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Research article

Function space properties of the Cauchy transform on the Sierpinski gasket

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Abstract: Let $S_j(z) = \varepsilon_j + (z - \varepsilon_j)/2$ be an iterated function system, where $\varepsilon_j = e^{2j\pi i/3}$ for $j = 0, 1, 2$. Then, there exists a uniform self-similar measure μ supported on a compact set K, which is the attractor of $\{S_j\}_{j=0}^{2}$. The Hausdorff dimension of the attractor K is $\alpha = \log 3/\log 2$. Let $F(z) = \int_K (z-\omega)^{-1} d\mu(\omega)$ be the Cauchy transform of μ. In this paper, we consider the Hardy space and the multiplier property of F. We prove that F' belongs to H^p for $0 < p < 1/(2 - \alpha)$ and that F is a multiplier of some class of function space.

Keywords: Cauchy transform; Sierpinski gasket; self-similar measure; Hardy space; multiplier
Mathematics Subject Classification: 28A80, 30C55, 30E20

1. Introduction

The Cauchy transform of a measure in the plane is a useful tool for geometric measure theory [1–3], and it has also important applications in solving integral equations [4,5]. If the measure is a self-similar measure, the Cauchy transform of it has very rich fractal behavior. Stricharz et. al. [6] initiated the study of the Cauchy transform $F(z) = \int_K (z - \omega)^{-1} d\mu(\omega)$ of a self-similar measure μ with compact support K, and they proved that F has a Holder continuous extension over K and showed how to compute the Laurent expansion of F in the complement of a disk containing K. Soon afterwards, more analytic and geometric properties of F were given by Dong and Lau [7–12]: for example, the asymptotic behavior of the Laurent coefficients of F and the region of starlikeness of F. They also gave estimates for the Taylor coefficients of the Cauchy transforms of some special Hausdorff measures [13,14]. For the special case that K is the Sierpinski gasket, and μ is the normalized Hausdorff measure on K, Dong and Lau [8–11] carried out a detailed study of the properties of the mapping of the Cauchy transform on K and investigated some open problems proposed in [6]. Away from K, F is well-behaved, but the image of F is chaotic near the boundary of

K and is difficult to catch [see 6,7,12]. In this paper, we will consider the properties of the function spaces of $F(z)$ near the Sierpinski gasket.

Let $S_j(z) = \varepsilon_j + (z - \varepsilon_j)/2$ be an iterated function system, where $\varepsilon_j = e^{2j\pi i/3}$ for $j = 0, 1, 2$. The attractor K of $\{S_j\}_{j=0}^{2}$ is just the Sierpinski gasket (Figure 1). It is well known that K is a compact set, $\mathbb{C} \setminus K$ is a multiply connected domain, and the Hausdorff dimension of K is $\alpha = \log 3/\log 2$. We denote unbounded connected region of $\mathbb{C} \setminus K$ by $\triangle_0$ and the triangular connected region of $\mathbb{C} \setminus K$ by $\triangle_n (n \geq 1)$. Then, $\mathbb{C} \setminus K = \cup_{n=0}^{\infty} \triangle_n$.

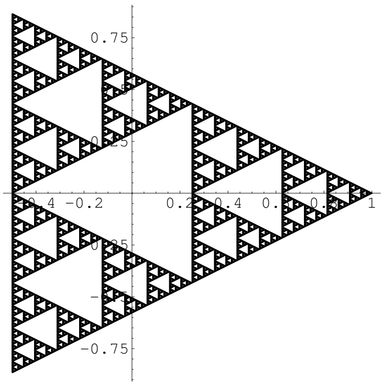

Figure 1. Sierpinski gasket.

