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# Self-consistent structure of a thin anisotropic current sheet

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**Abstract.** An approximate solution of the system of Vlasov-Maxwell equations for a collisionless current sheet with strongly anisotropic ion species (the bulk velocity outside the sheet is much greater than the thermal one) and sharply curved magnetic field lines (the curvature radius is much less than the ion gyroradius) is obtained in the form of a universal function with two asymptotics and an iterative procedure in between. The magnetic field spatial scaling suggested by *Francfort and Pellat* [1976] is confirmed analytically.

## 1. Introduction

There are two basic models of collisionless plasma current sheets with nonzero normal component of the magnetic field: isotropic sheet [*Schindler*, 1972; *Kan*, 1973] and anisotropic sheet with the bulk velocity of the ion species much greater than the thermal one [*Eastwood*, 1972]. In the first model the magnetic tension is balanced by the pressure gradient. In the second model, which is one-dimensional, it is balanced by the finite inertia of ions moving along the curved magnetic field lines of the sheet. The first model has been thoroughly investigated [e.g., *Cowley*, 1980, and references therein]. The corresponding system of Vlasov-Maxwell equations is resolved analytically in the framework of WKB approximation [*Schindler*, 1972; *Lembege and Pellat*, 1982], with the solution in the form of the local Harris distribution [*Harris*, 1962]. Because of the latter circumstance the isotropic equilibrium has been also intensively exploited in the analysis of sheet stability, quasi-stationary evolution scenarios, and models of multiscaled and hybrid equilibria. In the case of anisotropic sheet, only general estimates of the key sheet parameters [*Eastwood*, 1972; *Hill*, 1975; *Cowley*, 1980], non-self-consistent scaling laws [*Francfort and Pellat*, 1976; *Pritchett and Coroniti*, 1992], and various simulation results [e.g., *Alexeev and Malova* 1990; *Burkhart et al.*, 1992a; *Pritchett and Coroniti*, 1992; *Holland and Chen*, 1993] are available so far. The absence of a self-consistent solution with exact scalings and explicit profiles of density and current density still impedes the use of the anisotropic sheet model in applications.

In this work we present the solution of the system of Vlasov-Maxwell equations for an anisotropic current sheet (hereafter CS) with shifted Maxwellian velocity distribution outside the CS in terms of the magnetic field spatial profile. We make the treatment of *Francfort and Pellat* [1976] self-consistent on the basis of the quasi-adiabatic approach proposed by *Kropotkin and Domrin* [1995]. We present the solution in the form of a universal function including near-central-plane and far outside asymptotics as well as an iterative procedure to determine the profile in between. The dimensionless self-consistent equation for the magnetic field is shown explicitly to have the spatial scaling suggested initially by *Francfort and Pellat* [1976].

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## 2. Basic Theory

Consider the current sheet, which is homogeneous along the $X$ and $Y$ directions. The magnetic field is assumed to have two components: $\mathbf{B} = (B(z), 0, B_n)$. Since the constant and homogeneous electric field may be excluded by the transition to a deHoffmann-Teller frame of reference, we shall not take into account this field in the calculations without loss of generality. Then the only relevant Maxwell equation $\operatorname{curl} \mathbf{B} = (4\pi/c)\,\mathbf{J}$ may be reduced to the following scalar equation,

$$\frac{dB}{dz} = \frac{4\pi}{c} J \tag{1}$$

where the current density is determined by the ion distribution

$$J = e \int v_y f(z, \mathbf{v})\, d\mathbf{v} \tag{2}$$

To avoid the explicit solution of the Vlasov equation, we shall search for such of its solutions as depend upon the invariants of motion only. In the case of isotropic sheet these would be the particle energy and the dawn-dusk component of generalized momentum. We adopt here another set of invariants.

Consider a group of passing ions such that their velocity $v_0$ parallel to $\mathbf{B}$ outside the CS is much in excess of their perpendicular velocity $v_\perp$, $v_\perp/v_0 \ll 1$; however, their Larmor radius evaluated for the normal field $B_n$ far exceeds the field line minimum curvature radius,

$$\kappa = B_n / B_0 \sqrt{L/\rho_0} \ll 1$$

where $L$ is the sheet thickness and $\rho_0 = v_0/\omega_{B_0}$, $\omega_{B_0} = eB_0/mc$, with $B_0$ the field outside the CS. According to *Chen and Palmadesso* [1986] and *Büchner and Zelenyi* [1989], for a fixed field profile with the CS thickness $L$ the

motion of such particles inside the CS is quasi-periodic; it is adiabatic with the adiabatic invariant of the "fast" $z$ motion [*Sonnerup*, 1971],

$$I = \oint v_z dz \tag{3}$$

If the parameters $\kappa$ and $v_\perp/v_0$ are small enough, the particle motion inside the CS will be mainly rotation about the normal field $\mathbf{B}_n$, with only small oscillation about the central plane, so that $v_z \ll (v_x^2 + v_y^2)^{1/2} \simeq v_0$.

The question is whether those ions alone can form the equilibrium CS. If they can, the ion distribution function should be constant along the orbits of such ions; i.e., it should be a function $F(v, I)$ of $v$ and $I$, which are approximately constant along those orbits.

