Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 7196–7224.
DOI: 10.3934/math.2023362
Received: 06 October 2022
Revised: 28 December 2022
Accepted: 29 December 2022
Published: 12 January 2023

Research article

Multiple solutions for a class of BVPs of fractional discontinuous differential equations with impulses

Yang Wang[1], **Yating Li**[2] **and Yansheng Liu**[2,*]

[1] School of Information Engineering, Shandong Management University, Jinan 250357, Shandong, China

[2] School of Mathematics and Statistics, Shandong Normal University, Jinan, 250014, Shandong, China

* **Correspondence:** Email: ysliu@sdnu.edu.cn.

Abstract: In this paper, we mainly study the following boundary value problems of fractional discontinuous differential equations with impulses:

$$\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\Re}\Lambda(t) = \mathcal{E}(t)F(t,\Lambda(t)), \ a.e.\ t \in Q, \\ \triangle\Lambda|_{t=t_\kappa} = \Phi_\kappa(\Lambda(t_\kappa)), \ \kappa = 1,\ 2,\ \cdots,\ m, \\ \triangle\Lambda'|_{t=t_\kappa} = 0, \ \kappa = 1,\ 2,\ \cdots,\ m, \\ \vartheta\Lambda(0) - \chi\Lambda(1) = \int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon, \\ \zeta\Lambda'(0) - \delta\Lambda'(1) = \int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon, \end{cases}$$

where $\vartheta > \chi > 0$, $\zeta > \delta > 0$, $\Phi_\kappa \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathcal{E}$, ϱ_1, $\varrho_2 \geq 0$ a.e. on $Q = [0, 1]$, $\mathcal{E}$, ϱ_1, $\varrho_2 \in L^1(0, 1)$ and $F : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, +\infty)$. By using Krasnosel skii's fixed point theorem for discontinuous operators on cones, some sufficient conditions for the existence of single or multiple positive solutions for the above discontinuous differential system are established. An example is given to confirm the main results in the end.

Keywords: boundary value problems; discontinuous differential equations; fixed point theory; fractional differential equation; multiple solutions
Mathematics Subject Classification: 34B18, 34A34, 26A33

1. Introduction

Fractional calculus has been widely and deeply used in many fields, for example, continuum mechanics, control theory of dynamical systems, and so on. For this reason, fractional differential equations (FDEs, in short), as a useful tool to model the dynamics of numerous physical systems, have gained considerable popularity in physics, population dynamics, chemical technology, control of dynamical systems, etc. For further details on FDEs, see [1–3] and their references.

In the last few decades, as a significant branch of FDEs, impulsive differential equation (IDE, for short), which provides a natural description of observed evolution processes, has been emerging as a very meaningful research area. In addition, IDEs are also as important mathematical tools for better understanding real-world problems (see, for instance, [4–8]). Hence, many authors have used IDEs to describe some phenomenon with abrupt changes, such as, harvest, disease, control theory of dynamical systems and so on. For example, [9–11] researched some different types IDEs, which are nonlinear impulsive differential systems with infinite delays, impulsive neural networks, singularly perturbed nonlinear impulsive differential systems with delays of small parameter, respectively. Moreover, [12] studied persistence of delayed cooperative models by means of impulsive control method.

Meanwhile, boundary value problems (BVPs, for short) of IDEs have been researched extensively and deeply. Correspondingly, many scholars have studied some BVPs of fractional differential equations (FIDEs, for short) and obtain lots of important conclusions. For example, [13] researched singular semipositive BVPs of fourth-order differential systems with parameters. [14] studied a class of BVPs for nonlinear fractional Kirchhoff equations and obtained the existence of multiple sign-changing solutions.

As far as we know, continuity is a fundamental assumption in degree theory. However, there are a lot of discontinuous differential equations in many areas, such as, automatic control, neural network, etc. Because of the corresponding operators are not continuous, general topological degree theory is invalid to studying the existence of solutions for most discontinuous differential equations, such as, [20,21]. To overcome this problem, a new definition of topological degree for a class of discontinuous operators is introduced by R. Figueroa et al. Subsequently, a number of fixed point theorems for such operators are derived in [16], such as, Schauder-type and Krasnoselskii's theorem for discontinuous operators. Then they are used to solve discontinuous differential systems. For example, [17] considered the existence for a class of second-order discountious BVPs by constructing a closed-convex Krasovskij envelope and Schauder-type theorem for discontinuous operators. [18] researched a class of BVPs of second-order discontinuous differential equations with impulse effects by using the nonlinear alternative of Krasnoselskii's fixed point theorem for discontinuous operators on cones.

However, to our best knowledge, there are few studies on multiple solutions for integral boundary value problems of fractional discontinuous differential equations with impulse effects. The purpose of present paper is to fill this gap.

Motivated by the above discussions, this paper studies multiple solutions for the following boundary value problem:

$$\begin{cases} {}_{t}^{C}\mathcal{D}_{0^+}^{\Re}\Lambda(t) = \mathcal{E}(t)\mathcal{F}(t,\Lambda(t)), \ a.e.\ t \in Q', \\ \triangle\Lambda|_{t=t_\kappa} = \Phi_\kappa(\Lambda(t_\kappa)), \ \kappa = 1,\ 2,\ \cdots,\ m, \\ \triangle\Lambda'|_{t=t_i} = 0, \ \ \kappa = 1,\ 2,\ \cdots,\ m, \\ \vartheta\Lambda(0) - \chi\Lambda(1) = \int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon, \\ \zeta\Lambda'(0) - \delta\Lambda'(1) = \int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon, \end{cases} \tag{1.1}$$

where ${}_{t}^{C}\mathcal{D}_{0^+}^{\Re}$ is the Caputo fractional derivative with t, $1 < \Re < 2$, $\vartheta > \chi > 0$, $\zeta > \delta > 0$, $\mathcal{E}_k \in C(\mathbb{R}^+,\mathbb{R}^+)$, $\mathcal{E}$, ϱ_1, $\varrho_2 \geq 0$ a.e. on $J = [0,1]$, $Q' = Q \setminus \{t_1,\ \cdots,\ t_m\}$, $\mathcal{E}$, ϱ_1, $\varrho_2 \in L^1(0,1)$, F :

$Q \times \mathbb{R}^+ \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,+\infty)$, $0 < t_1 < t_2 < \cdots < t_m < 1$. $\triangle\Lambda|_{t=t_k}$, $\triangle\Lambda'|_{t=t_k}$ denote the jump of $\Lambda(t)$ and $\Lambda'(t)$ at $t = t_k$, respectively. This paper has the following innovations and features. Firstly, BVP (1.1) is of fractional discontinuous differential equations with instantaneous impulse effects. The nonlinearity $\mathcal{F}$ here is discontinuous over countable families of curve [22]. Secondly, the boundary value condition considered here is of integral type. It makes BVP (1.1) more widely applicable in solving practical problems. Thirdly, the used approach in this paper has certain advantages over some reference as above. In detail, the distinctive tool used here is multivalued analysis in the study of discontinuous problems and the novelty is the use of multivalued analysis to obtain results for single-valued operators. Compared with [18], we redefine the admissible continuous curves for the new system (1.1). At the same time, a suitable cone is established by researching properties of Green's function deeply. Therefore, the positive solutions can be obtained by means of Krasnoselskii's fixed point theorem for discontinuous operators on cones.

The rest of this paper is organized as follows. Some basic definitions and notations are contained in Section 2. Section 3 presents the main results. Finally, an illustrative example is given in Section 4.

2. Preliminaries

In this section, we first introduce some definitions and lemmas that are used in this paper.

Definition 2.1. [3] *The Riemann-Liouville fractional integral of order $\Re \in \mathbb{R}^+$ of a function F on interval (α,β) is defined as follows:*

$$(I_{0^+}^{\Re}F)(t) = \frac{1}{\Gamma(\Re)}\int_{\alpha}^{t}(t-\upsilon)^{\Re-1}F(\upsilon)d\upsilon.$$

Definition 2.2. [3] *The Caputo fractional derivative of order $\Re \in \mathbb{R}^+$ of a function F on interval (α,β) is defined as follows:*

$$({}_t^C\mathcal{D}_{0^+}^{\Re})F(t) = \frac{1}{\Gamma(n-\Re)}\int_{\alpha}^{t}(t-\upsilon)^{n-\Re-1}F^{(n)}(\upsilon)d\upsilon.$$

Let

$$\begin{aligned} PC(Q) &= \{\Lambda : [0,1] \to \mathbb{R}, \Lambda \in C(Q'),\ and\ \Lambda(t_\kappa^+),\ \Lambda(t_\kappa^-)\ exists,\\ &\quad and\ \Lambda(t_\kappa^-) = \Lambda(t_\kappa),\ 1 \le \kappa \le m\},\end{aligned}$$

and

$$\begin{aligned} PC^1(Q) &= \{\Lambda : [0,1] \to \mathbb{R}, \Lambda \in PC(Q),\ {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda \in PC(Q),\ {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t_\kappa^+),\ {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda'(t_\kappa^-)\\ &\quad exists,\ and\ {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t_\kappa^-) = {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t_\kappa),\ 1 \le \kappa \le m\}.\end{aligned}$$

Obviously, they are Banach spaces with the norm

$$\|\Lambda\|_0 = sup_{0\le t\le 1}|\Lambda(t)|$$

and

$$\|\Lambda\|_1 = max\{\|\Lambda\|_0,\ \|{}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda\|_0\},$$

respectively.

For the sake of simplicity, let $\mathfrak{A}_j = \int_0^1 \varrho_j(\upsilon)d\upsilon$, $\mathfrak{Q}_j = \dfrac{1}{\vartheta-\chi}\mathfrak{A}_j$ $(j = 1,2)$, $\mathfrak{P}_j = \int_0^1 \dfrac{(\vartheta-\chi)\upsilon+\chi}{(\vartheta-\chi)(\zeta-\delta)}\varrho_j(\upsilon)d\upsilon$, $\Gamma_1 = (1-\mathfrak{P}_2)(1-\mathfrak{Q}_1)-\mathfrak{P}_1\mathfrak{Q}_2$ and $\mathfrak{Q}_M = max\{\dfrac{\mathfrak{Q}_2}{\Gamma(\zeta-\delta)}, \dfrac{1-\mathfrak{Q}_1}{\Gamma(\zeta-\delta)}\}$.

Lemma 2.3. *If* $(1-\mathfrak{P}_2)(1-\mathfrak{Q}_1) \neq \mathfrak{P}_1\mathfrak{Q}_2$*, for* $\mathfrak{H} \in L(Q, \mathbb{R}^+)$*, the following boundary value problem*

$$\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\Re}\Lambda(t) = \mathfrak{H}(t), \ a.e.\ t \in Q', \\ \triangle\Lambda|_{t=t_\kappa} = \Phi_\kappa(\Lambda(t_\kappa)), \ \kappa = 1,\ 2,\ \cdots,\ m, \\ \triangle\Lambda'|_{t=t_\kappa} = 0, \ \kappa = 1,\ 2,\ \cdots,\ m, \\ \vartheta\Lambda(0) - \chi\Lambda(1) = \int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon, \\ \zeta\Lambda'(0) - \delta\Lambda'(1) = \int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon, \end{cases} \tag{2.1}$$

has a solution

$$\Lambda(t) = \int_0^1 \mathcal{H}_1(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{i=1}^m \mathcal{H}_2(t,t_i)\Phi_i(\Lambda(t_i)),$$

where

$$\mathcal{H}_1(t,\upsilon) = \aleph(t,\upsilon) + \sum_{n=1}^2 \varphi_n(t)\int_0^1 \aleph(\upsilon,t)\varrho_n(t)dt,$$

$$\mathcal{H}_2(t,t_i) = \begin{cases} \dfrac{\chi}{\vartheta-\chi} + \dfrac{\chi}{\vartheta-\chi}\displaystyle\sum_{n=1}^2 \mathfrak{A}_n\varphi_n(t), & 0 \le t \le t_i \le 1; \\ \dfrac{\vartheta}{\vartheta-\chi} + \dfrac{\vartheta}{\vartheta-\chi}\displaystyle\sum_{n=1}^2 \mathfrak{A}_n\varphi_n(t), & 0 \le t_i < t \le 1, \end{cases}$$

$$\varphi_1(t) = \frac{(\zeta-\delta)(1-\mathfrak{P}_2) + [\chi+(\vartheta-\chi)t]\mathfrak{Q}_2}{(\vartheta-\beta)(\zeta-\delta)\Gamma_1},$$

$$\varphi_2(t) = \frac{(\zeta-\delta)\mathfrak{P}_1 + [\chi+(\vartheta-\chi)t](1-\mathfrak{Q}_1)}{(\vartheta-\chi)(\zeta-\delta)\Gamma_1}.$$

and

$$\aleph(t,\upsilon) = \begin{cases} \dfrac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \dfrac{[\chi\delta+(\vartheta-\chi)\delta t](1-\upsilon)^{\Re-2}}{(\vartheta-\beta)(\zeta-\delta)\Gamma(\Re-1)} + \dfrac{(t-\upsilon)^{q-1}}{\Gamma(\Re)}, & 0 \le \upsilon \le t \le 1; \\ \dfrac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \dfrac{[\chi\delta+(\vartheta-\chi)\delta t](1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)}, & 0 \le t \le \upsilon \le 1. \end{cases}$$