Let μ be the uniform self-similar measure on K, i.e., μ is the restriction of the α -Hausdorff measure on K normalized to a probability measure. With slight abusing of notation, we let $\mathcal{H}^{\alpha}$ be the Hausdorff measure normalized on K. From the basic property of the Hausdorff measure [15], for $E \subset \mathbb{C}$, we have $\mathcal{H}^{\alpha}(\phi(E)) = \mathcal{H}^{\alpha}(E)$, where ϕ can be the complex conjugation or the rotation of $e^{i\theta}$. Also, for any $n \in \mathbb{Z}$, $\mathcal{H}^{\alpha}(2^n E) = 2^{\alpha n}\mathcal{H}(E)$. The Cauchy transform of $\mu = \mathcal{H}^{\alpha}|_K$ is

$$F(z) = \int_K \frac{d\mathcal{H}^{\alpha}(w)}{z - \omega}. \tag{1.1}$$

Our main consideration is on the dyadic points of $\partial\triangle_0$. With fixed k, for $1 \leq m \leq 2^k - 1$, let

$$z_{k,m} = \frac{m}{2^k}\varepsilon_1 + (1 - \frac{m}{2^k})\varepsilon_2 = -\frac{1}{2} + \frac{m - 2^{k-1}}{2^k}\sqrt{3}i.$$

These are the dyadic points on the line segment joining the two vertices ε_1 and ε_2. The dyadic points on the other two sides of $\partial\triangle_0$ can be obtained by $z_{k,m}$ multiplied by ε_j, $j = 1, 2$. It suffices to consider $z_{k,m}$ since $\varepsilon_j F(\varepsilon_j z) = F(z),\ j = 0, 1, 2$.

The paper is organized as follows. In Section 2, we introduce some necessary results and notations. In Section 3, we give an H^p space property of $F(1/z)$ on $|z| < 1$. In the final section, we study the multiplier property of $F(1/z)$ on $|z| < 1$.

2. Preliminaries

In this section, we first give some necessary notations and propositions firstly. Let $T = e^{\pi i}(K - 1)$ be a relocation of the Sierpinski gasket K. The new vertices are at 0, $\sqrt{3}e^{\pi i/6}$, $\sqrt{3}e^{-\pi i/6}$. Set $S_j K =$

K_j, $j = 0, 1, 2$. Let $T_j = e^{\pi i}(K_j - 1)$, $j = 0, 1, 2$, denote the three triangular components of T containing the respective vertices. We define the "Sierpinski cones" of T (Figure 2) as $A_0 = \bigcup_{n\in\mathbb{Z}} 2^n(T_1 \cup T_2)$. For $\ell = 1, \cdots, 5$, let $A_\ell = e^{\ell\pi i/3} A_0$, and

$$H_\ell(z) = \int_{A_\ell} \frac{d\mathcal{H}^\alpha(\omega)}{(z-\omega)^2}.$$

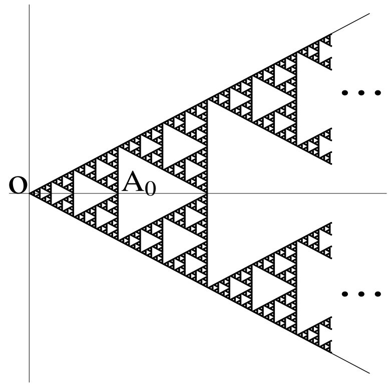

Figure 2. Sierpinski cones.

It is easy to check that $H_\ell(2z) = 2^{\alpha-2} H_\ell(z)$ by the scaling property of Hausdorff measure. In the sequel, we need the following propositions.

Proposition 2.1. *[9] There exists some constant $C > 0$ such that ,*

$$\max_{\mathrm{dist}(z,K)\geq t} |F'(z)| \leq C t^{\alpha-2}, \ t > 0.$$

Proposition 2.2. *[9] For $0 < \rho < 1$, there exists some constant $C > 0$ which depends on ρ such that for $|\arg z| < 5\pi/6$ and $0 < |z| \leq \rho\sqrt{3}$,*

$$|F'(1+z) + H_3(z)| \leq C.$$

For the details of the proof of the above two propositions, we can see [10].