The semi-qualitative treatment of *Francfort and Pellat* [1976] and the simulations done by *Burkhart et al.* [1992a] suggest a positive answer to that question. The basics of a self-consistent theory was presented by *Kropotkin and Domrin* [1995, 1996]. Here the properties of smallness will be postulated, first, for the ratio of frequencies of the "slow" and "fast" motions, which leads to the existence of the invariant $I$, and second, for the ratio $v_z/v_0$, which leads to the existence of an approximate invariant

$$v = \sqrt{v_x^2 + v_y^2} \simeq v_0 \tag{4}$$

The validity conditions of both these assumptions being related to the main ion population forming the current in the CS should be obtained afterwards, i.e., when the field profile is evaluated. This is done in section 4.

Here we note, however, that an additional assumption is needed, since we mean that the main, i.e., thermal, ion population possesses the two mentioned features. In general, there are two possible limiting cases for the equilibrium anisotropic CS formation. They differ by the thermal velocity spread of the incoming ion flows. Those flows may be "cold" in the sense that $v_T/v_0 \ll B_n/B_0 = b_n$, $v_T$ being the ion thermal speed. That case was studied in a number of papers beginning with that by *Eastwood* [1974] (see also *Pritchett and Coroniti* [1992]). In that case a strong angular scattering occurs inside CS, and the pitch angle of outgoing ions is much greater than that of incoming ones.

Our study deals with the opposite case, $\epsilon = v_T/v_0 \gg B_n/B_0$ (though $\epsilon \ll 1$). A series of studies beginning with those of *Sonnerup* [1971], *Alexeev and Kropotkin* [1970], (see also *Büchner and Zelenyi* [1991]) reveal that in this case the above mentioned strong scattering does not occur; to the contrary, the adiabatic invariance of the action integral during ion motion takes place both outside and inside the CS. This invariance is only slightly violated by the separatrix crossings, where the ion changes its orbital mode from gyromotion to oscillations about the central plane, and vice versa. These topics are discussed in more detail in section 4.

With $v = \text{const}$, the equation for the $Z$ motion reduces to the form

$$\frac{d^2 z}{dt^2} = -\frac{e}{mc} v B \cos\phi \tag{5}$$

where $\phi = \arctan(v_x/v_y)$ and the approximate invariant (3) may be calculated explicitly,

$$I = 4\sqrt{\frac{2ev\cos\phi}{mc}} \int_0^{z_1} dz' \sqrt{\int_{z'}^{z_1} B(z'')\, dz''} \tag{6}$$

Here, $z_1$ is the maximum deviation of the particle with the given parameter $I$ from the neutral plane $z = 0$ during its oscillatory motion. It is determined by the condition

$$\frac{ev\cos\phi}{c} \int_z^{z_1} B(z')\, dz' = \frac{mv_z^2}{2} \tag{7}$$

Note that with $v$ and $I$ being conserved we have in principle an opportunity to calculate $J$ everywhere for a given distribution function $f$ expressed through $v$ and $I$. However, equations (5) and (6) are valid only in the crossing orbit portions, with the regions very close to $\phi = \pm\pi/2$ being excluded. Thus technically, we cannot evaluate the contributions to $J$ arising from the crossing orbit portions close to the entrance and exit, at $\phi \approx \pm\pi/2$, and also from non-crossing orbit portions. However, both contributions are small, because only a small ($\sim v_\perp \sim \epsilon v_0 \ll v_0$) component of the ion velocity perpendicular to $\mathbf{B}_0$ is involved, as compared with the $y$—directed ion velocity $\sim v_0$ in the CS central domain, on the crossing orbit elements.

Assumptions (3) and (4) allow us to express the distribution function in terms of the invariants $I$ and $v$,

$$f(z, \mathbf{v})\, d\mathbf{v} = f(v, I(z, v_z, v, \phi))\, v dv dv_z d\phi$$

Then, turning in equation (2) to the integration over $z_1$ instead of $v_z$ and using relation (7), we obtain a self-consistent form of equation (1) [*Kropotkin and Domrin*, 1995],

$$\frac{dB}{dz} = \frac{4\pi e}{c}\sqrt{\frac{e}{2mc}} \int_0^\infty dv v^{5/2} \int_{-\pi/2}^{\pi/2} d\phi (\cos\phi)^{3/2} \times \int_z^\infty \frac{B(z_1) dz_1}{\sqrt{\int_z^{z_1} B(z') dz'}} f(v, I(v, \phi, z_1)) \tag{8}$$

with $I$ expressed in terms of $v$, $\phi$, and $z_1$ due to (6).

To infer the particular profile of the magnetic field from system (6), (8), we have to choose the particular form of the distribution function. At $z \to \pm\infty$ we take it in the form

$$f = \frac{n_0}{(\sqrt{\pi} v_T)^3} \exp\left[-\frac{(\mathbf{v} - \mathbf{v}_D)^2}{v_T^2}\right] \tag{9}$$

Here, $n_0$ is the plasma density outside CS, and $v_T = (2T/m)^{1/2}$ is the ion thermal velocity. The bulk velocity $v_D$ in case of thin sheets is approximately equal to the drift velocity of the deHoffmann-Teller system of reference along the $X$ direction. The latter drift is of a completely different nature in comparison with the $Y$ drift in the isotropic sheet.