Proof. Let Λ be a general solution on each interval $(t_\kappa, t_{\kappa+1}]$ $(\kappa = 0,\ 1,\ 2,\ \cdots,\ m)$. By integrating both sides of Eq (2.1), one can obtain that

$$\Lambda(t) = \frac{1}{\Gamma(\Re)}\int_0^t (t-\upsilon)^{\Re-1}\mathfrak{H}(\upsilon)d\upsilon - c_\kappa - dt,\ for\ t \in (t_\kappa, t_{\kappa+1}], \tag{2.2}$$

where $t_0 = 0,\ t_{m+1} = 1$. Then,

$$\Lambda'(t) = \frac{1}{\Gamma(\Re - 1)} \int_0^t (t - \upsilon)^{\Re - 2} \mathfrak{H}(\upsilon) d\upsilon - d,\ t \in (t_\kappa, t_{\kappa+1}].$$

In view of Eq (2.1), we get

$$-\vartheta c_0 - \chi[\frac{1}{\Gamma(\Re)} \int_0^1 (1 - \upsilon)^{\Re - 1} \mathfrak{H}(\upsilon) d\upsilon - c_m - d] = \int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon) d\upsilon, \tag{2.3}$$

$$-\zeta d - \delta[\frac{1}{\Gamma(\Re - 1)} \int_0^1 (1 - \upsilon)^{\Re - 2} \mathfrak{H}(\upsilon) d\upsilon - d] = \int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon) d\upsilon, \tag{2.4}$$

$$c_{\kappa-1} - c_\kappa = \Phi_\kappa(\Lambda(t_\kappa)), \tag{2.5}$$

and

$$d = -\frac{\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon) d\upsilon + \frac{\delta}{\Gamma(\Re - 1)} \int_0^1 (1 - \upsilon)^{\Re - 2} \mathfrak{H}(\upsilon) d\upsilon}{\zeta - \delta}. \tag{2.6}$$

From (2.3), (2.5) and (2.6), one can easily get that

$$\begin{aligned} c_0 \ =\ & -\frac{1}{\vartheta - \chi}[\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon) d\upsilon + \frac{\chi}{\Gamma(\Re)} \int_0^1 (1 - \upsilon)^{\Re - 1} \mathfrak{H}(\upsilon) d\upsilon + \chi \sum_{i=1}^m \Phi_i(\Lambda(t_i)) \\ & + \frac{\chi(\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon) d\upsilon + \frac{\delta}{\Gamma(\Re - 1)} \int_0^1 (1 - \upsilon)^{\Re - 2} \mathfrak{H}(\upsilon) d\upsilon)}{\zeta - \delta}], \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} c_\kappa \ &= c_0 - \sum_{i=1}^\kappa \Phi_i(\Lambda(t_i)) \\ &= -\frac{1}{\vartheta - \chi}[\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon) d\upsilon + \frac{\chi}{\Gamma(\Re)} \int_0^1 (1 - \upsilon)^{\Re - 1} \mathfrak{H}(\upsilon) d\upsilon + \chi \sum_{i=1}^m \Phi_i(\Lambda(t_i)) \\ & \quad + \frac{\chi(\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon) d\upsilon + \frac{\delta}{\Gamma(\Re - 1)} \int_0^1 (1 - \upsilon)^{\Re - 2} \mathfrak{H}(\upsilon) d\upsilon)}{\zeta - \delta}] - \sum_{i=1}^\kappa \Phi_i(\Lambda(t_i)). \end{aligned} \tag{2.8}$$

Hence, (2.7) and (2.8) imply that

$$\begin{aligned} c_\kappa + dt = & -\frac{1}{\vartheta - \chi}[\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon) d\upsilon + \frac{\chi}{\Gamma(\Re)} \int_0^1 (1 - \upsilon)^{\Re - 1} \mathfrak{H}(\upsilon) d\upsilon + \chi \sum_{i=1}^m \Phi_i(\Lambda(t_i)) \\ & + \frac{\chi(\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon) d\upsilon + \frac{\delta}{\Gamma(\Re - 1)} \int_0^1 (1 - \upsilon)^{\Re - 2} \mathfrak{H}(\upsilon) d\upsilon)}{\zeta - \delta}] - \sum_{i=1}^\kappa \Phi_i(\Lambda(t_i)) \end{aligned}$$

$$
\begin{aligned}
&+[-\frac{\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon + \frac{\delta}{\Gamma(\Re-1)}\int_0^1 (1-\upsilon)^{\Re-2}\mathfrak{H}(\upsilon)d\upsilon}{\zeta-\delta}]t \\
&= -\frac{\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon}{\vartheta-\chi} - \frac{(\vartheta-\chi)t+\chi}{(\vartheta-\chi)(\zeta-\delta)}\int_0^1 \varrho_2(s)\Lambda(\upsilon)d\upsilon \\
&\quad -\frac{\chi}{(\vartheta-\chi)\Gamma(\Re)}\int_0^1 (1-\upsilon)^{\Re-1}\mathfrak{H}(\upsilon)d\upsilon \\
&\quad -\frac{\delta[(\vartheta-\chi)t+\chi]}{(\vartheta-\chi)(\zeta-\delta)}\frac{1}{\Gamma(\Re-1)}\int_0^1 (1-\upsilon)^{\Re-2}\mathfrak{H}(\upsilon)d\upsilon \\
&\quad -\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) - \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)), \qquad (2.9)
\end{aligned}
$$

for $\kappa = 0,\ 1,\ 2,\ \cdots,\ m$. Now substituting (2.9) into (2.2), for $t \in Q_0 = [0, t_1]$,

$$
\begin{aligned}
\Lambda(t) &= \frac{1}{\Gamma(\Re)}\int_0^t (t-\upsilon)^{\Re-1}\mathfrak{H}(\upsilon)d\upsilon + \frac{\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon}{\vartheta-\chi} + \frac{(\vartheta-\beta)t+\chi}{(\vartheta-\chi)(\zeta-\delta)}\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon \\
&\quad +\frac{\chi}{(\vartheta-\chi)\Gamma(\Re)}\int_0^1 (1-\upsilon)^{\Re-1}\mathfrak{H}(\upsilon)d\upsilon + \frac{\delta[(\vartheta-\chi)t+\chi]}{(\vartheta-\chi)(\zeta-\delta)}\frac{1}{\Gamma(\Re-1)}\int_0^1 (1-\upsilon)^{\Re-2}\mathfrak{H}(\upsilon)d\upsilon \\
&\quad +\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) \\
&= \int_0^t [\frac{(t-\upsilon)^{\Re-1}}{\Gamma(\Re)} + \frac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \frac{\chi\delta+(\vartheta-\chi)\delta t}{(\vartheta-\chi)(\zeta-\delta)}\frac{(1-\upsilon)^{\Re-2}}{\Gamma(\Re-1)}]\mathfrak{H}(\upsilon)d\upsilon \\
&\quad +\int_t^1 [\frac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \frac{\chi\delta+(\vartheta-\chi)\delta t}{(\vartheta-\chi)(\zeta-\delta)}\frac{(1-\upsilon)^{\Re-2}}{\Gamma(\Re-1)}]\mathfrak{H}(\upsilon)d\upsilon \\
&\quad +\frac{1}{\vartheta-\chi}\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon + \frac{(\vartheta-\chi)t+\chi}{(\vartheta-\chi)(\zeta-\delta)}\int_0^1 \varrho_2(s)\Lambda(\upsilon)d\upsilon \\
&\quad +\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) \\
&= \int_0^1 \aleph(t,s)\mathfrak{H}(\upsilon)d\upsilon + \frac{1}{\vartheta-\chi}\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon + \frac{(\vartheta-\chi)t+\chi}{(\vartheta-\chi)(\zeta-\delta)}\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon \\
&\quad +\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_0^1 \varrho_1(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})h(\widetilde{t})d\widetilde{t}d\upsilon &= (1-\mathfrak{Q}_1)\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon - \mathfrak{P}_1\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon \\
&\quad -\mathfrak{A}_1\frac{\chi}{\vartheta-\chi}[\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))],
\end{aligned}
$$

$$\int_0^1 \varrho_2(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})h(\widetilde{t})d\widetilde{t}d\upsilon = -\mathfrak{Q}_2\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon + (1-\mathfrak{P}_2)\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon - \mathfrak{A}_2\frac{\chi}{\vartheta-\chi}[\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))],$$

Hence,

$$\begin{aligned}\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon &= \frac{1}{\Gamma_1}[(1-\mathfrak{P}_2)(\int_0^1 \varrho_1(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon + \mathfrak{A}_1\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)))\\ &+\mathfrak{P}_1(\int_0^1 \varrho_2(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon + \mathfrak{A}_2\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)))],\end{aligned}$$

$$\begin{aligned}\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon &= \frac{1}{\Gamma_1}[\mathfrak{Q}_2(\int_0^1 \varrho_1(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon + \mathfrak{A}_1\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)))\\ &+(1-\mathfrak{Q}_1)(\int_0^1 \varrho_2(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon + \mathfrak{A}_2\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)))],\end{aligned}$$

which show that

$$\begin{aligned}\Lambda(t) &= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \frac{1}{\vartheta-\chi}\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon + \frac{(\vartheta-\chi)t+\chi}{(\vartheta-\chi)(\zeta-\delta)}\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon\\ &+\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))\\ &= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \varphi_1(t)[\int_0^1 \varrho_1(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon + \mathfrak{A}_1\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))]\\ &+\varphi_2(t)[\int_0^1 \varrho_2(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon + \mathfrak{A}_2\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))]\\ &+\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))\\ &= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{n=1}^{2}\varphi_n(t)[\int_0^1 \varrho_n(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon\\ &+\mathfrak{A}_n\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))] + \frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))\\ &= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{n=1}^{2}\varphi_n(t)[\int_0^1 \varrho_n(\widetilde{t})\int_0^1 \aleph(\widetilde{t},\upsilon)\mathfrak{H}(\upsilon)d\upsilon d\widetilde{t}\\ &+\mathfrak{A}_n\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))] + \frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{n=1}^{2}\varphi_n(t)[\int_0^1 \mathfrak{H}(\upsilon)\int_0^1 \aleph(\widetilde{t},s)\varrho_n(\widetilde{t})d\widetilde{t}d\upsilon \\
&\quad +\mathfrak{A}_n\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))] + \frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) \\
&= \int_0^1 [\aleph(t,\upsilon) + \sum_{n=1}^{2}\varphi_n(t)\int_0^1 \aleph(\widetilde{t},s)\varrho_i(\widetilde{t})d\widetilde{t}]\mathfrak{H}(\upsilon)d\upsilon \\
&\quad +[\frac{\chi}{\vartheta-\chi} + (\sum_{n=1}^{2}\frac{\chi}{\vartheta-\chi}\mathfrak{A}_n\varphi_n(t))]\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) \\
&= \int_0^1 \mathcal{H}_1(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{i=1}^{m}\mathcal{H}_2(t,t_i)\Phi_i(\Lambda(t_i)).
\end{aligned}
$$

Similar to the above process, for $t \in Q_\kappa = (t_\kappa, t_{k+1}]$, we have

$$
\begin{aligned}
\Lambda(t) &= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \frac{1}{\vartheta-\chi}\int_0^1 \varrho_1(\upsilon)\Lambda(\upsilon)d\upsilon + \frac{(\vartheta-\chi)t+\chi}{(\vartheta-\chi)(\zeta-\delta)}\int_0^1 \varrho_2(\upsilon)\Lambda(\upsilon)d\upsilon \\
&\quad +\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)) \\
&= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \varphi_1(t)[\int_0^1 \varrho_1(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon \\
&\quad +\mathfrak{A}_1(\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)))] + \varphi_2(t)[\int_0^1 \varrho_2(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon \\
&\quad +\mathfrak{A}_2(\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)))] \\
&\quad +\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)) \\
&= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{n=1}^{2}\varphi_n(t)[\int_0^1 \varrho_n(\upsilon)\int_0^1 \aleph(\upsilon,\widetilde{t})\mathfrak{H}(\widetilde{t})d\widetilde{t}d\upsilon \\
&\quad +\mathfrak{A}_n(\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)))] + \frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)) \\
&= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{n=1}^{2}\varphi_n(t)[\int_0^1 \varrho_n(t)\int_0^1 \Phi(\widetilde{t},\upsilon)\mathfrak{H}(\upsilon)d\upsilon d\widetilde{t} \\
&\quad +\mathfrak{A}_n(\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)))] + \frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i)) + \sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)) \\
&= \int_0^1 \aleph(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon + \sum_{n=1}^{2}\varphi_n(t)[\int_0^1 \mathfrak{H}(\upsilon)\int_0^1 \Phi(\widetilde{t},\upsilon)\varrho_n(\widetilde{t})d\widetilde{t}d\upsilon
\end{aligned}
$$

$$
\begin{aligned}
&+\mathfrak{A}_n(\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))+\sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i)))]+\frac{\chi}{\vartheta-\chi}\sum_{i=1}^{m}\Phi_i(\Lambda(t_i))+\sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i))\\
= &\int_0^1[\aleph(t,\upsilon)+\sum_{n=1}^{2}\varphi_n(t)\int_0^1\aleph(\widetilde{t},\upsilon)\varrho_n(\widetilde{t})d\widetilde{t}]\mathfrak{H}(\upsilon)d\upsilon\\
&+[\frac{\chi}{\vartheta-\chi}+(\sum_{n=1}^{2}\frac{\chi}{\vartheta-\chi}\mathfrak{A}_n\varphi_n(t))]\sum_{i=k+1}^{m}\Phi_i(\Lambda(t_i))\\
&+[\frac{\vartheta}{\vartheta-\chi}+(\sum_{n=1}^{2}\frac{\vartheta}{\vartheta-\chi}\mathfrak{A}_n\varphi_n(t))]\sum_{i=1}^{\kappa}\Phi_i(\Lambda(t_i))\\
= &\int_0^1\mathcal{H}_1(t,\upsilon)\mathfrak{H}(\upsilon)d\upsilon+\sum_{i=1}^{m}\mathcal{H}_2(t,t_i)\Phi_i(\Lambda(t_i)).
\end{aligned}
$$

The proof is completed. □

We assume that the following condition is satisfied in this paper:
(H1) $\mathfrak{Q}_1<1,\ \mathfrak{P}_2<1,\ (1-\mathfrak{Q}_1)(1-\mathfrak{P}_2)>\mathfrak{P}_1\mathfrak{Q}_2$.