3. H^p property of $F(\frac{1}{z})$ on $|z| < 1$

In this section, we consider the function space property of $F'(\frac{1}{z})$ on $\mathbb{D} = \{z : |z| < 1\}$. The Hardy space H^p consists of analytic functions f in $\mathbb{D}$ such that

$$\| f \|_p = \sup_{0\leq r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p} < +\infty.$$

Theorem 3.1. *Let $g(z) = F(\frac{1}{z})$ for $z \in \mathbb{D}$. Then, $g'(z) \in H^p$ for $0 < p < \frac{1}{2-\alpha}$ and $g' \notin H^p$ for $p \geq \frac{1}{2-\alpha}$, where α is the Hausdorff dimension of K.*

Remark Similarly, we may prove that $g^{(k)}(z) \in H^p$ for $0 < p < \frac{1}{k+1-\alpha}$ and $g^{(k)}(z) \notin H^p$ for $p \geq \frac{1}{k+1-\alpha}$.

Proof. Note that $g(z)$ is analytic in $\mathbb{D}$, and $g'(e^{i\theta})$ exists for $\theta \notin \{0, 2\pi/3, 4\pi/3\}$. By Theorem 2.6 in [16, p. 21], we only need to prove $g'(e^{i\theta}) \in L^p$ for $0 < p < 1/(2-\alpha)$, and $g'(e^{i\theta}) \notin L^p$ for $p \geq 1/(2-\alpha)$.

For $-\pi/3 \leq \theta < 0$, let $z = e^{i\theta}$ and $z^* = \rho e^{-i\theta} \in \partial\triangle_0$, where $\rho > 0$. By the sine rule, we have

$$\begin{aligned} \operatorname{dist}(e^{-i\theta}, K) &= \sin(\frac{\pi}{6} - \theta)|e^{-i\theta} - z^*| \\ &= -\sin\frac{\theta}{2}(\sqrt{3}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}) \\ &\geq |\sin\frac{\theta}{2}|. \end{aligned} \tag{3.1}$$

From Proposition 2.1, there exists some constant $C > 0$ such that

$$|g'(z)| \leq C \operatorname{dist}(e^{-i\theta}, K)^{\alpha-2} \leq C|\theta|^{\alpha-2}, \ -\frac{\pi}{3} \leq \theta < 0.$$

Notice that $\mathcal{H}^\alpha$ and K are symmetric with respect to the real-axis. Then, $g'(\bar{z}) = \overline{g'(z)}$, and $\int_{-\pi/3}^{\pi/3} |g'(e^{i\theta})|^p d\theta = 2\int_{-\pi/3}^{0} |g'(e^{i\theta})|^p d\theta$. Hence, for $0 < p < 1/(2-\alpha)$,

$$\int_{-\pi}^{\pi} |g'(e^{i\theta})|^p d\theta = 6\int_{-\pi/3}^{0} |g'(e^{i\theta})|^p d\theta \leq C\int_0^{\pi/3} \theta^{p(\alpha-2)} d\theta < +\infty.$$

The above inequality gives $g'(e^{i\theta}) \in L^p$ for $0 < p < 1/(2-\alpha)$.

Next, we will prove $g'(e^{i\theta}) \notin L^{\frac{1}{2-\alpha}}$. For $0 < t \leq \sqrt{3}/2$ and $|\theta| < 5\pi/6$, from Proposition 2.2, we obtain

$$|F'(1 + te^{i\theta}) + 2^{(2-\alpha)N} H_3(2^N te^{i\theta})| \leq C_1, \tag{3.2}$$

where the positive integer N satisfies $1/2 \leq 2^N t < 1$. For $0 < t < 1$, let $1 + te^{i\theta} = e^{i\varphi}$. Then,

$$\varphi = \varphi(t) = \arctan\frac{t\sqrt{1 - t^2/4}}{1 - t^2/2} \quad \textit{and} \quad \theta = \theta(t) = \frac{\pi}{2} + \arcsin\frac{t}{2}. \tag{3.3}$$