To connect the asymptotic distribution (9) with that used in equation (8), we use the above mentioned condition $v_T \ll v_D$ and the relation between the sheet invariant $I$ and the

magnetic moment $\mu = mv_\perp^2/(2B)$ at the boundary between crossing and noncrossing parts of the transient orbit. The condition $v_T \ll v_D$ together with the condition $v_z \ll v_0$ allows us to reduce the expression

$$\frac{1}{\sqrt{\pi}v_T}\exp\left[-\frac{(v_\parallel - v_D)^2}{v_T^2}\right] \to \delta(v - v_D)$$

while the relation between the invariants

$$I = \frac{4\pi c}{e}\mu \tag{10}$$

allows us to transform the remainder of the distribution to obtain

$$f = \frac{n_0}{\pi v_T^2}\delta(v - v_D)\exp\left(-\frac{1}{2\pi}\frac{I\omega_{B_0}}{\epsilon^2 v_D^2}\right) \tag{11}$$

where $\omega_{B_0} = eB_0/(mc)$, with $B_0 = |B(z \to \pm\infty)|$, and $\epsilon = v_T/v_D$.

Using distribution (11) and introducing the dimensionless variables $z = \zeta\epsilon^{4/3}v_D/\omega_{B_0}$, $B = bB_0$, we arrive at the following dimensionless equation for the magnetic field:

$$\frac{db}{d\zeta} = \frac{\sqrt{2}}{\pi}\left(\frac{v_D}{v_A}\right)^2\int_0^{\pi/2} d\phi\,(\cos\phi)^{3/2} \times\int_\zeta^\infty \frac{b(\zeta_1)\,d\zeta_1}{\sqrt{\int_\zeta^{\zeta_1} b(\zeta')\,d\zeta'}} \times\exp\left(-\frac{2\sqrt{2\cos\phi}}{\pi}\int_0^{\zeta_1} d\zeta'\sqrt{\int_{\zeta'}^{\zeta_1} b(\zeta'')\,d\zeta''}\right) \tag{12}$$

Here, $v_A = B_0/(4\pi m n_0)^{1/2}$ is the Alfvén velocity outside CS, $z_1 = \zeta_1\epsilon^{4/3}v_D/\omega_{B_0}$ and $\zeta \geq 0$, with $b(-\zeta) = -b(\zeta)$. For simplicity we consider in further analysis only the domain $\zeta \geq 0$.

Equation (12) is a generalization of the estimates made by *Francfort and Pellat* [1976]. It may be reduced to their result for the current density and magnetic field profiles if we put $b(\zeta) = 1$ in the right-hand side of (12), for they used the wedge-shaped approximation for the magnetic field $B = B_0\,\mathrm{sign}(z)$ in describing the particle dynamics and calculating the current density. As a result we have demonstrated by equation (12) that the parameter $\epsilon^{4/3}v_D/\omega_{B_0}$ suggested by those authors as a scaling parameter for the anisotropic sheet is still valid in the self- consistent case. Previously, this result was demonstrated only by some particle-in-cell simulation results [*Burkhart et al.*, 1992a; *Holland and Chen*, 1993]. It appeared also as a result of the exponential approximation of the distribution function $f(I)$ when that distribution was inferred from (8) with a certain magnetic field profile $B(z)$ [*Kropotkin and Domrin*, 1995].

The parameter $(v_D/v_A)$ in equation (12) is not arbitrary. The dimensionless magnetic field $b(\zeta)$ should satisfy the following natural boundary conditions: $b(0) = 0$ and $|b(\pm\infty)| = 1$. These conditions allow us to determine the particular value of $(v_D/v_A)$. Integration of both sides of (12) yields

$$b(\infty) = \frac{\sqrt{2}}{\pi}\left(\frac{v_D}{v_A}\right)^2\int_0^{\pi/2} d\phi\,(\cos\phi)^{3/2} \times\int_0^\infty d\zeta\int_\zeta^\infty \frac{b(\zeta_1)\,d\zeta_1}{\sqrt{\int_\zeta^{\zeta_1} b(\zeta')\,d\zeta'}} \times\exp\left(-\frac{2\sqrt{2\cos\phi}}{\pi}\int_0^{\zeta_1} d\zeta'\sqrt{\int_{\zeta'}^{\zeta_1} b(\zeta'')\,d\zeta''}\right) \tag{13}$$

Changing the sequence of the integrations in (13), we obtain

$$\begin{aligned} b(\infty) &= \frac{\sqrt{2}}{\pi}\left(\frac{v_D}{v_A}\right)^2\int_0^{\pi/2} d\phi\,(\cos\phi)^{3/2} \times\int_0^\infty d\zeta_1\int_0^{\zeta_1} \frac{b(\zeta_1)\,d\zeta}{\sqrt{\int_\zeta^{\zeta_1} b(\zeta')\,d\zeta'}} \\ &\times\exp\left(-\frac{2\sqrt{2\cos\phi}}{\pi}\int_0^{\zeta_1} d\zeta'\sqrt{\int_{\zeta'}^{\zeta_1} b(\zeta'')\,d\zeta''}\right) \\ &= \frac{\sqrt{2}}{\pi}\left(\frac{v_D}{v_A}\right)^2 \times\int_0^{\pi/2} d\phi\,(\cos\phi)^{3/2}\int_0^\infty d\zeta_1\left(-\frac{\pi}{\sqrt{2\cos\phi}}\right) \\ &\times\frac{d}{d\zeta_1}\exp\left(-\frac{2\sqrt{2\cos\phi}}{\pi}\int_0^{\zeta_1} d\zeta'\sqrt{\int_{\zeta'}^{\zeta_1} b(\zeta'')\,d\zeta''}\right) \\ &= \left(\frac{v_D}{v_A}\right)^2 \end{aligned} \tag{14}$$

Hence $v_D = v_A$ as it should be on the basis of more general physical considerations [e.g., *Hill*, 1975]. As a result, equation (12) determines the universal function $b(\zeta)$, which controls all the features of the magnetic field in the anisotropic sheet

$$B = B_0 b(z/L) \tag{15}$$

where

$$L = \epsilon^{4/3}v_D/\omega_{B_0} \tag{16}$$

is the only space scale parameter of the solution.