Lemma 2.4. *The functions $\mathcal{H}_1$ and $\mathcal{H}_2$ have the following properties:*

(1) for all $t,\ \upsilon\in[0,1],\ i=1,\ \cdots,\ m$, $\mathcal{H}_1(t,\upsilon)\geq 0,\ \mathcal{H}_2(t,t_i)\ >0$;

(2) for all $t,\ \upsilon\in[0,1]$, $\mathfrak{d}_1\mathcal{M}(\upsilon)\ \leq\ m(\upsilon)\leq\ \mathcal{H}_1(t,\upsilon)\ \leq\ \mathcal{M}(\upsilon)$;

(3) for all $t\in[0,1],\ i=1,\ \cdots,\ m$, $\mathfrak{d}_2\mathcal{H}_2(1,0)\ \leq\ \mathcal{H}_2(t,t_i)\ \leq\mathcal{H}_2(1,0)$;

(4) for all $\upsilon\in[0,1]$, $max_{t\in[0,1]}\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t,\upsilon)\leq\dfrac{1}{\Gamma(3-\Re)}\mathcal{M}(\upsilon)$;

(5) $max_{t\in[0,1]}\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t,t_i)\leq\ \dfrac{1}{\Gamma(3-\Re)}\mathcal{H}_2(1,0),\ i=1,\ 2,\ \cdots,\ m$,
where

$$\mathcal{M}(\upsilon)=\mathfrak{g}(\upsilon)+\sum_{n=1}^{2}\varphi_n(1)\int_0^1\aleph(\upsilon,\widetilde{t})\varrho_n(\widetilde{t})d\widetilde{t},$$

$$m(\upsilon)=\mathfrak{d}_1\mathfrak{g}(\upsilon)+\sum_{n=1}^{2}\varphi_n(0)\int_0^1\aleph(\upsilon,\widetilde{t})\varrho_n(\widetilde{t})d\widetilde{t},$$

$$\mathfrak{g}(\upsilon)=\frac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)}+\frac{\vartheta\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)}+\frac{1}{\Gamma(\Re)},$$

$$\Pi=\frac{1+\sum\limits_{n=1}^{2}\mathfrak{A}_n\varphi_n(0)}{1+\sum\limits_{n=1}^{2}\mathfrak{A}_n\varphi_n(1)},\ \Pi_1=min_{\upsilon\in[0,1]}[\frac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)}+\frac{\chi\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)}],$$

and $\mathfrak{d}_1 = \dfrac{\chi\Gamma(\Re)\Pi_1}{\vartheta\Gamma(\Re)\Pi_1 + \chi}$, $\mathfrak{d}_2 = \dfrac{\chi}{\vartheta}\Pi.$

Proof. First, it is easy to see that

$$\mathcal{H}_1(t,s),\ \mathcal{H}_2(t,t_i) > 0,$$

for all $t,\ \upsilon \in [0,1],\ i = 1,\ 2,\ \cdots,\ m$. For given $\upsilon \in [0,1]$, we can get $\aleph(t,\upsilon)$ is increasing with respect to t for $t \in Q$ by the definition of $\aleph(t,\upsilon)$. Then,

$$\aleph(t,\upsilon) \le \frac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \frac{\vartheta\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)} + \frac{1}{\Gamma(\Re)} = \mathfrak{g}(\upsilon),$$

and

$$\begin{aligned}
\frac{\aleph(t,\upsilon)}{\mathfrak{g}(\upsilon)} &\ge \frac{\dfrac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \dfrac{\chi\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)} + \dfrac{(t-\upsilon)^{\Re-1}}{\Gamma(\Re)}}{\dfrac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \dfrac{\vartheta\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)} + \dfrac{1}{\Gamma(\Re)}} \\
&\ge \frac{\dfrac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \dfrac{\chi\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)}}{\dfrac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \dfrac{\vartheta\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)} + \dfrac{1}{\Gamma(\Re)}} \\
&\ge \frac{1}{\dfrac{\vartheta}{\chi} + \dfrac{1}{\Gamma(\Re)[\dfrac{\chi(1-\upsilon)^{\Re-1}}{(\vartheta-\chi)\Gamma(\Re)} + \dfrac{\chi\delta(1-\upsilon)^{\Re-2}}{(\vartheta-\chi)(\zeta-\delta)\Gamma(\Re-1)}]}} \\
&\ge \frac{1}{\dfrac{\vartheta}{\chi} + \dfrac{1}{\Gamma(\Re)\Pi_1}} = \mathfrak{d}_1.
\end{aligned}$$

Hence,

$$\mathfrak{d}_1\mathfrak{g}(\upsilon) \le \aleph(t,\upsilon) \le \mathfrak{g}(\upsilon),\ \text{for all } t,\ \upsilon \in\ [0,1],$$

and

$$\mathfrak{d}_1\mathcal{M}(\upsilon)\ \le\ m(\upsilon) \le\ \mathcal{H}_1(t,\upsilon)\ \le\ \mathcal{M}(\upsilon),\ \text{for all } t,\ \upsilon \in [0,1].$$

The proof of (3) is given below.

On the one hand, from the definition of $\mathcal{H}_2(t,t_i)$ and $\varphi_n(t)(n=1,2)$, for $0 \le t \le t_i \le 1$, it is easily to see that

$$\frac{\mathcal{H}_2(t,t_i)}{\mathcal{H}_2(1,0)} = \frac{\dfrac{\chi}{\vartheta-\chi} + \dfrac{\chi}{\vartheta-\chi}\displaystyle\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(t)}{\dfrac{\vartheta}{\vartheta-\chi} + \dfrac{\vartheta}{\vartheta-\chi}\displaystyle\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(1)}$$

$$
\begin{aligned}
&\geq \frac{\chi}{\vartheta}\left[\frac{1+\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(0)}{1+\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(1)}\right] \\
&= \frac{\chi}{\vartheta}\Pi = \mathfrak{d}_2.
\end{aligned}
$$

On the other hand, for $0 \leq t_i < t \leq 1$, we get

$$
\begin{aligned}
\frac{\mathcal{H}_2(t,t_i)}{\mathcal{H}_2(1,0)} &= \frac{\dfrac{\vartheta}{\vartheta-\chi}+\dfrac{\vartheta}{\vartheta-\chi}\displaystyle\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(t)}{\dfrac{\vartheta}{\vartheta-\chi}+\dfrac{\vartheta}{\vartheta-\chi}\displaystyle\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(1)} \\
&> \frac{1+\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(0)}{1+\sum_{n=1}^{2}\mathfrak{A}_n\varphi_n(1)} \\
&= \Pi.
\end{aligned}
$$

Therefore,

$$
\mathfrak{d}_2\mathcal{H}_2(1,0) \leq \mathcal{H}_2(t,t_i) \leq \mathcal{H}_2(1,0),
$$

for all $t \in [0,1],\ i = 1,\ 2,\ \cdots,\ m.$

Next, by calculation, one can obtain that

$$
{}_t^C\mathcal{D}_{0^+}^{\Re-1}\aleph(t,\upsilon) = \begin{cases} \dfrac{\delta(1-\upsilon)^{\Re-2}t^{2-\Re}}{(\zeta-\delta)\Gamma(\Re-1)\Gamma(3-\Re)}+1, & 0 \leq \upsilon < t \leq 1; \\ \dfrac{\delta(1-\upsilon)^{\Re-2}t^{2-\Re}}{(\zeta-\delta)\Gamma(\Re-1)\Gamma(3-\Re)}, & 0 \leq t \leq \upsilon \leq 1, \end{cases}
$$

$$
{}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t,\upsilon) = {}_t^C\mathcal{D}_{0^+}^{\Re-1}\aleph(t,\upsilon) + \sum_{n=1}^{2}[\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\varphi_n(t)]\int_0^1 \aleph(\upsilon,\widetilde{t})\varrho_n(\widetilde{t})d\widetilde{t},
$$

and

$$
{}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t,\upsilon) = \begin{cases} \dfrac{\chi}{\vartheta-\chi}\displaystyle\sum_{n=1}^{2}\mathfrak{A}_n[\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\varphi_n(t)], & 0 \leq t \leq t_i \leq 1; \\ \dfrac{\vartheta}{\vartheta-\chi}\displaystyle\sum_{n=1}^{2}\mathfrak{A}_n[\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\varphi_n(t)], & 0 \leq t_i < t \leq 1. \end{cases}
$$

Hence,

$$max_{t\in[0,1]}\, {}^{C}_{t}\mathcal{D}^{\Re-1}_{0^+}\aleph(t,\upsilon) \le \frac{1}{\Gamma(3-\Re)}\mathfrak{g}(\upsilon),\ for\ all\ \upsilon \in [0,1],$$

$$max_{t\in[0,1]}\, {}^{C}_{t}\mathcal{D}^{\Re-1}_{0^+}\mathcal{H}_1(t,\upsilon) \le \frac{1}{\Gamma(3-\Re)}\mathcal{M}(\upsilon),\ for\ all\ \upsilon \in [0,1],$$

$$max_{t\in[0,1]}\, {}^{C}_{t}\mathcal{D}^{\Re-1}_{0^+}\mathcal{H}_2(t,t_i) \le \frac{1}{\Gamma(3-\Re)}\mathcal{H}_2(1,0),\ i=1,\ 2,\ \cdots,\ m.$$

Hence, (4) and (5) are valid. □

Lemma 2.5. [19] *The set $\Upsilon \subset PC([0,1],R^n)$ is relatively compact if and only if*
(1) Υ is bounded, that is, $\|\phi\| \le C$ for each $\phi \in \Upsilon$ and some $C > 0$.
(2) Υ is quasi-equicontinuous in $(t_{\kappa-1}, t_\kappa](\kappa \in \mathbb{N})$, that is to say, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\phi(t_1) - \phi(t_2)| < \varepsilon$$

for all $\phi \in \Upsilon$, $t_1, t_2 \in (t_{\kappa-1}, t_\kappa]$ with $|t_1 - t_2| < \delta$.

Let Ω be a nonempty open subset of a Banach space $(X,\ \|\cdot\|)$. $\mathrm{T} : \overline{\Omega} \to X$ is an operator, where T may be discontinuous.

Definition 2.6. [15] *The closed-convex Krasovskij envelope (cc-envelope, for short) of an operator $\mathcal{T} : \overline{\Omega} \to X$ is the multivalued mapping $\mathbb{T} : \overline{\Omega} \to 2^X$ given by*

$$\mathbb{T}\Lambda = \bigcap_{\varepsilon>0} \overline{co}\mathcal{T}(\overline{B}_\varepsilon(\Lambda) \cap \overline{\Omega})\ for\ every\ \Lambda \in \overline{\Omega},$$

where $\overline{co}$ means closed convex hull, $\overline{B}_\varepsilon(\Lambda)$ is the closed ball centered at Λ and radius ε.

Lemma 2.7. [15] *$\widetilde{\Lambda} \in \mathbb{T}\Lambda$ if for every $\varepsilon > 0$ and every $\mathfrak{p} > 0$ there exist $m \in \mathbb{N}$ and a finite family of vectors $\Lambda_i \in \overline{B}_\varepsilon(\Lambda) \cap \overline{\Omega}$ and coefficients $\pi_i \in [0,1](i = 1,2,\cdots,m)$ such that $\sum_{i=1}^{m} \pi_i = 1$ and*

$$\|\widetilde{\Lambda} - \sum_{i=1}^{m} \pi_i T\Lambda_i\| < \mathfrak{p}.$$

Next, we introduce Krasnoselskii's fixed point theorems for discontinuous operators on cones. Let P be a cone of Banach space X. Then, P defines the partial ordering in given by $\Lambda \le \widetilde{\Lambda}$ if and only if $\widetilde{\Lambda} - \Lambda \in P$. For $\Lambda, \widetilde{\Lambda} \in P$, the set $[\Lambda, \widetilde{\Lambda}] = \{\widehat{\Lambda} \in P : \Lambda \le \widehat{\Lambda} \le \widetilde{\Lambda}\}$ is an order interval with $\Lambda \le \widetilde{\Lambda}$. Denote $P_R = \{\Lambda \in P : \|\Lambda\| < R\}$, for given $R > 0$.

