. The Blomqvist's beta and the asymptotic variance $\sigma^2_{R,C}$ for the copulas considered in Section 2 are discussed in the following subsections.

4.1. FGM copula

From Equation (29), we get $\beta(C_\theta) = \theta/4$. Moreover, $\beta \in [-1/4, 1/4]$. Therefore, θ is obtained by inversion of Blomqvist's beta as $\theta = 4\beta$. The explicit expression for the asymptotic variance for the estimator of β has been given in Genest *et al.* (2013) and it is easily verified, using Equation (31), that

$$\sigma^2_{\beta,C} = 1 - \frac{\theta^2}{16}.$$

For this copula, the function $g_\beta(\beta) = 4\beta$, hence $g'_\beta(\beta) = 4$. Therefore, the asymptotic variance of $\theta_{\beta,n}$ is given by

$$\sigma^2_{\theta,C} = \left\{g'_\beta(\beta)\right\}^2 \sigma^2_{\beta,C} = 16 - \theta^2.$$

The explicit expression for the asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$, for this copula, when the parameters of the margins are known, using the Delta method, which is given in Appendix 1.

4.2. AMH copula

From Equation (29), we get $\beta(C_\theta) = \theta/(4-\theta)$. Moreover, $\beta \in [-1/5, 1/3]$. Therefore, θ is obtained by inversion of Blomqvist's beta as $\theta = 4\beta/(1+\beta)$. The explicit expression for the asymptotic variance of β_n, has been given by Schmid and Schmidt (2007) and it is easily verified, using Equation (31), that

$$\sigma^2_{\beta,C} = \frac{16(\theta^4 - 7\theta^3 + 36\theta^2 - 80\theta + 64)}{(4-\theta)^5}.$$

For this copula, the function $g_\beta(\beta) = 4\beta/(1+\beta)$, hence $g'_\beta(\beta) = 4/(1+\beta)^2$. Therefore, the asymptotic variance of $\theta_{\beta,n}$ is given by

$$\sigma^2_{\theta,C} = \left\{g'_\beta(\beta)\right\}^2 \sigma^2_{\beta,C} = \frac{(\theta^4 - 7\theta^3 + 36\theta^2 - 80\theta + 64)}{(4-\theta)}.$$

The explicit expression for the asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$, for this copula, when the parameters of the margins are known, using the Delta method, which is given in Appendix 1.

4.3. Gumbel's bivariate exponential copula

From Equation (29), we get $\beta(C_\theta) = -1 + e^{-\theta(\ln 2)^2}$. Moreover, $\beta \in [-0.381497, 0]$. Therefore, θ is obtained by inversion of Blomqvist's beta as $\theta = -\ln(\beta+1)/(\ln 2)^2$.
In order to compute asymptotic variance $\sigma^2_{\beta,C}$ of β_n one can easily find $C_\theta(1/2, 1/2) = \frac{1}{4}e^{-\theta(\ln 2)^2}$ and $C_1(1/2, 1/2) = C_2(1/2, 1/2) = 1 - \frac{1}{2}(1 + \theta \ln 2)e^{-\theta(\ln 2)^2}$. Therefore, Equation (31) yields

$$\sigma^2_{\beta,C} = 4e^{-\theta(\ln 2)^2}\left[-1 + e^{-\theta(\ln 2)^2}\left(\frac{3}{4} + \theta\ln 2\right) + 2\left(1 - \frac{1}{2}(1+\theta\ln 2)\,e^{-\theta(\ln 2)^2}\right)^2\right].$$

Using the fact that, the function $g_\beta(\beta) = -\ln(\beta+1)/(\ln 2)^2$, hence $g'_\beta(\beta) = -\frac{1}{(\ln 2)^2(\beta+1)}$, the asymptotic variance of $\theta_{\beta,n}$ is given by

$$\begin{aligned}\sigma^2_{\theta,C} &= \left\{g'_\beta(\beta)\right\}^2 \sigma^2_{\beta,C} \\ &= \frac{4}{(\ln 2)^4 e^{-\theta(\ln 2)^2}}\left[-1 + e^{-\theta(\ln 2)^2}\left(\frac{3}{4} + \theta\ln 2\right) + 2\left(1 - \frac{1}{2}(1+\theta\ln 2)\,e^{-\theta(\ln 2)^2}\right)^2\right].\end{aligned}$$

The explicit expression for the asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$, for this copula, when the parameters of the margins are known, using the Delta method, which is given in Appendix 1.

4.4. GH copula

From Equation (29), we get $\beta(C_\theta) = -1 + 4e^{-2^{1/\theta}\ln 2}$. Moreover, $\beta \in [0, 1)$. Therefore, θ is obtained by inversion of Blomqvist's beta as $\theta = \ln 2/\ln\left[\ln(4/(\beta+1))/\ln 2\right]$.
The explicit expression for asymptotic variance of β_n has been given by Schmid and Schmidt (2007) and also is easily verified, using Equation (31), that

$$\sigma^2_{\beta,C} = 8h_\theta\left[1 - 2h_\theta + \left(2^{\frac{1}{\theta}+1}h_\theta - 1\right)^2\right]$$

where $h_\theta = exp\left(-2^{1/\theta}\ln 2\right)$. Using the fact that, the function $g_\beta(\beta) = \ln 2/\ln\left(\frac{\ln(4/(\beta+1))}{\ln 2}\right)$, hence $g'_\beta(\beta) = \frac{\ln 2}{(\beta+1)\ln\left(\frac{4}{\beta+1}\right)\left[\ln\left(\frac{\ln(4/(\beta+1))}{\ln 2}\right)\right]^2}$, the asymptotic variance of θ_β is given by

$$\sigma^2_{\theta,C} = \left\{g'_\beta(\beta)\right\}^2\sigma^2_{\beta,C} = \frac{(\ln 2)^2\left[1 - 2h_\theta + \left(2^{\frac{1}{\theta}+1}h_\theta - 1\right)^2\right]}{2h_\theta\left(\ln h_\theta\right)^2\left[\ln\left(-\ln h_\theta\right) - \ln(\ln 2)\right]^2}.$$

Due to unavailability of explicit expression for R, it was not possible to obtain an explicit expression for the asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$ for this copula. An estimate of $\sigma^2_{R,C}$ can be obtained using the bootstrap procedure.