To study the properties of this unique function, we shall change the variable in equation (12) from $\zeta$ to the dimensionless vector potential $Y$ component $a$ according to the relation $db/d\zeta = (db/da)(da/d\zeta) = (1/2)(db^2/da)$. After substitution of $\zeta \to \zeta(a)$ and integration over $a$, changing the sequence of the integrations, we obtain the following equation for the function $b(a)$ for $a \geq 0$:

$$b(a) = 2\frac{v_D}{v_A}\sqrt{\frac{\sqrt{2}}{\pi}\int_0^\infty \sqrt{a'}\,[F(a') - F(a' + a)]\,da'} \tag{17}$$

where

$$F(a') = \int_0^{\pi/2} d\phi \, (\cos\phi)^{3/2} \times \exp\left( -\frac{2\sqrt{2\cos\phi}}{\pi} \int_0^{a'} da'' \frac{\sqrt{a'-a''}}{b(a'')} \right) \quad (18)$$

Because of the boundary condition $b(\infty) = 1$, equation (17) can be rewritten in another form suitable for numerical solution,

$$b(a) = \sqrt{1 - \frac{\int_0^\infty \sqrt{a'} F(a+a')\, da'}{\int_0^\infty \sqrt{a'} F(a')\, da'}} \quad (19)$$

## 3. Numerical Calculations and Asymptotics

To find the function $b(a)$ numerically, we use the iterative procedure based on equation (19) with the $n$th iteration of $b^{(n)}(a)$ determined by the right-hand side of this equation as a function of $b^{(n-1)}(a)$. To control the convergence of the iterative procedure, we calculate the effective value of the parameter $v_D/v_A$,

$$\left(\frac{v_D}{v_A}\right)^{(n)}_{eff} = \sqrt{\frac{\pi}{4\sqrt{2}} \frac{1}{\int_0^\infty \sqrt{a'} F\left(a'; b^{(n)}\right) da'}} \quad (20)$$

The results of the calculation are represented by Figures 1–4. The sequences of the iterations $b^{(n)}(a)$ and $(v_D/v_A)^{(n)}_{eff}$ with $b^{(0)}(a) \equiv 1$, which are shown in Figure 1, demonstrate pretty good convergence of the procedure to the exact self-consistent solution. The solution resulting from 10 iterations is presented in Figures 2–4. The dimensionless dependence $b(a)$ is shown in Figure 2, while the dependences $b(\zeta)$ and $j_y(\zeta) = db(\zeta)/d\zeta$ are shown in Figure 3 and Figure 4, correspondingly. Two other curves in the figures are presented for comparison. The first one simulates the profile resulting from the non-self-consistent approach of *Francfort and Pellat* [1976] with the wedge-shaped approximation of the magnetic field, $B_x = B_0 \,\mathrm{sign}(z)$, $B_z = B_n$, when describing the particle dynamics and calculating the current density. It appears if we set $b(\zeta) \equiv 1$ in the right-hand side of (12). The second curve is one of the field pro-

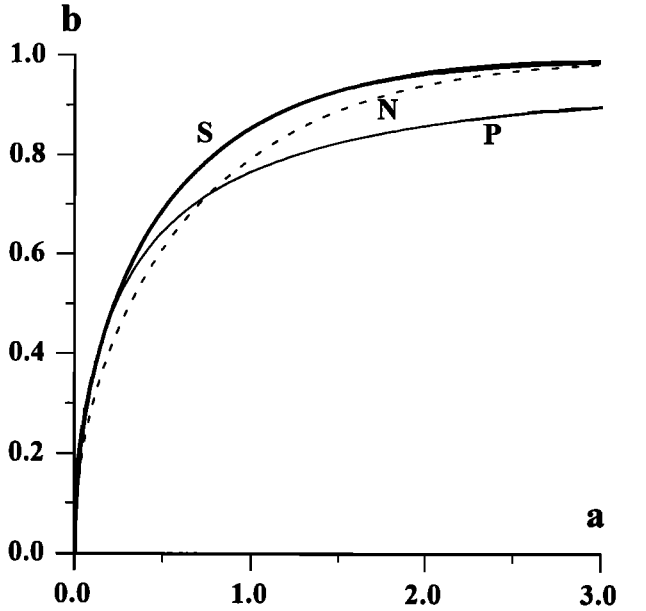


**Figure 2.** Self-consistent profile (S, bold curve) of the magnetic field in the dimensionless variables $b$ and $a$ as compared with the non-self-consistent profile (N, dashed curve) resulting from the formulae of *Francfort and Pellat* [1976] and the prescribed profile (P) for the inversed problem of *Kropotkin and Domrin* [1995].