Lemma 2.8. [16] *Let $R > 0$, $0 \in \Omega_i \subset P_R$ be relatively open subsets of P $(i = 1, 2)$. $\mathcal{T} : \overline{P}_R \to P$ is a mapping, where $\mathcal{T}\overline{P}_R$ is relatively compact and it fulfills condition*

$$\Lambda \cap \mathbb{T}\Lambda \subset \{\mathcal{T}\Lambda\} \tag{2.10}$$

in $\overline{P}_R$.
(a)For all $\Lambda \in \partial\Omega_1 (\lambda \geq 1)$, if $\lambda\Lambda \notin \mathbb{T}\Lambda$, then $i(\mathcal{T},\ \Omega_1,\ P) = 1$.
(b)For every $\ell \geq 0$ and all $\Lambda \in P$ with $\Lambda \in \partial\Omega_2$, if there exists $\ell \in P(\ell \neq 0)$ such that $\Lambda \notin \mathbb{T}\Lambda + \ell\omega$, then $i(\mathcal{T},\ \Omega_2,\ P) = 0$.

Lemma 2.9. [16] *Assume that one of the following two conditions holds:*
(i) $\widetilde{\Lambda} \ngeq \Lambda$ for all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ with $\Lambda \in P$ and $\|\Lambda\| = r_1$.
(ii) $\|\widetilde{\Lambda}\| < \|\Lambda\|$ for all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ and all $\Lambda \in P$ with $\|\Lambda\| = r_1$.
Then, Condition (a) in Lemma 2.8 is satisfied.

Analogously, if one of the following two conditions holds:
(i) $\widetilde{\Lambda} \nleq \Lambda$ for all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ with $\Lambda \in P$ and $\|\Lambda\| = r_1$.
(ii) If $\|\widetilde{\Lambda}\| > \|\Lambda\|$ for all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ and all $\Lambda \in P$ with $\|\Lambda\| = r_2$.
Then, assumption (b) in Lemma 2.8 holds.

For the discontinuous nonlinearities F, we define the admissible discontinuities curves.

Definition 2.10. *We say that $\hbar : Q \to \mathbb{R}^+$, $\hbar \in PC^1(Q)$ is an admissible discontinuity curve for the differential system (1.1) if $\hbar$ satisfies $\triangle\hbar'|_{t=t_i} = 0 (i = 1,\ \ldots,\ m)$, the boundary value conditions of (1.1) and one of the following conditions holds:*
(i)

$$\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\Re}\hbar(t) = \mathcal{E}(t)F(t, \hbar(t)),\ a.e.\ t \in Q', \\ \triangle\hbar|_{t=t_\kappa} = \Phi_\kappa(\hbar(t_\kappa)),\ \kappa = 1,\ \cdots,\ m, \end{cases} \tag{2.11}$$

(ii) there exist $\mathcal{G}$, $\overline{\mathcal{G}} \in L^1(J)$, $\mathcal{G}(t)$, $\overline{\mathcal{G}}(t) > 0$ a.e. for $t \in [0, 1]$, $S,\ \Theta \subset J$, $m(S \cap \Theta) = 0$, $m(S \cup \Theta) > 0$, and $\varepsilon > 0$ such that

$$\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\Re}\hbar(t) + \overline{\mathcal{G}}(t) < \mathcal{E}(t)F(t, x),\ a.e.\ t \in \Theta,\ x \in [\hbar(t) - \varepsilon, \hbar(t) + \varepsilon], \\ {}_t^C\mathcal{D}_{0^+}^{\Re}\hbar(t) - \mathcal{G}(t) > \mathcal{E}(t)F(t, x),\ a.e.\ t \in S,\ x \in [\hbar(t) - \varepsilon, \hbar(t) + \varepsilon], \\ {}_t^C\mathcal{D}_{0^+}^{\Re}\hbar(t) = \mathcal{E}(t)F(t, \hbar(t)),\ a.e.\ t \in Q' \setminus (S \cup \Theta), \\ \triangle\hbar|_{t=t_\kappa} = \Phi_\kappa(\hbar(t_\kappa)),\ k = 1,\ \cdots,\ m, \end{cases} \tag{2.12}$$

(iii) there exist $\kappa \in \{1,\ 2,\ \cdots,\ m\}$ such that

$$\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\Re}\hbar(t) = \mathcal{E}(t)F(t, \hbar(t)),\ a.e.\ t \in Q', \\ \triangle\hbar|_{t=t_\kappa} \neq \Phi_\kappa(\hbar(t_\kappa)), \end{cases} \tag{2.13}$$

(iv) there exists $\mathcal{G}$, $\overline{\mathcal{G}} \in L^1(\Theta)$, $\mathcal{G}(t)$, $\overline{\mathcal{G}}(t) > 0$ a.e. for $t \in [0, 1]$, $S,\ \widetilde{\Theta} \subset \Theta$, $m(S \cap \widetilde{\Theta}) = 0$, $m(S \cup \widetilde{\Theta}) > 0$, and $\varepsilon > 0$, $\kappa \in \{1,\ 2,\ \cdots,\ m\}$ such that

$$\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\Re}\hbar(t) + \overline{\mathcal{G}}(t) < \mathcal{E}(t)F(t,\Lambda),\ a.e.\ t \in \Theta,\ x \in [\hbar(t)-\varepsilon, \hbar(t)+\varepsilon], \\ {}_t^C\mathcal{D}_{0^+}^{q}\hbar(t) - \mathcal{G}(t) > \mathcal{E}(t)F(t,x),\ a.e.\ t \in S,\ x \in [\hbar(t)-\varepsilon, \hbar(t)+\varepsilon], \\ {}_t^C\mathcal{D}_{0^+}^{q}\hbar(t) = \mathcal{E}(t)F(t,\hbar(t)),\ a.e.\ t \in Q' \setminus (S \cup \Theta), \\ \triangle\hbar|_{t=t_\kappa} \neq \Phi_\kappa(\hbar(t_\kappa)). \end{cases} \tag{2.14}$$

Then, we assert that ħ is viable for BVP (1.1) if (i) is satisfied; we say that ħ is inviable if one of (ii)-(iv) is satisfied.

3. Existence results

Let $\Xi = PC^1[0,1]$, $\mathcal{P} := \{\Lambda \in \Xi : \Lambda(t) \geq \mathfrak{d}\|\Lambda\|_1,\ \forall t \in [0,1]\}(\mathfrak{d} = \Gamma(3-\Re)\mathfrak{d}_3,\ \mathfrak{d}_3 = min\{\mathfrak{d}_1,\ \mathfrak{d}_2\})$ and $\mathcal{P}_r := \{\Lambda \in \mathcal{P} :\ \|\Lambda\|_1 \leq r\}$. In order to apply Krasnoselskii's compression-expansion type fixed point theorems for discontinuous operators to BVP (1.1), we recall that if Λ is a solution of the following equation:

$$\Lambda(t) = \int_0^1 \mathcal{H}_1(t,s)\mathcal{E}(s)F(s,\Lambda(\upsilon))d\upsilon + \sum_{i=1}^{m} \mathcal{H}_2(t,t_i)\Phi_i(\Lambda(t_i)), \tag{3.1}$$

then $\Lambda \in \Xi$ is a solution of BVP (1.1).

Define an operator $\mathcal{T} : \mathcal{P} \to \Xi$ as follows:

$$\mathcal{T}\Lambda(t) := \int_0^1 \mathcal{H}_1(t,s)\mathcal{E}(s)F(s,\Lambda(\upsilon))d\upsilon + \sum_{i=1}^{m} \mathcal{H}_2(t,t_i)\Phi_i(\Lambda(t_i)),\ \Lambda \in \mathcal{P}. \tag{3.2}$$

For any $\Lambda \in \mathcal{P}$, $\mathcal{T}\Lambda$ is well defined by $\mathcal{E} \in L(0,1)$, the continuity of $\mathcal{H}_1$ and the assumption of F. One can see that the existence of positive fixed points of $\mathcal{T}$ implies the existence of positive solutions for BVP (1.1).

Subsequently, let

$$\mathcal{N}_1 = (\int_0^1 \mathcal{M}(\upsilon)\mathfrak{g}(\upsilon)d\upsilon)^{-1},\ \mathcal{N}_2 = (\int_0^1 m(\upsilon)\mathfrak{g}(\upsilon)d\upsilon)^{-1},$$

$$\mathcal{N}_3 = (sup_{t\in[0,1]} \int_0^1 {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t,\upsilon)\mathfrak{g}(\upsilon)d\upsilon)^{-1},\ \mathcal{N}_4 = (inf_{t\in[0,1]} \int_0^1 {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t,\upsilon)\mathfrak{g}(\upsilon)d\upsilon)^{-1},$$

$$\mathcal{N}_5 = sup_{t\in[0,1],\ i\in\{1,\ \cdots,m\}}\mathcal{H}_2(t,t_i),\ \mathcal{N}_6 = inf_{t\in[0,1],\ i\in\{1,\ \cdots,m\}}\mathcal{H}_2(t,t_i),$$

$$\mathcal{N}_7 = sup_{t\in[0,1],\ i\in\{1,\ \cdots,m\}}\, {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t,t_i),\ \mathcal{N}_8 = inf_{t\in[0,1],\ i\in\{1,\ \cdots,m\}}\, {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t,t_i).$$

Now, we are in position to give the assumptions satisfied throughout the paper.

(H2)$F : Q \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

(a) $t \in Q \mapsto F(\cdot,\Lambda)$ is measurable for any $\Lambda \in \mathbb{R}^+$;

(b)For a.e. $t \in Q$ and all $\Lambda \in [0, r]$, there exists $R > 0$ such that $F(t, \Lambda) \leq R$ for each $r > 0$.

(H3) $\mathcal{E}(t) \geq 0$ almost everywhere for $t \in [0, 1]$ and $\mathcal{E}$ is measurable.

(H4) Admissible discontinuity curves $\hbar_n : Q \to \mathbb{R}^+ (n \in N)$ satisfy that the function $\Lambda \mapsto F(t, \Lambda)$ is continuous in $[0, \infty) \setminus \bigcup_{n\in\mathbb{N}}\{\hbar_n(t)\}$ for $a.e.\ t \in Q$.

(H5) $\lim\limits_{\Lambda\to 0^+} \inf\limits_{t\in[0,1]} \dfrac{F(t,\Lambda)}{\Lambda} > \dfrac{5}{4\mathfrak{d}}[\dfrac{1}{\mathcal{N}_2} + m\mathcal{N}_6]^{-1}$, $\lim\limits_{\Lambda\to 0^+} \dfrac{\Phi_\kappa(\Lambda)}{\Lambda} > \dfrac{5}{4\mathfrak{d}}[\dfrac{1}{\mathcal{N}_2} + m\mathcal{N}_6]^{-1}$,
(H6) $\lim\limits_{\Lambda\to +\infty} \sup\limits_{t\in[0,1]} \dfrac{F(t,\Lambda)}{\Lambda} < \dfrac{5}{6}[\dfrac{1}{\mathcal{N}_1} + m\mathcal{N}_5]^{-1}$, $\lim\limits_{\Lambda\to +\infty} \dfrac{\Phi_\kappa(\Lambda)}{\Lambda} < \dfrac{5}{6}[\dfrac{1}{\mathcal{N}_1} + m\mathcal{N}_5]^{-1}$,
(H7) $\lim\limits_{\Lambda\to 0^+} \sup\limits_{t\in[0,1]} \dfrac{F(t,\Lambda)}{\Lambda} < \dfrac{5}{6}[\dfrac{1}{\mathcal{N}_1} + m\mathcal{N}_5]^{-1}$, $\lim\limits_{\Lambda\to 0^+} \dfrac{\Phi_\kappa(\Lambda)}{\Lambda} < \dfrac{5}{6}[\dfrac{1}{\mathcal{N}_1} + m\mathcal{N}_5]^{-1}$,
(H8) $\lim\limits_{\Lambda\to +\infty} \inf\limits_{t\in[0,1]} \dfrac{F(t,\Lambda)}{\Lambda} > \dfrac{5}{4\mathfrak{d}}[\dfrac{1}{\mathcal{N}_2} + m\mathcal{N}_6]^{-1}$, $\lim\limits_{\Lambda\to +\infty} \dfrac{\Phi_\kappa(\Lambda)}{\Lambda} > \dfrac{5}{4\mathfrak{d}}[\dfrac{1}{\mathcal{N}_2} + m\mathcal{N}_6]^{-1}$.

Lemma 3.1. *The operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is well-defined and maps bounded sets into relatively compact sets.*

Proof. In view of the nonnegativity of F, $\mathcal{H}_1$, $\mathcal{H}_2$, $\Phi_\kappa (\kappa = 1, \cdots, m)$ and $\mathcal{E}(t) \geq 0$ for $a.e.\ t \in Q$, we conclude that $\mathcal{T}\Lambda(t) \geq 0$ for $t \in [0, 1]$. Hence, $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is well-defined.