Since $F'(e^{i\varphi}) = -e^{-2i\varphi} g'(e^{-i\varphi})$ and $H_3(2z) = 2^{\alpha-2} H_3(z)$, we have

$$|e^{-2i\varphi} g'(e^{-i\varphi}) - 2^{(2-\alpha)N} H_3(2^N te^{i\theta})| \leq C_1 \tag{3.4}$$

by using (3.2). Define $\beta = \arcsin(t/2)$ and $b = b(t) := 2^N t$. Noting that $b = b(t) \in [\frac{1}{2}, 1)$ and $i(e^{i\beta} - 1) = -2\sin(\beta/2)e^{i\beta/2}$, we see that

$$\begin{aligned} H_3(bie^{i\beta}) &= \int_{A_3} \frac{d\mathcal{H}^\alpha(w)}{(bi - w + bi(e^{i\beta} - 1))^2} \\ &= \int_{A_3} \frac{d\mathcal{H}^\alpha(w)}{(bi - w)^2} + \sum_{k=1}^{\infty}(k+1)\int_{A_3} \frac{(2b\sin(\frac{\beta}{2}))^k e^{\frac{k\beta i}{2}} d\mathcal{H}^\alpha(w)}{(bi - w)^{k+2}} \\ &:= H_3(bi) + \varepsilon(t). \end{aligned} \tag{3.5}$$

To estimate $\varepsilon(t)$, we set $E_1 = -T_1$, $E_2 = -T_2$. Then,

$$\begin{aligned}
|\varepsilon(t)| &\leq \sum_{k=1}^{\infty}(k+1)\int_{A_3}\frac{(2b\sin(\beta/2))^k d\mathcal{H}^\alpha(w)}{|bi-w\,|^{k+2}}\\
&= \sum_{k=1}^{\infty}(k+1)\sum_{n=-\infty}^{\infty}3^n\int_{E_1\cup E_2}\frac{(2b\sin(\beta/2))^k d\mathcal{H}^\alpha(w)}{|bi-2^n w\,|^{k+2}}\\
&\leq \sum_{k=1}^{\infty}(k+1)\sum_{n=0}^{\infty}(\frac{3}{8})^n\int_{E_1\cup E_2}\frac{(2b\sin(\beta/2))^k d\mathcal{H}^\alpha(w)}{|2^{-n}bi-w\,|^{k+2}}\\
&\quad+\sum_{k=1}^{\infty}(k+1)\sum_{n=1}^{\infty}(\frac{1}{3})^n\int_{E_1\cup E_2}\frac{(2b\sin(\beta/2))^k d\mathcal{H}^\alpha(w)}{|bi-2^{-n}w\,|^{k+2}}.
\end{aligned}$$

With consideration of geometric factors, for $b\in[1/2,1)$, $n\geq 1$ and $w\in E_1\cup E_2$, the two inequalities $|w-2^{-n}bi\,|\geq 3/4$ and $|bi-2^{-n}w\,|\geq \sqrt{3}b/2$ hold. Hence,

$$|\varepsilon(t)|\leq \frac{16}{15}\sum_{k=1}^{\infty}(k+1)\Big(\frac{4}{3}\Big)^{k+2}\Big(2b\sin(\frac{\beta}{2})\Big)^k+\frac{1}{3}\sum_{k=1}^{\infty}(k+1)\Big(\frac{2\sqrt{3}}{3b}\Big)^{k+2}\Big(2b\sin(\frac{\beta}{2})\Big)^k.$$

By $\sin\beta = t/2$, it is easy to check that $\sin(\beta/2) = \sqrt{1-\sqrt{1-t^2/4}}\Big/\sqrt{2} < t/3$ for small $t>0$. This shows that we can find constants $C_2>0$ and $\delta>0$ such that

$$|\varepsilon(t)|\leq C_2 t,\quad 0<t\leq\delta. \tag{3.6}$$

From (3.4)–(3.6), we know that

$$\begin{aligned}
|g'(e^{-i\varphi})| &\geq 2^{(2-\alpha)N}(|H_3(bi)|-C_2t)-C_1\\
&\geq 2^{(2-\alpha)N}|H_3(bi)|-C,
\end{aligned}$$

where C is a positive constant. This implies that, for $0<t\leq\delta$, we have

$$(C+|g'(e^{-i\varphi})|)^{\frac{1}{2-\alpha}}\geq 2^N|H_3(bi)|^{\frac{1}{2-\alpha}}. \tag{3.7}$$