4.5. Asymptotic properties of an estimator of R, when α_1, α_2 and θ are unknown

Let $\{(X_i, Y_i), i = 1, 2, \cdots, n\}$ be a bivariate random sample from (X, Y). Let $\eta = (\alpha_1, \alpha_2, \theta)$. Let g, g_1, g_2 be as defined in Section 3.5 and $g_3 = \beta_n - \beta$.
Let $\hat{\eta} = (\hat{\alpha}_1, \hat{\alpha}_2, \theta_{\beta,n})$ be the estimator obtained by the two-stage estimation procedure. The inverse Godambe information matrix (Joe (1997), pp. 301) discussed in Section 3.5 is

$$V = D_g^{-1}M_g(D_g^{-1})^T$$

where D_g and M_g are given in Equations (27) and (28) respectively.
Now, since the n pairs $(X_i, Y_i), i = 1, \cdots, n$ are i.i.d, using the result from Schmid and Schmidt (2007) , Joe (1997) (pp. 301) and the Cramer Wold device (Billingsley (1995), pp. 383) it can be shown that

$$\left(\sqrt{n}\left(\hat{\alpha}_1 - \alpha_1\right), \sqrt{n}\left(\hat{\alpha}_2 - \alpha_2\right), \sqrt{n}\left(\beta_n - \beta\right)\right) \xrightarrow{d} N\left(0, V\right)), \qquad as \quad n \to \infty.$$

Next, the estimator of θ can be expressed as $\theta_{\beta,n} = g_\beta(\beta_n)$ and assuming $g_\beta^{'}(\beta)$ exists and is non zero, the Delta method gives the following asymptotic distribution

$$\left(\sqrt{n}\left(\hat{\alpha}_1 - \alpha_1\right), \sqrt{n}\left(\hat{\alpha}_2 - \alpha_2\right), \sqrt{n}\left(\theta_{\beta,n} - \theta\right)\right) \xrightarrow{d} N\left(0, V_1\right)), \qquad as \quad n \to \infty,$$

where

$$V_1 \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g_\beta^{'}(\beta) \end{bmatrix} V \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g_\beta^{'}(\beta) \end{bmatrix}^T.$$

Further, the estimator of R can be expressed as $R_n = h(\hat{\alpha}_1, \hat{\alpha}_2, \theta_{\beta,n})$ and $h(.)$ has continuous first order partial derivatives. Again using the Delta method we get

$$\sqrt{n}\left(R_n - R\right) \xrightarrow{d} N\left(0, V_2\right), \qquad as \quad n \to \infty,$$

where $V_2 = \left(\frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \alpha_2}, \frac{\partial R}{\partial \theta}\right) V_1 \left(\frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \alpha_2}, \frac{\partial R}{\partial \theta}\right)^T$.

5. Simulation study

To study the performance of the estimators we simulated different data sets for the expected values $(E(X), E(Y))$ corresponding to the parameters

$$(\alpha_1, \alpha_2) = \{(2, 3), (2, 5), (3, 2), (5, 2)\}$$

and different values of θ in the parameter space, for each of the four copulas discussed in earlier sections. For each combination of expected values and dependence parameter θ, 100 data sets were simulated with the sample size of each being 50. We first draw a random sample $(x_1, x_2, \cdots, x_n)$ from an exponential distribution with parameter α_1. Next, we simulate y_i from the conditional distribution of Y given $X = x_i$. For details about the conditional distribution of Y given $X = x_i$ see Appendix 2. Then parameters α_1 and α_2 of the margins are estimated using maximum likelihood (ML) estimation.
Let $\hat{\theta}$ denote the estimator of θ using a two-stage ML procedure and $\theta_{\beta,n}$ denote the estimator of θ using Blomqvist's beta. Let $\hat{R}$ denote the estimator of R using a two-stage ML procedure and $R_{\theta,n}$ denote the estimator of R using Blomqvist's beta.
We estimate the dependence parameter θ using a two-stage ML estimation. Further, given the estimates $\left(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta}\right)$, we get the estimate $\hat{R}$ of R. The results of the simulation study are reported in Table 1 for FGM and AMH copulas, in Table 2 for Gumbel's bivariate exponential copula and in Table 3 for GH copula. For each cell in Tables 1-3, the first row is the estimate $\hat{\theta}$ of θ, the second row is the mean square errors (MSE) for $\hat{\theta}$. It is observed that, from Tables 1-3, the estimates $\hat{\theta}$ are close to its true values.

Table 1: Estimates $\hat{\theta}$ for the FGM and the AMH copula with the sample size n=50. The values in the second row are MSE for $\hat{\theta}$.

		θ					
Copula	(α_1, α_2)	-0.9	-0.5	-0.1	0.1	0.5	0.9
FGM	(2, 3)	-0.8281	-0.4055	-0.1554	0.1047	0.4573	0.8365
		0.0565	0.1692	0.1537	0.1620	0.1898	0.0601
	(2, 5)	-0.8065	-0.4995	-0.03080	0.1025	0.4431	0.8156
		0.0893	0.1470	0.2120	0.2291	0.1298	0.0653
	(3, 2)	-0.8107	-0.4366	-0.0755	0.1379	0.4972	0.8281
		0.0868	0.1490	0.2163	0.1594	0.1644	0.0586
	(5, 2)	-0.8205	-0.4466	-0.0832	0.07200	0.4599	0.8173
		0.0816	0.1531	0.1647	0.2061	0.1488	0.0668
AMH	(2, 3)	-0.8113	-0.4295	-0.1740	0.0619	0.5151	0.8335
		0.0700	0.2261	0.2424	0.1927	0.08517	0.0588
	(2, 5)	-0.8155	-0.4147	-0.1068	0.0800	0.4313	0.8121
		0.0812	0.2179	0.2586	0.1393	0.1409	0.1836
	(3, 2)	-0.8209	-0.5535	-0.0887	0.0856	0.4662	0.8140
		0.1048	0.1460	0.2099	0.1749	0.0948	0.1167
	(5, 2)	-0.8018	-0.4721	-0.1034	0.0715	0.4606	0.8243
		0.08365	0.1927	0.2169	0.2145	0.08753	0.1214

Table 2: Estimates $\hat{\theta}$ for the Gumbel's bivariate exponential copula with the sample size n=50. The values in the second row are MSE for $\hat{\theta}$.

	θ				
(α_1, α_2)	0.1	0.3	0.5	0.7	0.9
(2, 3)	0.1216	0.3580	0.5487	0.6497	0.8662
	0.0130	0.0356	0.0271	0.0365	0.0276
(2, 5)	0.1159	0.2936	0.4963	0.6995	0.9274
	0.0092	0.0326	0.0471	0.0427	0.0856
(3, 2)	0.1258	0.3497	0.5226	0.6861	0.8891
	0.0154	0.0257	0.0716	0.0237	0.0217
(5, 2)	0.1171	0.2814	0.4868	0.6911	0.8950
	0.0108	0.0214	0.0383	0.0159	0.0073

Table 3: Estimates $\hat{\theta}$ for the GH copula with the sample size n=50. The values in the second row are MSE for $\hat{\theta}$.