Then, by calculation, for $\Lambda \in P$, it is easy to see

$$ {}_t^C\mathcal{D}_{0^+}^{\Re-1}(\mathcal{T}\Lambda)(t) = \int_0^1 [\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t, \upsilon)]\mathcal{E}(\upsilon)F(\upsilon, \Lambda(\upsilon))d\upsilon + \sum_{i=1}^m [\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t, t_i)]\Phi_i(\Lambda(t_i)), $$

where

$$ {}_t^C\mathcal{D}_{0^+}^{\Re-1}\aleph(t, \upsilon) = \begin{cases} \dfrac{\zeta(1-\upsilon)^{\Re-2}t^{2-\Re}}{(\zeta-\delta)\Gamma(\Re-1)\Gamma(3-\Re)} + 1, & 0 \leq \upsilon < t \leq 1; \\ \dfrac{\zeta(1-\upsilon)^{\Re-2}t^{2-\Re}}{(\zeta-\delta)\Gamma(\Re-1)\Gamma(3-\Re)}, & 0 \leq t \leq \upsilon \leq 1, \end{cases} $$

$$ {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t, \upsilon) = {}_t^C\mathcal{D}_{0^+}^{\Re-1}\aleph(t, \upsilon) + \sum_{n=1}^2 [\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\varphi_n(t)] \int_0^1 \aleph(\upsilon, \widetilde{t})\varrho_n(\widetilde{t})d\widetilde{t}, $$

and

$$ {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t, t_i) = \begin{cases} \dfrac{\chi}{\vartheta-\chi}\displaystyle\sum_{n=1}^2 \mathfrak{A}_n[\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\varphi_n(t)], & 0 \leq t \leq t_i \leq 1; \\ \dfrac{\vartheta}{\vartheta-\chi}\displaystyle\sum_{n=1}^2 \mathfrak{A}_n[\,{}_t^C\mathcal{D}_{0^+}^{\Re-1}\varphi_n(t)], & 0 \leq t_i < t \leq 1. \end{cases} $$

By Lemma 2.4, one can get that

$$ \mathfrak{d}\|{}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{T}\Lambda\|_0 \quad = \quad \mathfrak{d}\ max_{t\in[0,1]}[\int_0^1 ({}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t, \upsilon))\mathcal{E}(\upsilon)F(\upsilon, \Lambda(\upsilon))d\upsilon $$

$$
\begin{aligned}
&+\sum_{i=1}^{m}({}^{C}_{t}\mathcal{D}^{\mathfrak{R}-1}_{0^+}\mathcal{H}_2(t,t_i))\Phi_i(\Lambda(t_i))]\\
\leq\;& \mathfrak{d}_3[\int_0^1 \mathcal{M}(\upsilon)\mathcal{E}(\upsilon)F(\upsilon,\Lambda(\upsilon))d\upsilon+\sum_{i=1}^{m}\mathcal{H}_2(1,0)\Phi_i(\Lambda(t_i))]\\
\leq\;& \int_0^1 m(\upsilon)\mathcal{E}(\upsilon)F(\upsilon,\Lambda(\upsilon))d\upsilon+\sum_{i=1}^{m}\mathcal{H}_2(0,1)\Phi_i(\Lambda(t_i))\\
=\;& min_{t\in[0,1]}\mathcal{T}\Lambda(\tau).
\end{aligned}
$$

Thinking about it from the other side, we have

$$
\begin{aligned}
\mathfrak{d}\|\mathcal{T}\Lambda\|_0 \leq\;& \mathfrak{d}_3[\int_0^1 \mathcal{M}(\upsilon)\mathcal{E}(\upsilon)F(\upsilon,\Lambda(\upsilon))d\upsilon+\sum_{i=1}^{m}\mathcal{H}_2(1,0)\Phi_i(\Lambda(t_i))]\\
\leq\;& \int_0^1 m(\upsilon)\mathcal{E}(\upsilon)F(\upsilon,\Lambda(\upsilon))d\upsilon+\sum_{i=1}^{m}\mathcal{H}_2(0,1)\Phi_i(\Lambda(t_i))\\
=\;& min_{t\in[0,1]}\mathcal{T}\Lambda(t).
\end{aligned}
$$

Therefore,

$$min_{t\in[0,1]}\mathcal{T}\Lambda(t)\geq\mathfrak{d}(\upsilon)\ max\{\|\mathcal{T}\Lambda\|_0,\|{}^{C}_{t}\mathcal{D}^{\mathfrak{R}-1}_{0^+}\mathcal{T}\Lambda\|_0\}=\mathfrak{d}(\upsilon)\|\mathcal{T}\Lambda\|_1.$$

Next, we notice that there exists $\mathcal{M}_\kappa>0$ such that

$$\Phi_\kappa(\Lambda)\leq\mathcal{M}_\kappa,\ for\ \Lambda\in[0,r],$$

where $\kappa=1,\ 2,\ \cdots,\ m$ for each $r>0$. Therefore, $\mathcal{T}(\mathcal{P}_r)$ is bounded by (H2).

Moreover, we have

$${}^{C}_{t}\mathcal{D}^{q}_{0^+}\mathcal{T}\Lambda(t)=\mathcal{E}(t)F(t,\Lambda(t))\leq R\mathcal{E}(t),$$

for any $\Lambda\in\mathcal{P}_r$ and a.e. $t\in Q_\kappa$.

Therefore,

$$|{}^{C}_{t}\mathcal{D}^{\mathfrak{R}-1}_{0^+}(\mathcal{T}\Lambda)(\widehat{t_2})-{}^{C}_{t}\mathcal{D}^{\mathfrak{R}-1}_{0^+}(\mathcal{T}\Lambda)(\widehat{t_1})|\leq\int_{\widehat{t_1}}^{\widehat{t_2}}|{}^{C}_{r}D^{q}_{0^+}(\mathcal{T}\Lambda)(r)|dr\leq\int_{\widehat{t_1}}^{\widehat{t_2}}R\mathcal{E}(r)dr,$$

where $\widehat{t_1},\ \widehat{t_2}\in Q_\kappa$. Hence, $\mathcal{T}(\mathcal{P}_r)$ is relatively compact. □

Lemma 3.2. *Let $\mathbb{T}$ be the cc-envelope of the operator $\mathcal{T}:\mathcal{P}_R\to\mathcal{P}$. If (H4) is satisfied, then*

$$\Lambda\cap\mathbb{T}\Lambda\subset\{\mathcal{T}\Lambda\},\ for\ all\ \Lambda\in\mathcal{P}_R.$$

Proof. Let $\mathfrak{W}_n=\{t\in Q:\Lambda(t)=\hbar_n(t)\}(n\in\mathbb{N})$. Fix $\Lambda\in\mathcal{P}_R$ and we think about three cases below.

Case 1: $m(\mathfrak{W}_n)=0$ for all $n\in\mathbb{N}$.

If $\Lambda_\kappa \to \Lambda$ in $\mathcal{P}_R$, by (H4), it is easy to see that $F(t, \Lambda_\kappa(t)) \to F(t, \Lambda(t))$ for a.e. $t \in Q$. This, together with (H2) and (H3), implies that

$$\mathcal{T}\Lambda_\kappa \to \mathcal{T}\Lambda \text{ in } \mathcal{P}_R.$$

Hence $\mathcal{T}$ is continuous at Λ. Hence, $\mathbb{T}\Lambda = \mathcal{T}\Lambda$.

Case 2: there exists $n \in \mathbb{N}$ such that $\hbar_n$ is inviable and $m(\mathfrak{W}_n) > 0$. Let $\mathbb{B} = \{n : m(\mathfrak{W}_n) > 0,\ \hbar_n\ is\ inviable\}$. Case 2 will be demonstrated in three subcases.

Case 2.1: The above $\hbar_n$ satisfies (ii) in Definition 2.10.

By (ii) in Definition 2.10, there exist $\mathcal{G}$, $\overline{\mathcal{G}} \in L^1(Q')$, $\mathcal{G}(t)$, $\overline{\mathcal{G}}(t) > 0\ for\ a.e.\ t \in [0,1]$, S_n, $\Theta_n \subset Q$, $m(S_n \cap \Theta_n) = 0$, $m(S_n \cup \Theta_n) > 0$, and $\varepsilon > 0$ such that

$$\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\mathfrak{R}}\hbar(t) + \overline{\mathcal{G}}(t) < \mathcal{E}(t)F(t, \hbar_n(t)),\ a.e.\ t \in \Theta_n,\ \Lambda \in [\hbar_n(t) - \varepsilon, \hbar_n(t) + \varepsilon], \\ {}_t^C\mathcal{D}_{0^+}^{\mathfrak{R}}\hbar(t) - \mathcal{G}(t) > \mathcal{E}(t)F(t, \hbar_n(t)),\ a.e.\ t \in S_n,\ \Lambda \in [\hbar_n(t) - \varepsilon, \hbar_n(t) + \varepsilon], \\ {}_t^C\mathcal{D}_{0^+}^{\mathfrak{R}}\hbar(t) = \mathcal{E}(t)F(t, \hbar_n(t)),\ a.e.\ t \in Q' \setminus (S_n \cup \Theta_n), \\ \triangle\hbar_n|_{t=t_\kappa} = \Phi_\kappa(\hbar_n(t_\kappa)),\ \kappa = 1, \cdots, m. \end{cases} \tag{3.3}$$

(I) $m(\{t \in S_n \cup \Theta_n | \Lambda(t) = \hbar_n(t)\}) = 0$ for all $n \in \mathbb{B}$.

By $m(\{t \in S_n \cup \Theta_n | \Lambda(t) = \hbar_n(t)\}) = 0$, for $a.e.\ t \in \mathfrak{W}_n$, one can obtain that

$${}_t^C\mathcal{D}_{0^+}^{\mathfrak{R}}\hbar_n(t) = \mathcal{E}(t)F(\Lambda, \hbar_n(t)).$$

This is,

$${}_t^C\mathcal{D}_{0^+}^{\mathfrak{R}}\Lambda(t) = \mathcal{E}(t)F(t, \Lambda),\ t \in \bigcup_{n\in\mathbb{B}} \mathfrak{W}_n.$$

For each $\kappa \in \mathbb{N}$, on account of $\Lambda \in \mathbb{T}\Lambda$, there exist functions $\Lambda_{p,i} \in B_{\frac{1}{p}}(\Lambda) \cap \mathcal{P}_R$ and coefficients $\lambda_{p,i} \in [0,1] (i = 1,\ 2, \cdots,\ m(p))$ such that

$$\sum_{i=1}^{m(p)} \lambda_{p,i} = 1,$$

and

$$||\Lambda - \sum_{i=1}^{m(p)} \lambda_{p,i}\mathcal{T}\Lambda_{p,i}|| < \frac{1}{p},$$

by Lemma 2.7 with $\varepsilon = \mathfrak{p} = \dfrac{1}{p}$.

Denote $V_p = \sum\limits_{i=1}^{m(p)} \lambda_{p,i}\mathcal{T}\Lambda_{p,i}$. If $p \to\ \infty$ in Q, we can see that $V_p \to \Lambda$ uniformly .

For $a.e.\ t \in Q \setminus \bigcup\limits_{n\in\mathbb{B}} \mathfrak{W}_n$, one can see that $\mathcal{E}(t)F(t, \cdot)$ is continuous at $\Lambda(t)$. Consequently, for any $\varepsilon > 0$, there is some $p_0 = p(t) \in \mathbb{N}$ such that, for all $\kappa \in \mathbb{N}$, $p \geq p_0$, we have

$$|\mathcal{E}(t)F(t, \Lambda_{p,i}(t)) - \mathcal{E}(t)F(t, \Lambda(t))| < \varepsilon,$$

for all $i \in \{1,\ 2,\ \cdots,\ m(p)\}$. Then,

$$|{}_t^C\mathcal{D}_{0^+}^{\Re} V_p(t) - \mathcal{E}(t)F(t,\Lambda(t))| \leq \sum_{i=1}^{m(p)} \lambda_{p,i}|\mathcal{E}(t)F(t,\Lambda_{p,i}(t)) - \mathcal{E}(t)F(t,\Lambda(t))| < \varepsilon.$$

This is,

$${}_t^C\mathcal{D}_{0^+}^{\Re} V_p(t) \to \mathcal{E}(t)F(t,\Lambda(t)),\ \text{when}\ p \to \infty,$$

for *a.e.* $t \in Q \setminus \bigcup_{n\in\mathbb{B}} \mathfrak{W}_n$.

On the other hand,

$$\begin{aligned} |{}_t^C\mathcal{D}_{0^+}^{\Re} V_p(t) - {}_t^C\mathcal{D}_{0^+}^{\Re}\Lambda(t)| &= \frac{1}{\Gamma(\Re)}|\int_0^t (t-\upsilon)^{\Re-1}V_p(\upsilon)d\upsilon - \int_0^t (t-\upsilon)^{\Re-1}\Lambda(\upsilon)d\upsilon| \\ &\leq \frac{1}{\Gamma(\Re)}\int_0^t (t-\upsilon)^{\Re-1}|V_p(\upsilon) - \Lambda(\upsilon)|d\upsilon \\ &\leq \varepsilon_1(\frac{1}{\Gamma(\Re)}\int_0^t (t-\upsilon)^{\Re-1}d\upsilon) \\ &\leq \frac{1}{\Gamma(\Re+1)}\varepsilon_1, \end{aligned}$$

which guarantees that ${}_t^C\mathcal{D}_{0^+}^{\Re}\Lambda(t) = \mathcal{E}(t)F(t,\Lambda)$ for *a.e.* $t \in Q \setminus \bigcup_{n\in\mathbb{B}} \mathfrak{W}_n$. The process above implies $\Lambda = \mathcal{T}\Lambda$ if $\Lambda \in \mathbb{T}\Lambda$.