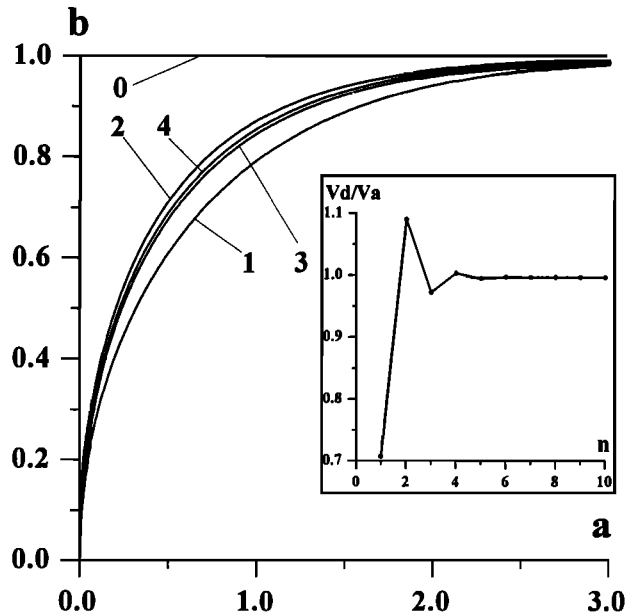


**Figure 1.** Sequence of iterations for the function $b(a)$: (a) iterations $b^{(n)}(a)$ (large plot) and (b) consistency test parameter $(v_D/v_A)^{(n)}_{eff}$ versus the iteration number $n$.

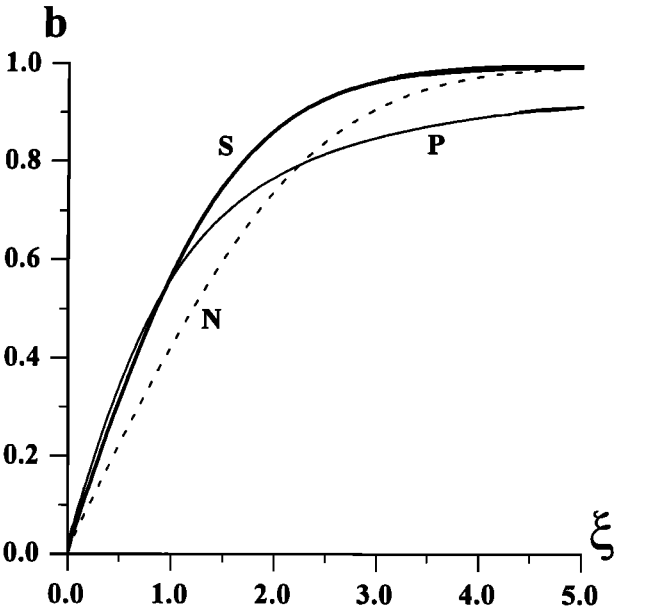


**Figure 3.** Magnetic field spatial profiles corresponding to self-consistent, non-self-consistent, and prescribed profiles drawn in Figure 2.

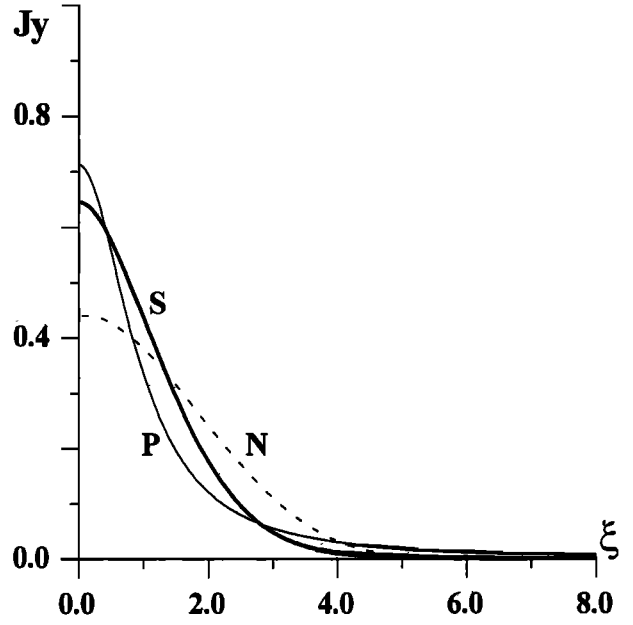


**Figure 4.** Current density spatial profiles for self-consistent, non-self-consistent, and prescribed profiles drawn in Figure 2.

files $b(a) = (a/(p+a))^{1/2}$ postulated by *Kropotkin and Domrin* [1995] to solve the inverse problem, i.e., to evaluate the distribution function corresponding to a prescribed field profile; the particular value of the curve parameter $p = 0.7$ is chosen to match the solution to the exponential distribution (11). The comparison shows that the approximation of *Kropotkin and Domrin* [1995] is quite consistent for $b(\zeta)$ with $0 \le \zeta \le 1$, while the relative error in the case of the current density profile within its half width reaches 30% for both compared techniques (non-self-consistent and inversed).

To complete the solution we calculate all the relevant asymptotics. In the limit $a \to 0$ we may expand the right-hand side of (17) in a power series to obtain

$$b(a) \simeq \alpha_0 \sqrt{a} \tag{21}$$

where

$$\alpha_0 = \sqrt{\frac{2\sqrt{2}}{\pi} \int_0^\infty \frac{F(a')}{\sqrt{a'}} da'} \tag{22}$$

To find the exact value of $\alpha_0$, we may either numerically approximate the exact solution $b(a)$ in the vicinity $a \simeq 0$ in the form (21) or calculate $\alpha_0$ directly, using (22) and the whole numerical array of $b(a)$. Both procedures give $\alpha_0 \simeq 1.12$.