	θ				
(α_1, α_2)	2	4	6	8	10
(2, 3)	2.0546	4.0439	6.0456	8.1217	9.4665
	0.0692	0.2523	0.5635	1.2066	0.5355
(2, 5)	2.0099	4.1780	5.9990	8.1075	9.5797
	0.0924	0.4710	0.6340	1.1877	0.4126
(3, 2)	2.0303	4.1132	6.1548	8.1388	9.5457
	0.0655	0.2072	0.7787	0.9372	0.5736
(5, 2)	2.0136	4.1692	6.0048	8.07732	9.5188
	0.0761	0.3110	0.6685	1.3253	0.5308

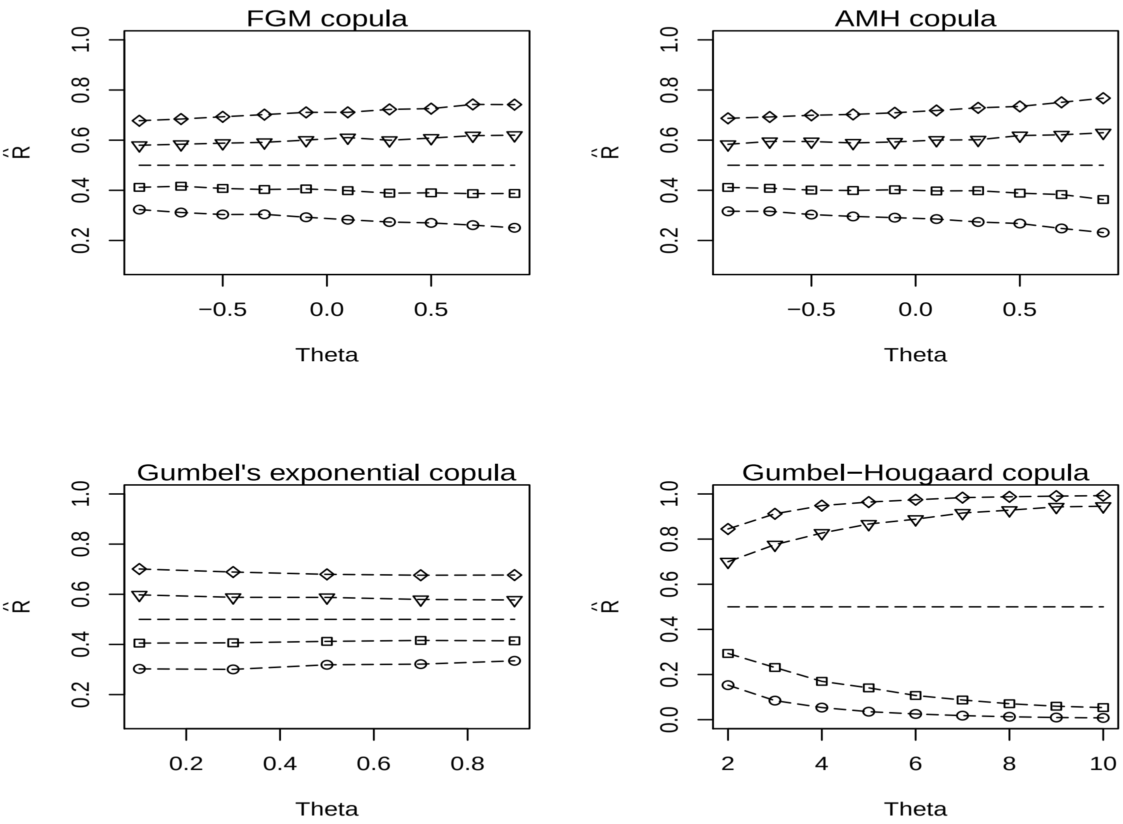

Figure 2: Variation in $\hat{R}$ versus dependence parameter θ (Theta) for the four pairs of $\{(E(X), E(Y)) = (0.5, 0.33)\bigtriangledown,\ (0.5, 0.2)\Diamond,\ (0.33, 0.5)\Box,\ (0.2, 0.5)\bigcirc\}$

Table 4: Estimates $\hat{R}$ for the FGM and the AMH copula with the sample size n=50. The values in the second row are MSE for $\hat{R}$.

				θ			
Copula	(α_1, α_2)	-0.9	-0.5	-0.1	0.1	0.5	0.9
FGM	(2, 3)	0.5797	0.5880	0.6003	0.6103	0.6084	0.6198
		0.0018	0.0022	0.0019	0.0021	0.0026	0.0020
	(2, 5)	0.6775	0.6933	0.7109	0.7111	0.7258	0.7418
		0.0014	0.0020	0.0022	0.0022	0.0017	0.0017
	(3, 2)	0.4118	0.4074	0.4056	0.3987	0.3900	0.3874
		0.0018	0.0021	0.0023	0.0026	0.0024	0.0020
	(5, 2)	0.3231	0.3035	0.2925	0.28287	0.2701	0.2506
		0.0014	0.0019	0.0019	0.0021	0.0018	0.0014
AMH	(2, 3)	0.5835	0.5946	0.5929	0.6003	0.6184	0.6296
		0.0017	0.0021	0.0023	0.0025	0.0032	0.0024
	(2, 5)	0.6871	0.6993	0.7092	0.7184	0.7346	0.7675
		0.0018	0.0018	0.0021	0.0022	0.0020	0.0018
	(3, 2)	0.4115	0.4008	0.4023	0.3975	0.3886	0.3633
		0.0033	0.0021	0.0024	0.0025	0.0024	0.0031
	(5, 2)	0.3164	0.3033	0.2910	0.2854	0.2676	0.2315
		0.0017	0.0025	0.0022	0.0027	0.0020	0.0019

Figure 2 illustrates the variation in $\hat{R}$ with respect to θ for the four copulas and for the four