(II) There exists $n \in \mathbb{B}$ such that $m(\{t \in S_n \cup \Theta_n | \Lambda(t) = \hbar_n(t)\}) > 0$.

Suppose $m(\{t \in S_n | \Lambda(t) = \hbar_n(t)\}) > 0$. Now we are in position to prove

$$\Lambda \notin \mathbb{T}\Lambda.$$

For *a.e.* $t \in Q$, by (H2), there exists $\mathcal{H}_R > 0$ such that $F(t,\Lambda(t)) < \mathcal{H}_R$. Let $F(t) = \mathcal{E}(t)\mathcal{H}_R$ and $\mathcal{A} = \{t \in S_n|\ \Lambda(t) = \hbar_n(t)\}(n \in \mathbb{N})$. There exists an interval $Q_{\kappa_0}(\kappa_0 \in \{1,\ \cdots,\ m\})$ such that $m(Q_{k_0} \cap \mathcal{A}) > 0$. Let $\mathbb{A} = Q_{k_0} \cap \mathcal{A}$. On account of $F \in L(Q)$ and Lemma 3.8 in [15], there is a measurable set $A_0 \subset \mathbb{A}$ with $m(A_0) = m(\mathbb{A}) > 0$ such that, we obtain

$$lim_{t\to\widehat{t_0}^+}\frac{2\int_{[\widehat{t_0},t]\setminus\mathbb{A}} F(\upsilon)d\upsilon}{\frac{1}{4}\int_{\widehat{t_0}}^t \mathcal{G}(\upsilon)d\upsilon} = 0 = lim_{t\to\widehat{t_0}^-}\frac{2\int_{[t,\widehat{t_0}]\setminus\mathbb{A}} F(\upsilon)d\upsilon}{\frac{1}{4}\int_t^{\widehat{t_0}} \mathcal{G}(\upsilon)d\upsilon}, \tag{3.4}$$

for all $\widehat{t_0} \in A_0$.

Moreover, by Corollary 3.9 in [15], there exists $A_1 \subset A_0$ with $m(A_0 \setminus A_1) = 0$ such that,

$$lim_{t\to\widehat{t_0}^+}\frac{\int_{[\widehat{t_0},t]\cap A_0} \mathcal{G}(\upsilon)d\upsilon}{\int_{\widehat{t_0}}^t \mathcal{G}(\upsilon)d\upsilon} = 1 = lim_{t\to\widehat{t_0}^-}\frac{\int_{[t,\widehat{t_0}]\cap A_0} \mathcal{G}(\upsilon)d\upsilon}{\int_{\widehat{t_0}}^t \mathcal{G}(\upsilon)d\upsilon}, \tag{3.5}$$

for all $\widehat{t_0} \in A_1$.

Fix a point $\widehat{t_0} \in A_1$. By (3.4) and (3.5), we konw that $t_- < \widehat{t_0}$, $t_+ > \widehat{t_0}$ exist with $t_+,\ t_- \to \widehat{t_0}$. Moreover, $t_+,\ t_-$ satisfies the following inequalities.

$$2\int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} F(\upsilon)d\upsilon < \frac{1}{4}\int_{\widehat{t_0}}^{t_+} \mathcal{G}(\upsilon)d\upsilon, \tag{3.6}$$

$$\int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} \mathcal{G}(\upsilon)d\upsilon \geq \int_{[\widehat{t_0},t^+]\cap A_0} \mathcal{G}(\upsilon)d\upsilon > \frac{1}{2}\int_{\widehat{t_0}}^{t_+} \mathcal{G}(\upsilon)d\upsilon, \tag{3.7}$$

$$2\int_{[t^-,\widehat{t_0}]\backslash\mathbb{A}} F(\upsilon)d\upsilon < \frac{1}{4}\int_{t_-}^{\widehat{t_0}} \mathcal{G}(\upsilon)d\upsilon, \tag{3.8}$$

$$\int_{[t^-,\widehat{t_0}]\cap\mathbb{A}} \mathcal{G}(\upsilon)d\upsilon > \frac{1}{2}\int_{t_-}^{\widehat{t_0}} \mathcal{G}(\upsilon)d\upsilon. \tag{3.9}$$

Now we will prove that $\Lambda \notin \mathbb{T}\Lambda$.

Claim: For every finite family $\Lambda_i \in B_\varepsilon(\Lambda) \cap \overline{B}_R$ and $\pi_i \in [0,1]$ $(i = 1, 2, \cdots, m_1)$, there exists $\mathfrak{p} > 0$ such that

$$\|\Lambda - \sum_{i=1}^{m_1} \pi_i \mathcal{T}\Lambda_i\| \geq \mathfrak{p},$$

where $\sum_{i=1}^{m_1} \pi_i = 1$.

Denote $V = \sum_{i=1}^{m_1} \pi_i \mathcal{T}\Lambda_i$. Then for a.e. $t \in \mathbb{A}$, we have

$${}^{C}_{t}\mathcal{D}^{\mathfrak{R}}_{0^+} v(t) = \sum_{i=1}^{m_1} \pi_i\, {}^{C}_{t}\mathcal{D}^{\mathfrak{R}}_{0^+}(\mathcal{T}\Lambda_i)(t) = \sum_{i=1}^{m_1} \pi_i \mathcal{E}(t) F(t, \Lambda_i(t)). \tag{3.10}$$

For every $i \in \{1,\ 2,\ \cdots,\ m_1\}$ and $t \in \mathbb{A}$, one can obtain that

$$|\Lambda_i(t) - \hbar_n(t)| = |\Lambda_i(t) - \Lambda(t)| < \varepsilon.$$

Then, for a.e. $t \in A$, we have

$${}^{C}_{t}\mathcal{D}^{\mathfrak{R}}_{0^+} V(t) = \sum_{i=1}^{m_1} \pi_i \mathcal{E}(t) F(t, \Lambda_i(t)) < \sum_{i=1}^{m_1} \pi_i ({}^{C}_{t}\mathcal{D}^{\mathfrak{R}}_{0^+} \hbar_n(t) - \mathcal{G}(t)) = {}^{C}_{t}\mathcal{D}^{\mathfrak{R}}_{0^+} \Lambda(t) - \mathcal{G}(t). \tag{3.11}$$

Now we compute

$$\begin{aligned}
{}^{C}_{t}\mathcal{D}^{\mathfrak{R}-1}_{0^+} V(t^+) - {}^{C}_{t}\mathcal{D}^{\mathfrak{R}-1}_{0^+} V(\widehat{t_0}) &= \int_{\widehat{t_0}}^{t^+} [{}^{C}_{s}\mathcal{D}^{\mathfrak{R}-1}_{0^+} V(\upsilon)]' d\upsilon = \int_{\widehat{t_0}}^{t^+} [{}^{C}_{s}\mathcal{D}^{\mathfrak{R}}_{0^+} V(\upsilon)] d\upsilon \\
&= \int_{[\widehat{t_0},t^+]\cap\mathbb{A}} [{}^{C}_{s}\mathcal{D}^{\mathfrak{R}}_{0^+} V(\upsilon)] d\upsilon + \int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} [{}^{C}_{t}\mathcal{D}^{\mathfrak{R}}_{0^+} V(\upsilon)] d\upsilon \\
&< \int_{[\widehat{t_0},t^+]\cap\mathbb{A}} {}^{C}_{s}\mathcal{D}^{\mathfrak{R}}_{0^+} \Lambda(\upsilon) d\upsilon - \int_{[\widehat{t_0},t^+]\cap\mathbb{A}} \mathcal{G}(\upsilon) d\upsilon \\
&\quad + \int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} F(\upsilon) d\upsilon
\end{aligned}$$

$$
\begin{aligned}
&= {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t^+) - {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(\widehat{t_0}) - \int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} {}_s^C\mathcal{D}_{0^+}^{\Re}\Lambda(\upsilon)d\upsilon \\
&\quad - \int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} \mathcal{G}(s)d\upsilon + \int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} F(\upsilon)d\upsilon \\
&\leq {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t^+) - {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(\widehat{t_0}) - \int_{[\widehat{t_0},t^+]\cap\mathbb{A}} \mathcal{G}(\upsilon)d\upsilon \\
&\quad +2\int_{[\widehat{t_0},t^+]\backslash\mathbb{A}} F(\upsilon)d\upsilon \\
&< {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t^+) - {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(\widehat{t_0}) - \frac{1}{4}\int_{\widehat{t_0}}^{t^+} \mathcal{G}(\upsilon)d\upsilon.
\end{aligned}
$$

Choosing

$$
\mathfrak{p} = min\{\frac{1}{4}\int_{t_-}^{\widehat{t_0}} \mathcal{G}(\upsilon)d\upsilon, \frac{1}{4}\int_{\widehat{t_0}}^{t_+} \mathcal{G}(\upsilon)d\upsilon\}. \tag{3.12}
$$

Hence, $\|\Lambda - V\|_1 \geq {}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t^+) - {}_t^C\mathcal{D}_{0^+}^{\Re-1}V(t^+) \geq \mathfrak{p}$, provided that ${}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(\widehat{t_0}) \geq {}_t^C\mathcal{D}_{0^+}^{\Re-1}V(\widehat{t_0})$.

Using t_- instead of t_+, we can get that

$$
{}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(\widehat{t_0}) \leq {}_t^C\mathcal{D}_{0^+}^{\Re-1}v(\widehat{t_0}),
$$

by similar progress. Hence, we have $\|\Lambda - V\|_1 \geq \mathfrak{p}$. The claim is proven.
By Lemma 2.7, one can see that $\Lambda \notin \mathbb{T}\Lambda$.

Case 2.2: The above $\hbar_n$ satisfies (iii) in Definition 2.10. Let $\mathbb{B}_1 = \{n : m(\mathfrak{W}_n) > 0,\ \hbar_n\ satisfies\ (iii)\ in\ Definition\ 2.10\}$.

Then, there exist $k \in \{1,\ 2,\ \cdots,\ m\}$ such that

$$
\begin{cases} {}_t^C\mathcal{D}_{0^+}^{\Re}\hbar(t) = \mathcal{E}(t)F(t,\hbar_n(t)),\ a.e.\ t \in Q'; \\ \triangle\hbar_n|_{t=t_\kappa} \neq \Phi_\kappa(\hbar_n(t_\kappa)),\ \kappa = 1,\ \cdots,\ m. \end{cases} \tag{3.13}
$$

We suppose that there exist $Y,\ \varepsilon > 0$ such that $\triangle\hbar_n|_{t=t_\kappa} + Y < \Phi_\kappa(z),\ z \in [\hbar_n(t_\kappa) - \varepsilon, \hbar_n(t_\kappa) + \varepsilon]$ by the continuity of Φ_κ.

(I) $\Lambda(t_\kappa) \neq \hbar_n(t_\kappa)$ or $\Lambda(t_\kappa^+) \neq \hbar_n(t_\kappa^+)$.

By (3.13), for $a.e.\ t \in \bigcup_{n\in\mathbb{B}_1}\mathfrak{W}_n$, we have ${}_t^C\mathcal{D}_{0^+}^{\Re}\Lambda(t) = \mathcal{E}(t)F(t,\Lambda(t))$. Similar to the proof of (I) in Case 2.1, it is easy to see that $\Lambda \notin \mathbb{T}\Lambda$ or $\Lambda = \mathcal{T}\Lambda$ if $\Lambda \in \mathbb{T}\Lambda$. Hence, $\Lambda \cap \mathbb{T}\Lambda \subset \{\mathcal{T}\Lambda\}$ for all $\Lambda \in \mathcal{P}_R$.

(II) When $\Lambda(t_\kappa) = \hbar_n(t_\kappa)$ and $\Lambda(t_\kappa^+) = \hbar_n(t_\kappa^+)$, we assert that $\Lambda \notin \mathbb{T}\Lambda$.

Claim: Let $\varepsilon > 0$ and $\mathfrak{p} = \dfrac{Y}{2}$, for every finite family $\Lambda_i \in B_\epsilon(\Lambda) \cap \mathcal{P}_R$ and $\pi_i \in [0,1](i = 1, 2, \cdots, m_1)$ with $\sum_{i=1}^{m_1} \pi_i = 1$, we have

$$\|\Lambda - \sum_{i=1}^{m_1} \pi_i \mathcal{T}\Lambda_i\| \geq \mathfrak{p}.$$

For simplicity, denote $V = \sum_{i=1}^{m_1} \pi_i \mathcal{T}\Lambda_i$. In view of $|\Lambda_i(t_\kappa) - \Lambda(t_\kappa)| = |\Lambda_i(t_\kappa) - \hbar_n(t_\kappa)| < \varepsilon_1$, one can get

$$\begin{aligned}
\triangle V|_{t=t_\kappa} &= \sum_{i=1}^{m_1} \pi_i(\triangle \mathcal{T}\Lambda_i|_{t=t_\kappa}) = \sum_{i=1}^{m_1} \pi_i(\Phi_\kappa(\Lambda_i(t_\kappa))) \\
&> \sum_{i=1}^{m_1} \pi_i(\triangle \hbar_n|_{t=t_\kappa} + Y) \\
&= \triangle \hbar_n|_{t=t_\kappa} + Y \\
&= \triangle \Lambda|_{t=t_\kappa} + Y,
\end{aligned}$$

which implies that

$$V(t_\kappa^+) - \Lambda(_\kappa^+) > V(t_\kappa) - \Lambda(t_\kappa) + \Lambda \geq -|V(t_\kappa) - \Lambda(t_\kappa)| + Y.$$

That is

$$\|\Lambda - V\|_1 \geq \frac{Y}{2}.$$

The claim is proven.