Let the positive integer N_0 satisfy $2^{-N}\leq\delta$ for all $N\geq N_0$. Note that $\varphi'(t)\geq c_1>0$ for $0<t\leq\delta$. According to (3.7), we obtain

$$\begin{aligned}
\int_{\varphi(2^{-N-1})}^{\varphi(2^{-N})}(C+|g'(e^{-i\varphi})|)^{\frac{1}{2-\alpha}}d\varphi &\geq 2^N\int_{\varphi(2^{-N-1})}^{\varphi(2^{-N})}|H_3(bi)|^{\frac{1}{2-\alpha}}d\varphi\\
&\geq c_1 2^N\int_{2^{-N-1}}^{2^{-N}}|H_3(2^N ti)|^{\frac{1}{2-\alpha}}dt\\
&= c_1\int_{1/2}^{1}|H_3(xi)|^{\frac{1}{2-\alpha}}dx\\
&:= c_2.
\end{aligned}$$

We can check that $H_3(z)$ is non-constant analytic in $|\arg z| < 5\pi/6$. This gives $c_2 > 0$. Noting that $\varphi(2^{-N-1}) \to 0^+$ as $N \to \infty$, we have

$$\int_0^{\varphi(2^{-N_0})} (C + |g'(e^{-i\varphi})|)^{\frac{1}{2-\alpha}} d\varphi = \sum_{N=N_0}^{+\infty} \int_{\varphi(2^{-N-1})}^{\varphi(2^{-N})} (C + |g'(e^{-i\varphi})|)^{\frac{1}{2-\alpha}} d\varphi = +\infty.$$

By using $(a+b)^p \le 2^p(a^p + b^p)$ for $a > 0, b > 0$ and $p > 0$, we have

$$\int_0^{\varphi(2^{-N_0})} |g'(e^{-i\varphi})|^{\frac{1}{2-\alpha}} d\varphi = +\infty,$$

which implies that $g'(z) \notin H^p$ for $p \ge 1/(2-\alpha)$.

□

4. Multiplier property of $F(\frac{1}{z})$ on $|z| < 1$

In this section , we consider the multiplier property of $g(z)$. Let Λ denote the set of complex-valued Borel measures on $\mathbb{T} = \{z : |z| = 1\}$, let $k_\lambda(z) = (1-z)^{-\lambda}$ for $\lambda > 0$, and $k_\lambda(z) = \log \frac{1}{1-z} + 1$ for $\lambda = 0$. Here, we choose the branch of $k_\lambda(z)$ which equals 1 when $z = 0$. Let $\mathfrak{F}_\lambda$ denote the family of functions h for which there exists $\mu \in \Lambda$ such that

$$h(z) = \int_{\mathbb{T}} k_\lambda(\zeta z) d\mu(\zeta), \ |z| < 1. \tag{4.1}$$

Each $\mathfrak{F}_\lambda$ is a Banach space with respect to the norm defined by

$$\| h \|_{\mathfrak{F}_\lambda} = \inf\{\| \mu \| : \mu \in \Lambda \text{ such that (4.1) holds}\},$$

where $\| \mu \|$ denotes the total variation of the measure μ. The spaces $\mathfrak{F}_\lambda$ were introduced in [17,18], and some roperties of functions in $\mathfrak{F}_\lambda$ were obtained in [19,20].

An analytic function $\upsilon(z)$ in $\mathbb{D}$ is called a multiplier of $\mathfrak{F}_\lambda$ provided that $\upsilon(z)h(z) \in \mathfrak{F}_\lambda$ for all $h \in \mathfrak{F}_\lambda$. Let $\mathcal{M}_\lambda$ denote the set of all multipliers of $\mathfrak{F}_\lambda$. $\mathcal{M}_\lambda$ is a Banach space with respect to the norm defined by

$$\| \upsilon \|_{\mathcal{M}_\lambda} = \sup\{\| \upsilon h \|_{\mathfrak{F}_\lambda} : \ h \in \mathfrak{F}_\lambda, \| h \|_{\mathfrak{F}_\lambda} \le 1\}.$$

The family $\mathcal{M}_\lambda$ has been studied in [19–21]. In this section, we will consider the multiplier property of $g(z) = F(1/z)$ with respect to $\mathfrak{F}_\lambda$.