The corresponding asymptotics $b(\zeta)$ at $\zeta \to 0$ has the form

$$b(\zeta) \simeq \frac{1}{2}\alpha_0^2 \zeta \tag{23}$$

To obtain the asymptotics at $a \to \infty$, we may start either from (17) or from (19) to have

$$b(a) \simeq 1 - \frac{2\sqrt{2}}{\pi} \int_0^\infty \sqrt{a'} F(a+a')\, da' \tag{24}$$

To simplify the expression (24), we take into account that at $a \to \infty$ we may set $b(a) \equiv 1$ in (18). Consequently,

$$F(a+a') \simeq \int_0^{\pi/2} d\phi\, (\cos\phi)^{3/2} \times \exp\left(-\frac{4\sqrt{2\cos\phi}}{3\pi}(a+a')^{3/2}\right) \tag{25}$$

with the main contribution to the right-hand side of (25) coming from the vicinity $\phi \simeq \pi/2$. As a result,

$$F(a+a') \simeq \int_0^\infty d\theta \theta^{3/2} \exp\left(-\frac{4\sqrt{2\theta}}{3\pi}(a+a')^{3/2}\right) \simeq \left(\frac{3\pi}{4\sqrt{2}}\right)^5 \frac{48}{(a+a')^{15/2}} \tag{26}$$

and after a simple integration we obtain finally

$$b(a) \simeq 1 - \frac{\alpha_\infty}{a^6}, \qquad b(\zeta) \simeq 1 - \frac{\alpha_\infty}{\zeta^6} \tag{27}$$

with $\alpha_\infty \simeq 31.5$.

## 4. A Posteriori Theory Justification Based on Ion Dynamics

In this section we present an a posteriori justification of the assumptions adopted in section 2. Using dimensionless variables $\mathbf{r} \to \mathbf{r}/\sqrt{\rho_0 L}$, $\mathbf{v} \to \mathbf{v}/v_0$, and $t \to tv_0/\sqrt{\rho_0 L}$, where $\rho_0 = v_0/\omega_{B_0}$, one can present the Hamiltonian of a single ion in the field $\mathbf{B} = (B(z), 0, B_n)$, with $B$ given by (15), in the form

$$H = \frac{v_x^2}{2} + \frac{v_z^2}{2} + \frac{1}{2}\left[\delta^2 a\left(\frac{z}{\delta}\right) - \kappa x\right]^2 = \frac{1}{2} \tag{28}$$

where $\delta = (L/\rho_0)^{1/2}$, $\kappa = b_n \delta$ and $b_n = B_n/B_0$. Here the conservation of the $Y$ component of the canonical momentum is taken into account to reduce the effective number of degrees of freedom. Thus the Hamiltonian (28) describes $X$ and $Z$ motions with different timescales $\omega_x^{-1}$ and $\omega_z^{-1}$, respectively, with these parameters calculated only for that portion of the whole infinite orbit of the untrapped particle where the particle crosses the central plane $z = 0$.

The use of the new invariant $I$ given by (3) as an action variable for $Z$ oscillations becomes reasonable under the condition $\omega_x/\omega_z \ll 1$. Let's check this relation for our sheet model. In the limit $z_{\max}/\delta \ll 1$ the Hamiltonian (28) due to (23) describes the particle dynamics in parabolic magnetic field approximation controlled by the single parameter $\kappa$ [*Chen and Palmadesso*, 1986; *Büchner and Zelenyi*, 1989]. Then, according to *Büchner and Zelenyi* [1989] in the limit $I \to 0$, far from the separatrix of the $Z$ motion the ratio $\omega_x/\omega_z$ tends to const $\cdot \kappa$, and hence it is the parameter $\kappa$ that controls the accuracy of the invariant $I$ conservation.

Consider now the case of the main ion population that forms the plasma sheet, i.e., those ions with $z_{\max}/\delta \sim 1$.

From (16) we see that the dimensionless spatial scale of the self-consistent field $B_x(z)$ is $\delta = \epsilon^{2/3} \ll 1$. The dimensionless equation of the $x$ motion induced by (28),

$$\frac{dv_x}{dt} = \kappa \left[\delta^2 a\left(\frac{z}{\delta}\right) - \kappa x\right] \tag{29}$$

reduces for those ions to the equation of motion of a harmonic oscillator with the frequency $\omega_x \simeq \kappa$, since the dimensionless potential $a(\xi) \sim 1$ at $\xi \sim 1$. On the other hand, since the thickness of the CS is equal to $\delta$, we obtain

$$I = \oint v_z dz \sim z_{\max}^2 \omega_z \tag{30}$$

Hence for the main CS population we have $I \sim \delta^2 \omega_z \sim \epsilon^{4/3}\omega_z$. However, the adiabatic invariant may be estimated also on the basis of formula (10) with $v_\perp$ set equal to the thermal speed $v_T$. In our dimensionless variables this yields $I \sim \epsilon^2/\delta \sim \epsilon^{4/3}$, so that $\omega_z \sim 1$. Thus we obtain the ratio $\omega_x/\omega_z \sim \kappa \ll 1$.