Case 2.3: The above $\hbar_n$ satisfies (iv) in Definition 2.10.

Hence, one can also obtain that

$$\Lambda \cap \mathbb{T}\Lambda \subset \{\mathcal{T}\Lambda\}, \ for\ all\ \Lambda \in \mathcal{P}_R.$$

by the process similar to proving Case 2.1 and Case 2.2.

Case 3: $m(\{\mathfrak{W}_n\}) > 0$ for $n \in \mathbb{N}$ such that $\hbar_n$ is viable.

For each $n \in \mathbb{N}$ and a.e. $t \in \mathfrak{W}_n$,

$${}^{C}_{t}\mathcal{D}^{\Re}_{0^+}\Lambda(t) = {}^{C}_{t}\mathcal{D}^{\Re}_{0^+}\hbar_n(t) = \mathcal{E}(t)F(t, \hbar_n(t)) = \mathcal{E}(t)F(t, \Lambda(t)).$$

Therefore,

$${}^{C}_{t}\mathcal{D}^{\Re}_{0^+}\Lambda(t) = \mathcal{E}(t)F(t, \Lambda(t)) \text{ a.e. in } \mathbb{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{W}_n.$$

If $\Lambda \in \mathbb{T}\Lambda$, we can obtain that

$${}^{C}_{t}\mathcal{D}^{\Re}_{0^+}\Lambda(t) = \mathcal{E}(t)F(t, \Lambda(t)) \text{ a.e. in } Q \setminus \mathbb{B},$$

by the process of proving (I) in Case 2.1. Hence, $\Lambda = \mathcal{T}\Lambda$. □

Theorem 3.3. *If (H1)–(H6) hold, then BVP (1.1) admits at least one positive solution.*

Proof. Claim 1: For all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ and $\Lambda \in P$, there exists $r_1 > 0$ such that $\widetilde{\Lambda} \nleq \Lambda$, where $\|\Lambda\| = \mathfrak{r}_1$.

In fact, the condition (H5) means that there exist $\widetilde{\varepsilon}_0$, $\mathfrak{r}_1 > 0$ such that

$$F(t,\Lambda) > (\lambda + \widetilde{\varepsilon}_0)\Lambda,\ \Phi_\kappa(\Lambda) > (\lambda + \widetilde{\varepsilon}_0)\Lambda,\ t \in [0,1],\ \Lambda \in [0, \frac{6}{5}\mathfrak{r}_1]. \tag{3.14}$$

Suppose $\Lambda \in P$ with $\|\Lambda\|_1 = \mathfrak{r}_1$. For every finite family $\Lambda_i \in B_\epsilon(\Lambda) \cap P$ and $\pi_i \in [0,1] (i = 1,\ 2,\ \cdots,\ m_2)$, with $\sum_{i=1}^{m_2} \pi_i = 1$, and $\epsilon \in [0, \frac{\mathfrak{r}_1}{5}]$, one can obtain that

$$\begin{aligned}
\widetilde{\Lambda}(t) &= \sum_{i=1}^{m_2} \pi_i \mathcal{T}\Lambda_i(t) \\
&= \sum_{i=1}^{m_2} \pi_i [\int_0^1 \mathcal{H}_1(t,\upsilon)\mathcal{E}(\upsilon)F(\upsilon, \Lambda_i(\upsilon))d\upsilon + \sum_{i=1}^{m} \mathcal{H}_2(t,t_i)\Phi_i(\Lambda_i(t_i))] \\
&> \sum_{i=1}^{m_2} \pi_i(\lambda + \widetilde{\varepsilon}_0)[\int_0^1 \mathcal{H}_1(t,\upsilon)\mathfrak{g}(\upsilon)\Lambda_i(\upsilon)d\upsilon + \sum_{i=1}^{m} \mathcal{H}_2(t,t_i)\Lambda_i(t_i)] \\
&\geq \sum_{i=1}^{m_2} \pi_i(\lambda + \widetilde{\varepsilon}_0)[\frac{\mathfrak{d}(\upsilon)\|\Lambda_i\|_1}{\mathcal{N}_2} + m\mathcal{N}_6\mathfrak{d}(\upsilon)\|\Lambda_i\|_1] \\
&\geq \mathfrak{d}(\upsilon)(\|\Lambda\|_1 - \epsilon)(\lambda + \widetilde{\varepsilon}_0)[\frac{1}{\mathcal{N}_2} + m\mathcal{N}_6] \\
&> \mathfrak{r}_1 = \|\Lambda\|_1.
\end{aligned}$$

This implies that $\widetilde{\Lambda} \nleq \Lambda$ for all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ with $\Lambda \in \mathcal{P}$ and $\|\Lambda\|_1 = \mathfrak{r}_1$. By Lemma 2.8 and 2.9, we get

$$i(\mathcal{T}, \mathcal{P} \cap \partial B_{\mathfrak{r}_1}, \mathcal{P}) = 0. \tag{3.15}$$

Claim 2: There exists $\mathfrak{R}_1 > \mathfrak{r}_1 > 0$ such that $\|\widetilde{\Lambda}\|_1 < \|\Lambda\|_1$ for all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ and all $\Lambda \in P$ with $\|\Lambda\|_1 = \mathfrak{R}_1$.

In fact, the assumption (H6) implies that there exists $0 < \varepsilon_1 < \widetilde{\lambda}$ such that

$$F(t,\Lambda) < (\widetilde{\lambda} - \varepsilon_1)\Lambda,\ \Phi_\kappa(\Lambda) < (\widetilde{\lambda} - \varepsilon_1)\Lambda,\ t \in [0,1],\ \Lambda \geq \frac{4}{5}\mathcal{R}_1.$$

Choosing $\mathfrak{R}_1 > max\{\mathfrak{r}_1, \frac{4\mathcal{R}_1}{5\mathfrak{d}(\upsilon)}\}$, for $\Lambda \in \partial\mathcal{P}_{\mathfrak{R}_1}$, one can see that

$$\Lambda(t) \geq \mathfrak{d}(\upsilon)\|\Lambda\|_1 = \mathfrak{d}(\upsilon)\mathfrak{R}_1 > \frac{4}{5}\mathcal{R}.$$

Suppose $\Lambda \in P$ with $\|\Lambda\|_1 = \mathfrak{R}_1$. For $\pi_i \in [0,1] (i = 1,\ 2,\ \cdots,\ m_3)$, with $\sum_{i=1}^{m_3} \pi_i = 1$ and every finite family $\Lambda_i \in B_\epsilon(\Lambda) \cap P$, $\epsilon \in [0, \frac{\mathfrak{r}_1}{5}]$, one can see that

$$\widetilde{\Lambda}(t) = \sum_{i=1}^{m_3} \pi_i \mathcal{T}\Lambda_i(t)$$

$$
\begin{aligned}
&= \sum_{i=1}^{m_3}\pi_i[\int_0^1 \mathcal{H}_1(t,\upsilon)\mathcal{E}(\upsilon)F(\upsilon,\Lambda_i(\upsilon))d\upsilon + \sum_{i=1}^{m}\mathcal{H}_2(t,t_i)\Phi_i(\Lambda_i(t_i))]\\
&< \sum_{i=1}^{m_3}\pi_i[\int_0^1 \mathcal{H}_1(t,\upsilon)\mathfrak{g}(\upsilon)(\widetilde{\lambda}-\varepsilon_1)\Lambda_i(\upsilon)d\upsilon + \sum_{i=1}^{m_3}\mathcal{H}_2(t,t_i)](\widetilde{\lambda}-\varepsilon_1)\Lambda_i(t_i)\\
&\leq (\Re_1+\epsilon)(\widetilde{\lambda}-\varepsilon_1)[\frac{1}{\mathcal{N}_1}+m\mathcal{N}_5]\\
&< \Re_1 = \|\Lambda\|_1,
\end{aligned}
$$

and

$$
\begin{aligned}
{}_t^C\mathcal{D}_{0^+}^{\Re-1}\widetilde{\Lambda}(t) &= \sum_{i=1}^{m_3}\pi_i({}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{T}\Lambda_i)(t)\\
&= \sum_{i=1}^{m_3}\pi_i[\int_0^1 {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t,\upsilon)\mathcal{E}(\upsilon)F(\upsilon,\Lambda_i(\upsilon))d\upsilon\\
&\quad + \sum_{i=1}^{m} {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t,t_i)\Phi_i(\Lambda_i(t_i))]\\
&< \|\Lambda_i\|_1(\widetilde{\lambda}-\varepsilon_1)[\frac{1}{\mathcal{N}_3}+m\mathcal{N}_7]\\
&\leq (\Re_1+\epsilon)(\widetilde{\lambda}-\varepsilon_1)[\frac{1}{\mathcal{N}_3}+m\mathcal{N}_7]\\
&< \Re_1 = \|\Lambda\|_1.
\end{aligned}
$$

Hence, $\|\widetilde{\Lambda}\|_1 < \|\Lambda\|_1$, for all $\widetilde{\Lambda} \in \mathbb{T}\Lambda$ and all $\Lambda \in P$ with $\|\Lambda\|_1 = \Re_1$. By Lemma 2.8 and 2.9, we get

$$i(\mathcal{T},\mathcal{P}\cap\partial B_{\Re_1},\mathcal{P}) = 1. \tag{3.16}$$

Together with (3.15), we have

$$i(\mathcal{T},\mathcal{P}\cap(B_{\Re_1}\backslash\overline{B}_{\mathfrak{r}_1}),\mathcal{P}) = 1-0 = 1. \tag{3.17}$$

Hence, BVP (1.1) admits at least one positive solution. □

Theorem 3.4. *Assume that (H1)–(H4), (H7) and (H8) hold. In addition, suppose that the following condition is satisfied.*
($H9$) *There exist* $R > 0$ *such that* $F^R < \frac{\mathcal{N}_1}{2}$ *and* $\sum_{k=1}^{m}\Phi_\kappa^R < \frac{1}{2\mathcal{N}_5}$, *where*

$$\Phi_\kappa^R := sup_{\,0\leq\|\Lambda\|\leq\frac{6R}{5}}\{\frac{\Phi_\kappa(\Lambda)}{R}\},\ F^R := sup_{t\in[0,1],\ 0\leq\|\Lambda\|\leq\frac{6R}{5}}\{\frac{F(t,\Lambda)}{R}\}.$$

Then, BVP (1.1) admits at least two positive solutions.

Proof. We will prove that $\mathcal{T}$ has at least two positive fixed points.

First, by the condition (H7), one can see that there exist $\mathfrak{r}_2, \widetilde{\varepsilon}_2 \in (0, \nu)$. Moreover, $\mathfrak{r}_2, \widetilde{\varepsilon}_2$ satisfy

$$F(t, \Lambda) < (\nu - \widetilde{\varepsilon}_2)\Lambda,\ \Phi_\kappa(\Lambda) < (\nu - \widetilde{\varepsilon}_2)\Lambda,\ t \in [0, 1],\ \Lambda \in [0, \frac{6}{5}\mathfrak{r}_2].$$

We claim that

$$\mu\Lambda \notin \mathbb{T}\Lambda,\ \forall \Lambda \in \mathcal{P} \cap \partial B_{\mathfrak{r}_2}, \tag{3.18}$$

for $\mu \geq 1$. In fact, on the contrary, if there exist $\Lambda \in \mathcal{P} \cap \partial B_{\mathfrak{r}_2}$, $\mu \geq 1$ such that $\mu\Lambda(t) = \mathcal{T}\widetilde{\Lambda}(t)$ for some $\widetilde{\Lambda} \in \overline{B}_\epsilon(\Lambda) \cap P,\ i.e.,$

$$\begin{aligned}
\mu\Lambda(t) &= \int_0^1 \mathcal{H}_1(t, \upsilon)\mathcal{E}(\upsilon)F(\upsilon, \widetilde{\Lambda}(\upsilon))d\upsilon + \sum_{i=1}^{m} \mathcal{H}_2(t, t_i)\Phi_i(\widetilde{\Lambda}(t_i)) \\
&< (\nu - \widetilde{\varepsilon}_2)(\|\Lambda\|_1 + \epsilon)[(\frac{1}{\mathcal{N}_1} + m\mathcal{N}_5] \\
&< \mathfrak{r}_2.
\end{aligned}$$

Then,

$$\begin{aligned}
\mu({}_t^C\mathcal{D}_{0^+}^{\Re-1}\Lambda(t)) &= \int_0^1 {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_1(t, \upsilon)\mathcal{E}(\upsilon)F(\upsilon, \widetilde{\Lambda}(\upsilon))d\upsilon + \sum_{i=1}^{m} {}_t^C\mathcal{D}_{0^+}^{\Re-1}\mathcal{H}_2(t, t_i)\Phi_i(\widetilde{\Lambda}(t_i)) \\
&< (\nu - \widetilde{\varepsilon}_2)(\|\Lambda\|_1 + \epsilon)[\frac{1}{\mathcal{N}_3} + m\mathcal{N}_7] \\
&\leq (\nu - \widetilde{\varepsilon}_2)(\|\Lambda\|_1 + \epsilon)[(\frac{1}{\mathcal{N}_1} + m\mathcal{N}_5] \\
&< \mathfrak{r}_2.
\end{aligned}$$

Over $t \in [0, 1]$, we obtain

$$\mu\|\Lambda\|_1 = \mu\mathfrak{r}_2 < \mathfrak{r}_2, \tag{3.19}$$

by taking the supremum, which is a contradiction.