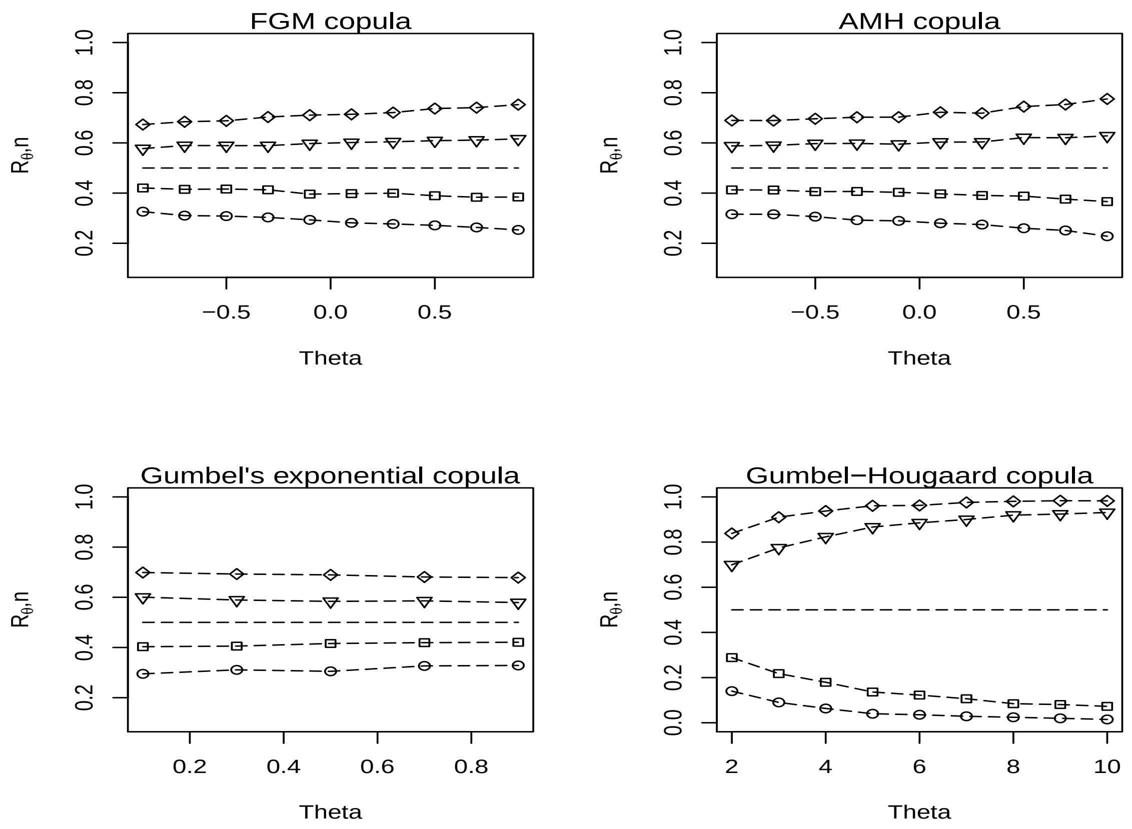

Figure 3: Variation in $R_{\theta,n}$ versus dependence parameter θ (Theta) for the four pairs of $\{(E(X), E(Y)) = (0.5, 0.33)\bigtriangledown,\quad (0.5, 0.2)\Diamond,\quad (0.33, 0.5)\Box,\quad (0.2, 0.5)\bigcirc\}$

different pairs of $(E(X), E(Y))$ considered in Section 2.5. From the Figure, the pattern of variation in estimates of R with respect to θ is found to be the same as the pattern of variation in true R with respect to θ. The values of $\hat{R}$ and MSEs for $\hat{R}$ reported in Tables 4-6. MSEs for $\hat{R}$ in Tables 4-6 are considerably small.

Example of Godambe information: To find the Godambe information matrix, we obtained expected values $E\left(\frac{\partial g_3}{\partial \alpha_1}\right)$, $E\left(\frac{\partial g_3}{\partial \alpha_2}\right)$ and $E\left(\frac{\partial g_3}{\partial \theta}\right)$ in D_g and $E\left(g_3^2\right)$, $E\left(g_1 g_3\right)$ and $E\left(g_2 g_3\right)$ in M_g using a MC methods. For example, we consider $(\alpha_1, \alpha_2, \theta) = (2, 3, 0.5)$ for FGM, AMH, and Gumbel's bivariate exponential copula and $(\alpha_1, \alpha_2, \theta) = (2, 3, 5)$ for GH copula. The inverse Godambe information matrix V are found to be respectively

$$\underset{\text{FGM}}{\begin{bmatrix} 0.08 & 0.2591 & -0.00440 \\ 0.2591 & 0.18 & 0.02448 \\ -0.00440 & 0.024448 & 0.71364 \end{bmatrix}} \qquad \underset{\text{AMH}}{\begin{bmatrix} 0.08 & -0.08403 & 0.39004 \\ -0.08403 & 0.18 & -0.18882 \\ 0.3910 & -0.18882 & 12.4109 \end{bmatrix}}$$

$$\underset{\text{Gumbel's bivariate exponential copula}}{\begin{bmatrix} 0.08 & -0.022165 & -0.62412 \\ -0.02216 & 0.18 & -0.91373 \\ -0.62412 & -0.91373 & 5.32552 \end{bmatrix}} \qquad \underset{\text{GH}}{\begin{bmatrix} 0.08 & 0.00476 & -0.0005 \\ 0.00476 & 0.18 & -0.0004 \\ -0.0005 & -0.0004 & 0.01589 \end{bmatrix}}.$$

The rank-based moment estimates of β_n is obtained using Equation (30). For large n, the estimator β_n is normally distributed with asymptotic variance $\sigma^2_{\beta,C}$ given in Equation (31). The estimates $\theta_{\beta,n}$ of dependence parameter is then obtained by solving the equation $\beta(C_\theta) =$

Table 5: Estimates $\hat{R}$ for the Gumbel's bivariate exponential copula with the sample size n=50. The values in the second row are MSE for $\hat{R}$.

	θ				
(α_1, α_2)	0.1	0.3	0.5	0.7	0.9
(2, 3)	0.5977	0.5880	0.5876	0.5794	0.5773
	0.0015	0.0016	0.0020	0.0019	0.0039
(2, 5)	0.7008	0.6887	0.6795	0.6760	0.6769
	0.0035	0.0029	0.0028	0.0031	0.0039
(3, 2)	0.4053	0.4064	0.4122	0.4158	0.4144
	0.0021	0.0078	0.0020	0.0021	0.0041
(5, 2)	0.3026	0.3005	0.3188	0.3215	0.3346
	0.0047	0.0033	0.003	0.0022	0.0049

Table 6: Estimates $\hat{R}$ for the GH copula with the sample size n=50. The values in the second row are MSE for $\hat{R}$.

	θ				
(α_1, α_2)	2	4	6	8	10
(2, 3)	0.6997	0.8269	0.8883	0.9282	0.9458
	0.0025	0.0009	0.0005	0.0003	0.0001
(2, 5)	0.8448	0.9485	0.9744	0.9875	0.9926
	0.0009	0.0002	0.0001	0.00002	0.00001
(3, 2)	0.2933	0.1697	0.1071	0.0706	0.0535
	0.0021	0.0008	0.0005	0.0002	0.0001
(5, 2)	0.1529	0.0536	0.0253	0.0129	0.0077
	0.0010	0.0002	0.0001	0.00003	0.000004

β_n, using Equations (29) and (30), for θ. Further, given the estimates $(\hat{\alpha}_1, \hat{\alpha}_2, \theta_{\beta,n})$, we get estimate $R_{\theta,n}$.