Theorem 4.1. *For each $\beta \ge 0$, $g(z) \in \mathcal{M}_\beta$. For any small $\varepsilon > 0$, $g'(z) \in \mathfrak{F}_{2-\alpha+\varepsilon}$ and $g'(z) \notin \mathfrak{F}_{2-\alpha-\varepsilon}$, where α is the Hausdorff dimension of K.*

Proof. Since $g' \in H^p$ for some $p > 1$, we have $g \in \mathcal{M}_\beta$ for each $\beta \ge 0$ by Theorem 3.1 in [21, p. 621]. It follows from the remark of Theorem 3.1 that $g''(z) \in H^{1/(3-\alpha)} \subset H^{1/(3-\alpha+\varepsilon)}$. Together with Theorem 3 in [17, p. 116], we see that $H^{1/p} \subset \mathfrak{F}_p$ for $p \ge 1$. Hence, $g'' \in \mathfrak{F}_{3-\alpha+\varepsilon}$. Note that $f \in \mathfrak{F}_\lambda$ if and only if $f' \in \mathfrak{F}_{\lambda+1}$ [17, p. 112]. Consequently $g'(z) \in \mathfrak{F}_{2-\alpha+\varepsilon}$ follows. From [7, p. 70], we obtain that $g(z) = z + \sum_{n=1}^{\infty} a_{3n+1} z^{3n+1}$ for $|z| < 1$, and $c_1 n^{-\alpha} \le a_{3n+1} \le c_2 n^{-\alpha}$ for $n \ge 1$, with constants $c_1 > 0$ and $c_2 > 0$. Assume that $g'(z) \in \mathfrak{F}_{2-\alpha-\varepsilon}$. Since every complex measure on $\mathbb{T}$ is of bounded variation, it follows easily that there exists some constant $c > 0$ such that $|(3n+1)a_{3n+1}| \le cn^{1-\alpha-\varepsilon}$, $n \ge 1$. This is a contradiction. Then, the result follows. □

In view of Theorem 4.1, an interesting question is to determine if $g'(z) \in \mathfrak{F}_{2-\alpha}$, which is equivalent to ([17, p. 115]).

$$h(z) = \sum_{n=0}^{\infty} \frac{n+1}{d_n(2-\alpha)} \int_K \omega^n d\mathcal{H}^\alpha(\omega) z^n \in \mathfrak{F}_1, \tag{4.2}$$

where $d_n(\lambda) = \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\Gamma(\lambda)}$ is defined by $(1-x)^{-\lambda} = \sum_{n=0}^{\infty} d_n(\lambda)x^n$. It follows from Stirling's formula that $d_n(\lambda)(n+1)^{1-\lambda} = \Gamma(\lambda)^{-1} + c(\lambda)(n+1)^{-1} + O((n+1)^{-2})$. Then,

$$\frac{n+1}{d_n(2-\alpha)d_n(\alpha+1)} = \frac{(n+1)^{1-\alpha}(n+1)^\alpha}{d_n(2-\alpha)d_n(\alpha+1)} = c_0 + \frac{c_1}{n+1} + c_n,$$

where $|c_n| \le C(n+1)^{-2}$. If we substitute this into (4.2), then we have

$$\begin{aligned} h(z) &= c_0 \int_K \frac{d\mathcal{H}^\alpha(\omega)}{(1-z\omega)^{\alpha+1}} + c_1 \sum_{n=0}^{\infty} \frac{d_n(\alpha+1)}{n+1} \int_K \omega^n d\mathcal{H}^\alpha(\omega) z^n + \\ & \quad \sum_{n=0}^{\infty} c_n d_n(\alpha+1) \int_K \omega^n d\mathcal{H}^\alpha(\omega) z^n \\ &:= c_0 h_1(z) + c_1 h_2(z) + h_3(z). \end{aligned}$$