Then, to prove $\mu\Lambda \notin co(T(B_{\widetilde{\varepsilon}}(\Lambda) \cap \mathcal{P}))$, we consider two cases: $\mu = 1$ and $\mu > 1$. If $\mu = 1$, we obtain by the reasonings done above that $\Lambda \neq \mathcal{T}\Lambda$. This together with condition $\Lambda \cap \mathbb{T}\Lambda \subset \{\mathcal{T}\Lambda\}$ implies $\Lambda \notin \mathbb{T}\Lambda$. If $\mu > 1$, by inequality (3.19), it is a contradiction.

Next, the condition (H8) means that there exist $\widetilde{\varepsilon}_3 > 0, \mathcal{R} > r_2$. They satisfy

$$F(t, \Lambda) > (\widetilde{\nu} + \widetilde{\varepsilon}_3)\Lambda,\ \Phi_\kappa(\Lambda) > (\widetilde{\nu} + \widetilde{\varepsilon}_3)\Lambda,\ t \in [0, 1],\ \Lambda \geq \frac{4}{5}\mathcal{R}.$$

Choosing $\Re_2 > max\{\mathfrak{r}_1, \dfrac{4\mathcal{R}}{5\mathfrak{d}(s)}\}$, for any $\Lambda \in \partial\mathcal{P}_{\Re_2}$, we have

$$\Lambda(t) \geq \mathfrak{d}\|\Lambda\|_1 = \mathfrak{d}(s)\Re_2 > \frac{4}{5}\mathcal{R}.$$

We claim that

$$\Lambda \notin \mathbb{T}\Lambda + \mu e,\ e(t) \equiv 1,\ t \in [0, 1],$$

for all $\Lambda \in \mathcal{P} \cap \partial B_{\Re_2}$ and $\mu \geq 0$.

In fact, on the contrary, suppose that there exist $\Lambda \in P \cap \partial B_{\Re_2}$, $\mu \geq 0$ such that $\Lambda = \mathcal{T}\widetilde{\Lambda} + \mu e$ for some $\widetilde{\Lambda} \in \overline{B}_{\epsilon}(\Lambda) \cap \mathcal{P}$, *i.e.*,

$$\begin{aligned}\Lambda(t) &= \int_0^1 \mathcal{H}_1(t,\upsilon)\mathcal{E}(\upsilon)F(\upsilon,\widetilde{\Lambda}(\upsilon))d\upsilon + \sum_{i=1}^{m}\mathcal{H}_2(t,t_i)\Phi_i(\widetilde{\Lambda}(t_i)) + \mu \\ &\geq (\Re_2 - \epsilon)\eth(\upsilon)(\widetilde{\nu} + \widetilde{\varepsilon}_3)[\frac{1}{\mathcal{N}_2} + m\mathcal{N}_6] + \mu \\ &> \Re_2 + \mu.\end{aligned}$$

This together with the definition of $\|\cdot\|_1$ guarantees that

$$\Re_2 = \|\Lambda\|_1 \geq max_{t\in\,[0,1]}\Lambda(t) > \Re_2 + \mu, \tag{3.20}$$

which is a contradiction for $\mu \geq 0$.

For $p \in \mathbb{N}$, one can see that $\Lambda \neq \sum\limits_{i=1}^{p}\pi_i T\widetilde{\Lambda}_i + \mu e(\mu \geq 0)$ for $\pi_i \in [0,1](i = 1, \cdots, p)$ and $v_i \in B_{\widetilde{\varepsilon}}(\Lambda) \cap P$, where $\sum\limits_{i=1}^{p}\pi_i = 1$. Hence, $\Lambda \notin co(\mathcal{T}(B_{\varepsilon}(\Lambda) \cap \mathcal{P})) + \mu e(\mu \geq 0)$.

Now we are in a position to prove that $\Lambda \notin \mathbb{T}\Lambda + \mu e$. If $\mu = 0$, we obtain by the reasonings done above that $\Lambda \neq \mathcal{T}\Lambda$.This together with condition $\Lambda \cap \mathbb{T}\Lambda \subset \{\mathcal{T}\Lambda\}$ implies $\Lambda \notin \mathbb{T}\Lambda$. If $\mu > 0$, in view of inequality (3.20), it is a contradiction.

By Lemma 2.8, one can get that $i(\mathcal{T},\ \mathcal{P} \cap \partial B_{\mathfrak{r}_2},\ \mathcal{P}) = 1$ and $i(\mathcal{T},\ \mathcal{P} \cap \partial B_{\Re_2},\ \mathcal{P}) = 0$. Hence,

$$i(\mathcal{T},\ \mathcal{P} \cap (B_{\Re_2} \backslash \overline{B}_{\mathfrak{r}_2},\ \mathcal{P}) = 0 - 1 = -1. \tag{3.21}$$

Third, (H9) implies that there exist $\Re_3 > \Re_2$ and $\epsilon \in [0, \frac{\mathfrak{r}_2}{5}]$ such that $F^{\Re_3} < \frac{\mathcal{N}_1}{2}$ and $\sum\limits_{k=1}^{m}\Phi_{\kappa}^{\Re_3} < \frac{1}{2\mathcal{N}_5}$.

Similar to the process above, there exist $\Re_3 > \Re_2$ such that

$$i(\mathcal{T},\ \mathcal{P} \cap \partial B_{\Re_3}, \mathcal{P}) = 1.$$

Hence,

$$i(\mathcal{T},\ \mathcal{P} \cap (B_{R_3} \backslash \overline{B}_{\Re_2}, \mathcal{P}) = 1 - 0 = 1.$$

Together with (3.21), BVP (1.1) admits at least two positive solutions in $\mathcal{P} \cap (B_{\overline{\Re}_2} \backslash \overline{B}_{\mathfrak{r}_2})$ and $\mathcal{P} \cap (B_{\overline{\Re}_3} \backslash \overline{B}_{\Re_2})$, respectively. □

4. Example

Example 4.1. Consider the following BVP

$$\begin{cases} {}^{C}_{t}\mathcal{D}^{1.5}_{0^+}\Lambda(t) = F(t,\Lambda),\ a.e.\ t \in [0,1], \\ \triangle\Lambda|_{t=t_1} = \Phi_1(\Lambda(t_1)), \\ \triangle\Lambda'|_{t=t_1} = 0, \\ 3\Lambda(0) - \Lambda(1) = \int_0^1 \frac{1}{2}\Lambda(\upsilon)d\upsilon, \\ 3\Lambda'(0) - \Lambda'(1) = \int_0^1 \Lambda(\upsilon)d\upsilon, \end{cases} \tag{4.1}$$

where $0 < t_1 < 1$, $\Phi_1(\Lambda) = \dfrac{\Lambda^2}{10^3}$ and

$$F(t,\Lambda) = \begin{cases} \dfrac{[\Gamma(2.5)]^2}{4}\dfrac{\Lambda^2}{10^3}[\cos^2(\dfrac{\Gamma(2.5)}{2t^{1.5}-\Gamma(2.5)\Lambda})+1], \ \Lambda \neq \dfrac{2t^{1.5}}{\Gamma(2.5)}, \ 0 \le t \le 1; \\ \dfrac{t^3}{500}, \ \Lambda = \dfrac{2t^{1.5}}{\Gamma(2.5)}, \ 0 \le t \le 1. \end{cases}$$

Conclusion: BVP (4.1) has at least two positive solutions.

Proof. First, F satisfies condition (H2) by it's expression. On the other hand, the function $\Lambda \to F(t,\Lambda)$ is continuous on

$$\mathbb{R}^+ \setminus \bigcup_{t\in Q}\{\hbar_n(t)\},$$

where for each $n \in \mathbb{Z}\setminus\{0\}$ and a.e. $t \in Q$. The curves $\hbar_n(t) = \dfrac{2t^{1.5}}{\Gamma(2.5)} - n^{-1}$ and $\hbar_0(t) = \dfrac{2t^{1.5}}{\Gamma(2.5)}$ are admissible discontinuity curves satisfying

$$1 = {}_t^C\mathcal{D}_{0^+}^{1.5}\hbar_n(t) - 1 > F(t,z)$$

where $z \in [\hbar_n(t)-1, \hbar_n(t)+1]$, $t \in [0,1]$.

By Lemma 2.3, one can obtain that $\mathfrak{A}_1 = \dfrac{1}{2}$, $\mathfrak{A}_2 = 1$, $\mathfrak{P}_1 = \mathfrak{Q}_1 = \dfrac{1}{4}$, $\mathfrak{P}_2 = \mathfrak{Q}_2 = \dfrac{1}{2}$, $\Gamma_1 = \dfrac{1}{8} > 0$, $\varphi_1(t) = 2t+2$, $\varphi_2(t) = 3t + \dfrac{5}{2}$,

$$\aleph(t,\upsilon) = \begin{cases} \dfrac{(t-\upsilon)^{0.5}}{\Gamma(1.5)} + \dfrac{(1-\upsilon)^{0.5}}{2\Gamma(1.5)} + \dfrac{(1+2t)(1-\upsilon)^{-0.5}}{4\Gamma(0.5)}, \ 0 \le \upsilon \le t \le 1; \\ \dfrac{(1-\upsilon)^{0.5}}{2\Gamma(1.5)} + \dfrac{(1+2t)(1-\upsilon)^{-0.5}}{4\Gamma(0.5)}, \ 0 \le t \le \upsilon \le 1, \end{cases}$$

$$\mathcal{H}_2(t,t_i) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2}(4t+\dfrac{7}{2}), \ 0 \le t \le t_i \le 1; \\ \dfrac{3}{2} + \dfrac{3}{2}(4t+\dfrac{7}{2}), \ 0 \le t_i < t \le 1. \end{cases} \tag{4.2}$$

Thus, by calculation, we can get that $(\mathcal{N}_1)^{-1} \approx 10.458$, $(\mathcal{N}_2)^{-1} \approx 4.375$, $(\mathcal{N}_3)^{-1} \approx 5.333$, $(\mathcal{N}_4)^{-1} \approx 4.333$, $\mathcal{N}_5 = \dfrac{51}{4}$, $\mathcal{N}_6 = \dfrac{9}{4}$, $\mathcal{N}_7 = 6$, $\mathcal{N}_8 = 2$. Choosing $\nu = 0.03$ and $\widetilde{\nu} = 2$, which satisfies $5\nu(\dfrac{1}{\mathcal{N}_1} + mN_5) \le 4$ and $3\mathfrak{d}\widetilde{\nu}(\dfrac{1}{\mathcal{N}_2} + mN_6) \ge 4$.

Therefore,

$$\lim_{\Lambda\to 0^+}\frac{\Phi_\kappa(\Lambda)}{\Lambda} = 0 < \nu, \ \lim_{\Lambda\to 0^+}\sup_{t\in[0,1]}\frac{F(t,\Lambda)}{\Lambda} = 0 < \nu.$$

$$\lim_{\Lambda\to +\infty}\frac{\Phi_\kappa(\Lambda)}{\Lambda} = +\infty > \widetilde{\nu}, \ \lim_{\Lambda\to +\infty}\inf_{t\in[0,1]}\frac{F(t,\Lambda)}{\Lambda} = +\infty > \widetilde{\nu}.$$

Moreover, we have $(\mathcal{N}_1)^{-1} \approx 10.458$, $\mathcal{N}_5 = \dfrac{51}{4}$ and let $R_3 = 10$. Then, (H9) is satisfied.

Hence, all conditions in Theorem 3.4 are satisfied. The proof is completed.

5. Conclusions

In this work, we studies the existence of positive and multiple positive solutions for a class of BVPs of fractional discontinuous differential equations with impulse effects. The main results are obtained by means of the multivalued analysis and Krasnoselskii's fixed point theorem for discontinuous operators on cones.

For our subsequent work, the following issues will continue to be focused on:
(i) The system is studied on this topic more extensive and complicated. Therefore, it is valuable to investigate FDEs with generalized derivatives or hybrid FDEs with delay.
(ii) With the development of the theoretical study on FDEs, the application area of FDEs with generalized derivatives in reality needs to be investigated in depth.

Acknowledgments

The authors are thankful to the editor and anonymous referees for their valuable comments and suggestions. This research was funded by NNSF of P.R. China (12271310), Natural Science Foundation of Shandong Province (ZR2020MA007), and Doctoral Research Funds of Shandong Management University(SDMUD202010), QiHang Research Project Funds of Shandong Management University(QH2020Z02).

Conflict of interest

The authors declare that there are no conflicts of interest.

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