The results of the simulation study are reported in Table 7 for FGM and AMH copulas, in Table 8 for Gumbel's bivariate exponential copula and in Table 9 for GH copula. For each cell in Tables 7-9 the first row is the estimate $\theta_{\beta,n}$ of θ, the second row is the MSE for $\theta_{\beta,n}$ and the values in the brackets are 95 percent confidence limits for θ based on normal approximation. From Tables 7-9, it is observed that the estimates $\theta_{\beta,n}$ of θ are close to its true values except for $\theta = -0.9, 0.9$ in the case of AMH copula. The MSE for $\theta_{\beta,n}$ are quite large and increases with θ in the case of GH copula.

Figure 3 illustrates the variation in $R_{\theta,n}$ with respect to θ for the four copulas and for four different pairs of $(E(X), E(Y))$ considered in Section 2.5. From the Figure, the pattern of variation in estimates $R_{\theta,n}$ of R with respect to θ is found to be the same as the pattern of variation in true R with respect to θ. The values of $R_{\theta,n}$ and MSEs for $R_{\theta,n}$ are reported in Tables 10-12.

6. Example

We use the data set of stress-strength measurements presented in Dargahi-Noubary and Nanthakumar (1992) and reproduced in Table 13. In these data, we have 15 pairs of measurements on strength X subject to stress Y. Basu (1981) and Dargahi-Noubary and Nanthakumar (1992) have found an estimate $\hat{R} = 0.9639$ and $\hat{R} = 0.948$ respectively assuming X and Y are independent exponential random variables. The Kendall's tau, Blomqvist's beta and the correlation coefficient for the data used are: -0.238, -0.3333 and -0.385 respectively. Now, from the range of Kendall's tau (τ) and Blomqvist's beta (β) reported in the introduction we

Table 7: Estimates $\theta_{\beta,n}$ for the FGM and the AMH copula with the sample size n=50. The values in the second row are MSE for $\theta_{\beta,n}$. The values in the brackets are the 95 percent confidence limits for θ.

		θ					
Copula	(α_1, α_2)	-0.9	-0.5	-0.1	0.1	0.5	0.9
FGM	(2, 3)	-0.8672	-0.4224	-0.0640	0.0960	0.5248	0.8320
		0.2873	0.3203	0.2509	0.3256	0.3011	0.3942
		(-0.97,-0.76)	(-0.53,-0.31)	(-0.16,0.035)	(-0.016,0.21)	(0.42, 0.63)	(0.71,0.96)
	(2, 5)	-0.8672	-0.6016	-0.0672	0.0896	0.5248	0.9376
		0.3549	0.3866	0.3160	0.4108	0.3093	0.2524
		(-0.98,-0.75)	(-0.72,-0.48)	(-0.18,0.043)	(-0.037,0.22)	(0.42,0.63)	(0.84,1.00)
	(3, 2)	-0.8640	-0.4032	-0.1024	0.1664	0.4992	0.9536
		0.3195	0.3402	0.2814	0.3072	0.3273	0.2959
		(-0.98,-0.75)	(-0.52,-0.29)	(-0.21,0.00)	(0.057,0.28)	(0.39, 0.61)	(0.85,1.00)
	(5, 2)	-0.8928	-0.5184	-0.1440	0.1120	0.5280	0.9120
		0.3283	0.2996	0.3182	0.2854	0.2035	0.3182
		(-1.00,-0.78)	(-0.63,-0.41)	(-0.26,-0.03)	(0.00,0.22)	(0.44,0.62)	(0.80,1.00)
AMH	(2, 3)	-0.7203	-0.4065	-0.1286	0.0951	0.4233	0.7652
		0.1624	0.2978	0.2791	0.2910	0.1793	0.0892
		(-0.80,-0.64)	(-0.51,-0.30)	(-0.23,-0.024)	(-0.01,0.20)	(0.34,0.51)	(0.71,0.82)
	(2, 5)	-0.7078	-0.5198	-0.1842	0.06572	0.4451	0.7710
		0.1669	0.2458	0.2903	0.2817	0.1760	0.0941
		(-0.79,-0.63)	(-0.62,-0.42)	(-0.29,-0.08)	(-0.04,0.17)	(0.36,0.53)	(0.71,0.83)
	(3, 2)	-0.6709	-0.4480	-0.1397	-0.0165	0.4353	0.7877
		0.2013	0.2653	0.2792	0.2563	0.1785	0.0704
		(-0.76,-0.58)	(-0.55,-0.35)	(-0.24,-0.04)	(-0.12,0.08)	(0.35,0.52)	(0.74,0.84)
	(5, 2)	-0.7046	-0.3969	-0.1795	0.0698	0.4705	0.7830
		0.1687	0.2339	0.2987	0.2520	0.1503	0.0859
		(-0.79,-0.62)	(-0.49,-0.30)	(-0.29,-0.072)	(-0.03,0.17)	(0.39,0.55)	(0.73,0.84)

Table 8: Estimates $\theta_{\beta,n}$ for the Gumbel's bivariate exponential copula with the sample size n=50. The values in the second row are MSE for $\theta_{\beta,n}$. The values in the brackets are the 95 percent confidence limits for θ.

	θ				
(α_1, α_2)	0.1	0.3	0.5	0.7	0.9
(2, 3)	0.1503	0.3627	0.5466	0.7283	0.9587
	0.0711	0.1111	0.1308	0.1625	0.2196
	(0.09,0.20)	(0.297,0.43)	(0.48,0.62)	(0.65,0.81)	(0.87,1)
(2, 5)	0.1522	0.3506	0.5153	0.7289	0.8682
	0.0752	0.1237	0.1378	0.1845	0.1676
	(0.09,0.21)	(0.28,0.42)	(0.44,0.59)	(0.64,0.81)	(0.79,0.95)
(3, 2)	0.1643	0.3418	0.5077	0.6900	0.9281
	0.1593	0.1004	0.1314	0.2182	0.1985
	(0.09,0.24)	(0.28,0.40)	(0.44,0.58)	(0.60,0.78)	(0.84,1)
(5, 2)	0.1707	0.3645	0.4950	0.7649	0.8926
	0.1676	0.1516	0.1125	0.1732	0.1631
	(0.09,0.25)	(0.29,0.44)	(0.43,0.56)	(0.68,0.85)	(0.81,0.97)

Table 9: Estimates $\theta_{\beta,n}$ for the GH copula with the sample size n=50. The values in the second row are MSE for $\theta_{\beta,n}$. The values in the brackets are the 95 percent confidence limits for θ.