The adiabaticity of the crossing motion at those phases that lie far from the central plane is also important for our model, because the condition $z_{\max} \sim \delta$ corresponds to the part of the particle orbit with $\sin\phi \approx 0$, while the whole region of crossing motion ($-\pi/2 < \phi < \pi/2$) extends over a much thicker layer, $z_{\max} \leq \delta\epsilon^{-1/3}$ (or $z \leq v_T/\omega_{B_0}$ in usual variables). Since according to (30) (with $\delta \sim \epsilon^{2/3}$) in this region $I \sim \epsilon^{2/3}\omega_z$, we estimate the frequency value at $\phi \approx \pm\pi/2$ as $\omega_z \sim \delta$. Consequently, the ratio $\omega_x/\omega_z$ may approach the value $\sim \kappa/\delta = b_n$. We now see that the sufficient requirement for the $x$ motion to be "slow"(i.e., slowly varying) in comparison with the $z$ motion is $b_n \ll 1$. Note that near the points of change of the particle motion type, where the motion transforms from crossing the plane $z = 0$ to a noncrossing one and vice versa, the adiabatically evaluated $\omega_z$ may even pass through zero. The role of those so-called exit and entry regions will be separately analyzed below.

Let's estimate the accuracy of the conservation of the invariant $I$ in more detail. In the approximation of "$1\,{}^1/_2$ degrees of freedom" (with $x$ and $v_x$ being considered as "slow" parameters for the $Z$ motion) the variation of $I$ is described by the equation [e.g., *Neishtadt*, 1987]

$$\frac{dI}{dt} \approx \frac{v_x}{\omega_z}\left[\frac{\partial H}{\partial x} - \left\langle\frac{\partial H}{\partial x}\right\rangle\right] = \frac{\kappa\delta^2 v_x}{\omega_z}(a - \langle a\rangle) \tag{31}$$

where the angle brackets denote the averaging over the "fast" $z$ motion. Since $I \sim \epsilon^2/\delta \sim \delta^2$ and $a \sim 1$, we can estimate the normalized variation of $I$ as follows:

$$\frac{\Delta I}{I} = \frac{1}{\omega_z I}\frac{dI}{dt} \sim \frac{\kappa v_x}{\omega_z^2}(a - \langle a\rangle) \tag{32}$$

As a result, in spite of the fact that the dimensionless value $I \ll 1$, the normalized variation $\Delta I/I$ is also much less than unity in the portion of the crossing motion with $z_{\max} \sim \delta$ ($a \sim 1$, $\omega_z \sim 1$, $v_x \leq 1$) provided that $\kappa \ll 1$. However, the smallness of $\Delta I/I$ within the whole layer of crossing motion, $z_{\max} \leq \delta\epsilon^{-1/3}$, though far from the entrance and exit regions, is, according to (32), determined by a more severe constraint: $\Delta I/I \ll b_n/\epsilon$ (at $\omega_z \sim \delta$, $v_x \leq 1$ and $|a - \langle a\rangle| \sim z_{\max}/\delta \sim \epsilon^{-1/3}$). Note that in most of the crossing region, if it is averaged over the "fast" $Z$ oscillation period, the invariant $I$ is conserved with much better accuracy, because the oscillations (31) are then averaged to zero.

The accuracy of the conservation of $I$ described above may be deteriorated, because the averaging of (31) over the period of the $Z$ motion becomes invalid near the entrance and exit regions where the so-called separatrix crossing of this motion takes place and the frequency $\omega_z$ turns to zero. However, this effect may enlarge estimate (32) only by a factor of $\sim \ln(2|\cos\theta_{sep}|)$, where $\theta_{sep}$ is the phase of the $Z$ motion near the separatrix [e.g., *Neishtadt*, 1987; *Büchner and Zelenyi*, 1989, and references therein]. Hence at $\Delta I/I \ll 1$, any considerable variation of $I$ ($\Delta I/I \sim 1$) due to separatrix crossing takes place only in the exponentially small region of the phase space.

Another point is the requirement appearing to verify the smallness of $v_z$. Since $I$ defined by (30) may be estimated also as $\sim v_z^2/\omega_z$ and, on the other hand, we have seen that $I \sim \epsilon^{4/3}$, we immediately find out that the dimensionless $v_z$ is

$$v_z \sim \epsilon^{2/3}\omega_z^{1/2} \leq \epsilon^{2/3} \tag{33}$$

for the whole class of crossing motions with $z_{\max} \leq \delta\epsilon^{-1/3}$. Remember that $v$ is normalized here by $v_0$. Therefore with $\epsilon \ll 1$ it is clear that during the ion oscillations about the CS central plane the $z$ component of the ion velocity vector is small, so that the value $v = (v_x^2 + v_y^2)^{1/2}$ may be treated as an approximate invariant of motion.