Since $|d_n(\alpha+1)\int_K \omega^n d\mathcal{H}^\alpha(\omega)| \le C$ by [7], it follows that $h_2(z) \in H^2$ and $h_3(z) \in H^\infty$, which imply $c_1 h_2(z) + h_3(z) \in \mathfrak{F}_1$ as $H^\infty \subset H^2 \subset H^1 \subset \mathfrak{F}_1$. Consequently,

$$g'(z) \in \mathfrak{F}_{2-\alpha} \Longleftrightarrow h_1(z) = \int_K \frac{d\mathcal{H}^\alpha(\omega)}{(1-z\omega)^{\alpha+1}} \in \mathfrak{F}_1. \tag{4.3}$$

In view of (4.3), by [18], we know that $g'(z) \in \mathfrak{F}_{2-\alpha}$ if and only if $\int_0^z h_1(t)dt \in \mathfrak{F}_0$. This leads us to consider

$$f_\varepsilon(z) = \int_K \frac{d\mathcal{H}^\alpha(\omega)}{(1-\omega z)^{\alpha-\varepsilon}}, \ |z| < 1. \tag{4.4}$$

Theorem 4.2. *$f_\varepsilon \in \mathcal{M}_\beta$ for each $\beta \ge 0$ if $\varepsilon > 0$, and $f_0(z) \notin \mathcal{M}_\beta$ for each $\beta \ge 0$.*

Proof. For the first assertion, we only need to show $f'_\varepsilon(z) \in H^p$ for some $p > 1$ by Theorem 3.1 in [21, p. 621]. Noting that $(1-x)^{-\lambda} = \sum_{n=0}^{\infty} d_n(\lambda)x^n$ for $|x| < 1$, it follows easily by the Hölder inequality that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |f'_\varepsilon(re^{i\theta})|^p d\theta &\le \frac{1}{2\pi}\int_0^{2\pi} \Big(\int_K \frac{|\omega|}{|1-re^{i\theta}\omega|^{(\alpha+1-\varepsilon)}} d\mathcal{H}^\alpha(\omega)\Big)^p d\theta \\ &\le C\int_K \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}\omega|^{p(\alpha+1-\varepsilon)}} d\mathcal{H}^\alpha(\omega) \\ &= C\int_K \frac{1}{2\pi}\int_0^{2\pi} |\sum_{n=0}^{\infty} d_n(\frac{p}{2}(\alpha+1-\varepsilon)) r^n e^{in\theta}\omega^n|^2 d\theta d\mathcal{H}^\alpha(\omega) \\ &= C\sum_{n=0}^{\infty} d_n^2(\frac{p}{2}(\alpha+1-\varepsilon)) \int_K |\omega|^{2n} d\mathcal{H}^\alpha(\omega) r^{2n}. \end{aligned}$$

It follows from Proposition 4.2 in [7] that $\int_K |\omega|^n d\mathcal{H}^\alpha(\omega) \le Cn^{-\alpha}$. Combining this with $d_n(\lambda) \sim \Gamma(\lambda)^{-1} n^{\lambda-1} (n \to \infty)$, we get that

$$\frac{1}{2\pi}\int_0^{2\pi} |f_\varepsilon'(re^{i\theta})|^p d\theta \le C\sum_{n=1}^{\infty} n^{p(\alpha+1-\varepsilon)-2-\alpha} r^{2n}.$$

Notice that $p(\alpha+1-\varepsilon)-2-\alpha \to -\varepsilon-1$ as $p \to 1$, we can choose $p > 1$ such that $p(\alpha+1-\varepsilon)-2-\alpha < -1-\varepsilon/2$. Hence, $f_\varepsilon' \in H^p$ for $p > 1$.

For the second assertion, it is sufficient to prove that $f_0(z)$ is unbounded in $\mathbb{D}$. By Theorem 5.2 in [7], we get that $\int_K \omega^n d\mathcal{H}^\alpha(\omega) = 0$ for $n \ne 3k$, and there exists some constant $c_1 > 0$ such that $\int_K \omega^{3k} d\mathcal{H}^\alpha(\omega) \ge c_1 k^{-\alpha}$ for all $k \ge 1$. Note that $d_n(\alpha) \ge c_2 n^{\alpha-1}$ for some constant $c_2 > 0$ and all $n \ge 1$. It follows that there exists some constant $c_3 > 0$ such that

$$\begin{aligned} f_0(x) &= \sum_{n=0}^{\infty} d_n(\alpha)\int_K \omega^n d\mathcal{H}^\alpha(\omega) x^n \\ &= 1+\sum_{n=1}^{\infty} d_{3n}(\alpha)\int_K \omega^{3n} d\mathcal{H}^\alpha(\omega) x^{3n} \\ &\ge c_3\sum_{n=1}^{\infty}\frac{x^{3n}}{n} \to \infty,\ x \to 1^-. \end{aligned}$$