	θ				
(α_1, α_2)	2	4	6	8	10
(2, 3)	2.0230	4.4042	6.9499	8.8062	9.4377
	0.2872	4.5181	11.3702	11.8908	10.1487
	(1.92,2.13)	(3.98,4.83)	(6.28,7.62)	(8.11,9.51)	(8.78,10.10)
(2, 5)	1.9741	4.4892	6.3003	8.2294	9.3814
	0.2204	5.8700	10.1177	10.5345	10.6858
	(1.88,2.07)	(4.01,4.97)	(5.64,6.95)	(7.57,8.89)	(8.74,10.03)
(3, 2)	2.1448	4.4512	6.5722	8.5858	9.5477
	0.2492	4.8213	10.2863	12.0253	10.4583
	(2.05,2.24)	(4.02,4.88)	(5.93,7.21)	(7.87,9.31)	(8.91,10.18)
(5, 2)	2.2263	4.3738	6.6710	7.8541	9.3968
	0.3394	4.2880	11.1465	11.9336	10.5811
	(2.11,2.34)	(3.97,4.78)	(6.01,7.33)	(6.65,8.00)	(8.76,10.04)

conclude that Gumbel's bivariate exponential copula is the only copula that can model the dependency between the variables among the four copulas considered. The mles of the parameters of exponential margins for this data are found to be $(\hat{\alpha}_1, \hat{\alpha}_2) = (0.735168, 19.6463)$. The two-stage likelihood-based estimate $\hat{\theta}$, estimate $\theta_{\beta,n}$ based on Blomqvist's beta and corresponding estimates $\hat{R}$, $R_{\theta,n}$ of R for the Gumbel's bivariate exponential copula model are: 0.81, 0.84, 0.941303, and 0.940478 respectively. We note that both the estimates of R are slightly less than the estimates obtained assuming independence.
The AIC measure is given by

$$AIC = -2 \ln L(\hat{\theta},\ \hat{\theta}\ is\ the\ mle) + 2(number\ \ of\ \ model\ \ parameters).$$

We replace the mles by the two-stage consistent estimators. For stress-strength measurements data set, the estimate of AIC measure for product (independent) copula is found to be -16.10708 and AIC measure for Gumbel's bivariate exponential copula is found to be -55.824773.

Table 10: Estimates $R_{\theta,n}$ for the FGM and the AMH copula with the sample size n=50. The values in the second row are MSE for $R_{\theta,n}$.

		θ					
Copula	(α_1, α_2)	-0.9	-0.5	-0.1	0.1	0.5	0.9
FGM	(2, 3)	0.5774	0.5893	0.5974	0.6016	0.6089	0.6159
		0.5684	0.0024	0.0022	0.0024	0.0024	0.0023
	(2, 5)	0.6733	0.6880	0.7108	0.7140	0.7370	0.7527
		0.0018	0.6880	0.0018	0.0026	0.0020	0.0021
	(3, 2)	0.4204	0.4161	0.3959	0.3979	0.38956	0.3845
		0.0021	0.0024	0.0022	0.0027	0.0022	0.0026
	(5, 2)	0.3260	0.3084	0.2933	0.2814	0.2715	0.2534
		0.0020	0.0019	0.0017	0.0015	0.0020	0.0026
AMH	(2, 3)	0.5881	0.5973	0.5949	0.6032	0.6213	0.6280
		0.0023	0.0019	0.0026	0.0021	0.0023	0.0022
	(2, 5)	0.6895	0.6963	0.7026	0.7222	0.7186	0.7752
		0.0016	0.0014	0.0015	0.0016	0.0017	0.0019
	(3, 2)	0.4128	0.4057	0.4031	0.3969	0.3880	0.3661
		0.0016	0.0018	0.0022	0.0023	0.0029	0.0032
	(5, 2)	0.3157	0.3062	0.2894	0.2799	0.2599	0.2281
		0.0016	0.0016	0.0017	0.0013	0.0015	0.0016

Table 11: Estimates $R_{\theta,n}$ for the Gumbel's bivariate exponential copula with the sample size n=50. The values in the second row are MSE for $R_{\theta,n}$.

	θ				
(α_1, α_2)	0.1	0.3	0.5	0.7	0.9
(2, 3)	0.6004	0.5893	0.5837	0.5858	0.5790
	0.0025	0.0023	0.0023	0.0016	0.0026
(2, 5)	0.6988	0.6927	0.6894	0.6809	0.6785
	0.0019	0.0020	0.0021	0.0018	0.0019
(3, 2)	0.4031	0.4055	0.4158	0.4192	0.4208
	0.0029	0.0016	0.0022	0.0023	0.0027
(5, 2)	0.2950	0.3108	0.3050	0.3266	0.3284
	0.0018	0.0017	0.0025	0.0016	0.0019

7. Conclusions

This article aims to study the expression for the reliability $R = P(Y < X)$ when the variables are dependent with exponential margins and study the effect of dependency on it. The dependency is modeled through copula functions. The copula function to be used depends on the underlying situation. We have studied the expression for R for four important copula functions and have studied also its variations with respect to the dependence parameter θ. It is seen that the variation in R with respect to θ is moderate in the case of FGM copula, AMH copula, and Gumbel's bivariate exponential copula. For the GH copula, the variation in R with respect to θ is more than that for the other three copulas. If $\alpha_1 = \alpha_2$, R equals 1/2 for every θ in case of the FGM, the AMH, and the Gumbel's bivariate exponential copulas. However, it depends on θ for the GH copula. We were not able to obtain an explicit expression for R in the case of GH copula and used the MC technique for its computation. For this copula, there seems to be a fairly large variation in R with respect to θ compared to the other three copulas considered. The performance of the estimators of the dependence parameter θ and R are studied via simulations. In general, it is observed that the estimates of θ are close to

Table 12: Estimates $R_{\theta,n}$ for the GH copula with the sample size n=50. The values in the second row are MSE for $R_{\theta,n}$.