Conservation of $I$ and $v$ outside CS for noncrossing parts of the orbit (ordinary Larmor rotation with $z \neq 0$) is also of some importance in our model, because we build distribution (11) on the basis of the asymptotic distribution (9) and implicit use of the Liouville theorem. The conservation of $v$ at $\epsilon \ll 1$ may be guaranteed in that region by the condition $b_n \ll 1$. The parameter $I$ can be considered approximately constant outside CS as long as the $z$ projection of the parallel velocity $v_\parallel$ is less than the velocity of "fast" motion along the $Z$ axis ($\approx v_\perp$) [e.g., *Savenkov et al.*, 1991] or $b_n \ll \epsilon$. However, as usual, when averaged over the "fast" $Z$ oscillation period (in fact, Larmor period), $I$ is conserved with much better accuracy and coincides approximately with the ordinary magnetic moment $\mu$. Better conservation of the latter invariant is provided by the smallness of the ratio of the particle Larmor radius $\rho$ to the curvature radius $\rho_c$ of magnetic field line, with the latter condition being much more easily fulfilled because $\rho \to \rho_0 = \text{const}$ outside CS, while $\rho_c$ tends to infinity. As a result the averaged invariant $\langle I\rangle$ is conserved with much better accuracy, separately for crossing and noncrossing regions.

Thus we can conclude that the invariants $I$ and $v$ are valid for a class of particles given by the condition

$$b_n \ll \epsilon \ll 1 \tag{34}$$

along the whole orbit of each particle. This condition allows us not only to describe the motion inside CS in terms of $I$ and $v$ but also to connect the distributions inside and outside CS

via the Liouville theorem. Note, however, that alternatively, one can postulate the distribution (11) inside CS ad hoc and then search for its transformation outside CS. Under such a formulation of the problem the validity of $I$ and $v$ as invariants of motion would require much less severe restrictions on the class of orbits involved, $\kappa \ll 1$ and $\epsilon \ll 1$.

## 5. Conclusion

We have obtained a self-consistent profile of the magnetic field of the thin anisotropic sheet with Maxwellian ion beams outside the CS and quasi-adiabatic ion dynamics, in the form of a universal function with the spatial scaling initially suggested by *Francfort and Pellat* [1976]. The universal function is studied in detail both analytically and numerically. Thus the profile may be considered as a reference solution like the well-known Harris profile in the case of isotropic sheets with zero normal component of the magnetic field [*Harris*, 1962]. It may be used further to study the stability and quasi-stationary evolution of anisotropic sheets as well as more complicated equilibria with embedded substructures [*Burkhart et al.*, 1992b; *Krallmann et al.*, 1994].

The condition $v_T/v_D \ll 1$ seems to be hardly valid for quiet time magnetotail near the Earth. However, this condition, as well as (34), is quite probable in the far tail or near the dayside magnetopause with reconnection and ion anisotropy involving field-aligned bulk flows counterstreaming with double Alfvén speed [e.g., *Fuselier et al.*, 1991]. They may be appropriate also in the description of transient sheet equilibria providing magnetic merging during substorm activations. An important though indirect indication of such transient anisotropic structure formation is the statistics of the bursty bulk flow events, which show an increase of the occurrence rate from 6% during quiet conditions to ~20% during active ($AE \sim 500$ nT) conditions [*Angelopoulos et al.*, 1994].

Note that magnetic merging in a thin anisotropic CS appears if the CS is considered in a frame with $E_y \neq 0$ such that one of the counterstreaming ion flows outside the CS has a zero velocity $x$ component while another has $v_x \simeq 2v_A$. The first one may be interpreted as the background plasma ions convecting slowly toward CS, and the second can be interpreted as an ion flow accelerated up to $2v_A$ speed by the $E$ field inside the CS.

But how can such a transient sheet equilibrium arise? In the case of a CS with a (weak) dependence on $X$ an equilibrium CS may involve both weak and strong anisotropy outside the CS, depending on the relative role of that anisotropy and $(\nabla p)_x$ in balancing the magnetic tension. Note that in the case of a Schindler-type equilibrium, with a prevailing role of $(\nabla p)_x$ a relation $b_n \sim L_z/L_x$ should be valid. Assume now that $b_n$ becomes very small (for example, a condition near the forming $X$ line), with $L_x$ being fixed as determined by the large-scale equilibrium condition of the magnetospheric system. Then the $L_z$ value should also diminish. It is easily seen that with $L_z \sim \rho_0$ the current velocity of ions in a locally Harris-type CS reaches the ion thermal speed $v_T$, and with smaller $L_z$ it should exceed $v_T$. That means that ions become accelerated inside the CS by an inductive electric field acting in such a transient situation. Thus anisotropy becomes inevitable for a very thin CS, and the Schindler-type equilibrium is no longer valid. However, in the limiting case, when the anisotropy rather than $(\nabla p)_x$ prevails in the momentum balance over $X$, we arrive at the relation $v_D/v_T \approx v_A/v_T = \beta^{-1/2}$. Then we see that the condition $\beta \ll 1$ outside the CS defines the upper boundary over $v_D/v_T$, as adopted in our model. It implies a hot ($\beta \geq 1$) plasma sheet, self-consistently confined by the magnetic field, and such a sheet may be considered a very natural current sheet version. The opposite condition $v_D/v_T \leq 1$ outside the CS would describe a CS embedded in a hot plasma not fully confined by the magnetic field produced by the thin CS. Of course, that case of weakly anisotropic current sheet embedded in a thick plasma sheet [e.g., *Holland and Chen*, 1993] is also very interesting and important. However, it will be considered elsewhere. On the other hand, one should be reminded that the condition $v_T/v_D \gg b_n$, defining the lower boundary of our model over the parameter $v_D/v_T$, is also quite natural, since it means that only subsonic plasma convective flows toward the CS are permissible.

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