	θ				
(α_1, α_2)	2	4	6	8	10
(2, 3)	0.6994	0.8232	0.8853	0.9192	0.9311
	0.0033	0.0041	0.0040	0.0028	0.0021
(2, 5)	0.8383	0.9376	0.9624	0.9804	0.9825
	0.0018	0.0013	0.0008	0.00033	0.00029
(3, 2)	0.2884	0.1792	0.1225	0.0844	0.0726
	0.0027	0.0049	0.0039	0.0030	0.0021
(5, 2)	0.1400	0.0633	0.0354	0.0243	0.0147
	0.0017	0.0014	0.0009	0.0005	0.0003

Table 13: Strength X and stress Y

X	1.77	0.9457	1.8985	2.6121	1.0929	0.0362	1.0615	2.3895
Y	0.0352	0.0397	0.0677	0.0233	0.087	0.1156	0.0286	0.0200
X	0.0982	0.7971	0.8316	3.2304	0.4373	2.5648	0.6377	
Y	0.0793	0.0072	0.0245	0.0251	0.0469	0.0838	0.0796	

its true value using the conditional likelihood procedure as compared to using Blomqvist's beta except for the FGM copula. The estimation procedure based on Blomqvist's beta is computationally simpler and the resulting estimates are fairly good. The pattern of variation in the estimates of R with respect to θ is found to be the same as the pattern of variation in true R with respect to θ for both the methods of estimation of θ. Mean square errors for estimates of R are considerably small. Next, we have studied the asymptotic properties of all the estimators. We have also, obtained an estimate of the corresponding Godambe information matrix. Finally, we apply our results to a real data set. It is found that Gumbel's bivariate exponential copula is the only copula that can model the dependency between the variables among the four copulas considered, for the given data set.

Appendix 1

1. The asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$ for the FGM copula, from Equation (8), is given by

$$\begin{aligned}\sigma^2_{R,C} &= \left\{h'_\theta(\theta)\right\}^2 \sigma^2_{\theta,C} \\ &= \left\{\frac{\alpha_1\alpha_2(-\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)}\right\}^2 \left\{16-\theta^2\right\}\end{aligned} \tag{32}$$

2. The asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$ for the AMH copula is given by

$$\sigma^2_{R,C} = \left\{h'_\theta(\theta)\right\}^2 \sigma^2_{\theta,C} \tag{33}$$

where, from Equation (11),

$$\begin{aligned} h'_\theta(\theta) &= \frac{\alpha_1}{(\alpha_1+\alpha_2)\theta}\left\{\frac{1}{(1-\theta)^2} - Hypergeometric2F1\left[2, \frac{\alpha_1}{\alpha_1+\alpha_2}, \frac{2\alpha_1+\alpha_2}{\alpha_1+\alpha_2}, \theta\right]\right\} \\ &+ \frac{\alpha_1}{(\alpha_1+\alpha_2)}\left\{\frac{1}{-1+\theta} - \frac{1+\theta}{(-1+\theta)^2}\right\} + \frac{\alpha_1(\alpha_1+2\alpha_2)Gamma\left(\frac{\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}\right)}{(\alpha_1+\alpha_2)^2 \;\; Gamma\left(\frac{2\alpha_1+3\alpha_2}{\alpha_1+\alpha_2}\right)} \\ &\times \left[\frac{1}{(1-\theta)^2} - Hypergeometric2F1\left[2, \frac{\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}, \frac{2\alpha_1+3\alpha_2}{\alpha_1+\alpha_2}, \theta\right]\right] \\ &+ \frac{Gamma\left(\frac{\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}\right) Hypergeometric2F1\left[2, \frac{\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}, \frac{2\alpha_1+3\alpha_2}{\alpha_1+\alpha_2}, \theta\right]}{Gamma\left(\frac{2\alpha_1+3\alpha_2}{\alpha_1+\alpha_2}\right)}, \end{aligned} \tag{34}$$

and $\sigma^2_{\theta,C}$ is defined in Section 4.2.

3. The asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$ for the Gumbel's bivariate exponential copula, from Equation (14), is given by

$$\begin{aligned} \sigma^2_{R,C} &= \left\{h'_\theta(\theta)\right\}^2 \sigma^2_{\theta,C} \\ &= \left\{\frac{(\alpha_1^2-\alpha_2^2)}{8\alpha_1\alpha_2\theta^2} + \frac{e^{\frac{(\alpha_1+\alpha_2)^2}{4\alpha_1\alpha_2\theta}}\sqrt{\pi}(\alpha_1-\alpha_2)Erfc\left[\frac{\alpha_1+\alpha_2}{2\sqrt{\alpha_1\alpha_2\theta}}\right]}{8\theta\sqrt{\alpha_1\alpha_2\theta}}\left[1+\frac{(\alpha_1+\alpha_2)^2}{2\alpha_1\alpha_1\theta}\right]\right\}^2 \sigma^2_{\theta,C}\ , \end{aligned} \tag{35}$$

where $\sigma^2_{\theta,C}$ is defined in Section 4.3.

Appendix 2

1. **FGM copula:** From Equation (7) the conditional distribution function of Y given $X = x$ is given by

$$F_{Y|X}(y|x) = \left(1-e^{-\alpha_2 y}\right)\left[1+\theta e^{-\alpha_2 y}\left(-1+2e^{-\alpha_1 x}\right)\right], \qquad x>0;\ y>0. \tag{36}$$

Therefore, using the inverse transformation, we get

$$y = -\frac{1}{\alpha_2}\ln\left[\frac{(1+\theta-2\theta e^{-\alpha_1 x}) - \sqrt{(1+\theta-2\theta e^{-\alpha_1 x})^2 - 4\theta(1-2e^{-\alpha_1 x})(1-W)}}{2\theta(1-2e^{-\alpha_1 x})}\right] \tag{37}$$

where $W \sim U(0,1)$.

2. **AMH copula:** From Equation (10) the conditional distribution function of Y given $X = x$ is given by

$$F_{Y|X}(y|x) = \frac{(1-e^{-\alpha_2 y})(1-\theta e^{-\alpha_2 y})}{(1-\theta e^{-\alpha_1 x}e^{-\alpha_2 y})^2}, \quad x>0;\ y>0. \tag{38}$$

Therefore, using the inverse transformation, we get

$$y = -\frac{1}{\alpha_2}\ln\left[\frac{-(1+\theta-2\theta We^{-\alpha_1 x}) - \sqrt{(1+\theta-2\theta We^{-\alpha_1 x})^2 + 4\theta(\theta We^{-2\alpha_1 x}-1)(1-W)}}{2\theta(\theta We^{-2\alpha_1 x}-1)}\right] \tag{39}$$

where $W \sim U\,(0,1)$.

3. **Gumbel's bivariate exponential copula:** From Equation (13) the conditional distribution function of Y given $X = x$ is given by

$$F_{Y|X}(y|x) = 1 - e^{-\theta\alpha_1\alpha_2 xy - \alpha_2 y}\left(1 + \theta\alpha_2 y\right), \quad x > 0;\; y > 0. \tag{40}$$

The inverse transformation, for this copula, is not possible for θ. We, therefore, use rejection method (Ross 1997) to draw a sample from Y. For rejection method, we consider the density function

$$g(y) = \alpha_2 e^{-\alpha_2 y} \quad y > 0; \quad \alpha_2 > 0 \tag{41}$$

as a basis, i.e., simulate a value from $g(y)$ and accept this simulated value with probability proportional to $f(y|X = x)/g(y)$ where $f(y|X = x)$ is given by Equation (23). We note that

$$\frac{f(y|X = x)}{g(y)} = e^{-\theta\alpha_1\alpha_2 xy}\left[(1 + \theta\alpha_1 x)(1 + \theta\alpha_2 y) - \theta\right]$$

is bounded for all y by the constant k given as

$$k = (1 + \theta\alpha_1 x)\left(1 + \frac{1}{\alpha_1 x}\right). \tag{42}$$

4. ***GH copula*:** From Equation (16) the conditional distribution function of Y given $X = x$ is given by

$$\begin{aligned} F_{Y|X}(y|x) \quad = \quad & \frac{\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^{\theta - 1}}{\left(1 - e^{-\alpha_1 x}\right)}\left[\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^{\theta} + \left(-\ln\left[1 - e^{-\alpha_2 y}\right]\right)^{\theta}\right]^{\frac{1}{\theta} - 1} \\ & \times Exp\left(-\left[\left(-\ln\left[1 - e^{-\alpha_1 x}\right]\right)^{\theta} + \left(-\ln\left[1 - e^{-\alpha_2 y}\right]\right)^{\theta}\right]^{\frac{1}{\theta}}\right), \qquad x > 0;\; y > 0. \end{aligned}$$

Then, using the inverse transformation, we get

$$y = -\frac{1}{\alpha_2}\ln\left[1 - e^{-\left\{-(E_1)^{\theta} + \left[(-1+\theta) ProductLog[z]\right]^{\theta}\right\}^{(1/\theta)}}\right] \tag{43}$$

where $E_1 = -\ln\left[1 - e^{-\alpha_1 x}\right]$, $z = \frac{\left[W\left(1 - e^{-\alpha_1 x}\right)(E_1)^{(1-\theta)}\right]^{\frac{1}{1-\theta}}}{-1+\theta}$, $W \sim U(0, 1)$ and ProductLog[z] gives the principal solution for $z = we^{w}$.

References

Basu AP (1981). "The Estimation of P (x < y) for Distributions Useful in Life Testing." *Naval Research Logistics Quarterly*, **3**, 383–392. doi:https://doi.org/10.1002/nav.3800280304.

Billingsley P (1995). *Probability and Measure*. John Wiley.

Blomqvist N (1950). "On a Measure of Dependence Between Two Random Variables." *The Annals of Mathematical Statistics*, **21**, 593–600. doi:https://doi.org/10.1214/aoms/1177729754.

Caillat L, Dutang C, Larrieu V, Nguyen T (2018). "Package 'gumbel', Version 1.10-2." *See web-page: https://cran.r-project.org/web/packages/gumbel/gumbel.pdf.*

Dargahi-Noubary GR, Nanthakumar A (1992). "Stress-strength Reliability for Designs Based on Large Historic Values of Stress." *Journal of Computational and Applied Mathematics*, **40**, 43–53. doi:https://doi.org/10.1016/0377-0427(92)90040-5.

Domma F, Giordano S (2013). "Copula-based Approach to Account for Dependence in Stress-strength Models." *Statistical Papers*, **54**(3), 807–826. doi:https://doi.org/10.1007/s00362-012-0463-0.

Genest C, Aguirre A, Harvey F (2013). "Copula Parameter Estimation Using Blomqvist's Beta." *Journal de la Societe Francaise de Statistique*, **154**(1), 5–24.

Gumbel EJ (1960b). "Distributions Des Valeurs Extremes En Plusiers Dimensions." *Publications de l'Institut de Statistique de l'Université de Paris*, **9**, 171–173.

Hanagal DD (1994). "Testing Reliability in a Bivariate Exponential Stress-strength Model." *Journal of the Indian Statistical Association*, **33**, 41–45.

Hougaard P (1986). "A Class of Multivanate Failure Time Distributions." *Biometrika*, **73**, 671–678. doi:https://doi.org/10.2307/2336531.

Hutchinson TP, Lai CD (1990). *Continuous Bivariate Distributions, Emphasising Applications*. Rumsby Scientific Publishing, Adelaide. doi:https://doi.org/10.1002/qre.4680090515.

Jana PK (1994). "Estimation of P [Y< X] in the Bivariate Exponential Case Due to Marshall-Olkin." *Journal of the Indian Statistical Association*, **32**, 35–37.

Joe H (1997). *Multivariate Models and Dependence Concepts, Monographs on Statistics and Applied Probability*. Chapman Hall, London. doi:https://doi.org/10.1201/9780367803896.

Kotz S, Lumelskii Y, Pensky M (2003). *The Stress-strength Model and Its Generalizations: Theory and Applications*. World Scientific Publishing Co. Pte. Ltd., London. doi:https://doi.org/10.1142/5015.

Kundu D, Gupta RD (2005). "Estimation of P [Y< X] for Generalized Exponential Distribution." *Metrika*, **61**(3), 291–308. doi:https://doi.org/10.1007/s001840400345.

Kundu D, Gupta RD (2006). "Estimation of P [Y< X] for Weibull Distributions." *IEEE Transactions on Reliability*, **55**(2), 270–280. doi:http://dx.doi.org/10.1109/TR.2006.874918.

Nadarajah S (2003). "Reliability for Lifetime Distributions." *Mathematical and Computer Modelling*, **37**, 683–688. doi:https://doi.org/10.1016/S0895-7177(03)00074-8.

Nadarajah S, Kotz S (2006). "Reliability for Some Bivariate Exponential Distributions." *Mathematical Problems in Engineering*, pp. 1–14. doi:https://doi.org/10.1155/MPE/2006/41652.

Nelsen RB (1999). *An Introduction to Copulas*. Springer, New York.

Ross SM (1997). *Introduction to Probability Models*. Academic Press.

Schmid F, Schmidt R (2007). "Nonparametric Inference on Multivariate Versions of Blomqvist's Beta and Related Measures of Tail Dependence." *Metrika*, **66**, 323–354. doi:https://doi.org/10.1007/s00184-006-0114-3.

Sklar A (1959). "Fonctions De Repartition à n Dimensions Et Leurs Marges." *Publications de l'Institut de Statistique de l'Université de Paris*, **8**, 229–231.

Affiliation:

Dipak D. Patil
Department of Statistics,
H. V. Desai College, 596 Budhawar Peth,
Pune-411002, India
E-mail: dipakpatilatstats@gmail.com

U. V. Naik-Nimbalkar
Savitribai Phule Pune
University,
Pune-411007, India
E-mail: uvnaik@gmail.com

M. M. Kale
Savitribai Phule Pune
University,
Pune-411007, India
E-mail: mohan.kale80@gmail.com

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 51 *Submitted:* 2021-03-21
July 2022 *Accepted:* 2021-08-02