□

Although we can not prove $f_0(z) = \int_0^z h_1(t)dt \in \mathfrak{F}_0$(or $g'(z) \in \mathfrak{F}_{2-\alpha}$), yet we can prove $f_0(z) \in$ BMOA, which consists of all functions $f \in H^1$ satisfying

$$\| f \|_{\mathrm{BMOA}} = \sup_{I\subset\mathbb{T}} \frac{1}{|I|}\int_I |f(\zeta) - f_I||d\zeta| < \infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$ with $|I| = \int_I |d\zeta|$ and $f_I = |I|^{-1}\int_I f(\zeta)|d\zeta|$. It should be noted that $\mathfrak{F}_0 \subset$ BMOA $\subset H^p$ for all $p > 0$ [21, p. 617].

Theorem 4.3. $f_0(z) \in$ BMOA.

Proof. We first prove that there exists some positive constant C such that

$$|f_0'(z)| \le \frac{C}{|1-z^3|},\ |z| < 1. \tag{4.5}$$

It is equivalent to prove that $p(z) := (1-z^3)f_0'(z)$ is bounded for $|z| < 1$. It is easy check that $p(z)$ is continue on $\{z : |z| \le 1/2\}$. Hence, $\max_{|z|\le 1/2}|p(z)| < \infty$. Next, we prove $p(z)$ is bounded for $1/2 < |z| < 1$. Let $\Omega = \{re^{i\theta} : 1/2 < r < 1, -\pi/3 \le \theta \le 0\}$. For $z \in \Omega$, let $d = \mathrm{dist}(z^{-1}, K)$. Obviously, $d > 0$ as $1 < |z|^{-1} < 2$. Noting that $p(e^{2\pi i/3}z) = p(z)$, $|p(\bar{z})| = |p(z)|$, and we can check that there exists some positive constant C_1 such that

$$|f_0'(z)| \le C_1\int_K \frac{d\mathcal{H}^\alpha(\omega)}{|\frac{1}{z}-\omega|^{\alpha+1}} \le \frac{C_1}{d}.$$

With consideration of geometry, we find that there exists some constant $C_2 > 0$ such that $d = \mathrm{dist}(z^{-1}, K) \geq C_2|1 - z|$ for $z \in \Omega$. Hence,

$$|p(z)| \leq C_1 C_2^{-1} |1 - z^3| |1 - z|^{-1} \leq C_3, \ z \in \Omega.$$

Note that $|p(e^{2\pi i/3} z)| = |p(z)|$, $|p(\bar{z})| = |p(z)|$. We obtain that $p(z)$ is bounded for $1/2 < |z| < 1$, and (4.5) follows.

It is known that an analytic function $\psi(z)$ on $\mathbb{D}$ belongs to BMOA if and only if $|\psi'(z)|^2(1 - |z|^2)dxdy/\pi$ is a Carleson measure [22, p. 240]. By using (4.5), we have $|f_0'(z)| \leq C|1 - z^3|^{-1}$. Hence, for any small sector $S_h(\theta_0) = \{re^{i\theta} : 1 - h \leq r < 1, |\theta - \theta_0| \leq h\}$,

$$\sup_{h>0} \frac{1}{h} \int_{S_h(\theta_0)} |f_0'(z)|^2 (1 - |z|^2) \frac{dxdy}{\pi} \leq C \sup_{h>0} \frac{1}{h} \int_{S_h(0)} \frac{1 - |z|^2}{|1 - z^3|^2} dxdy \leq C'.$$

This shows that $|f_0'(z)|^2(1 - |z|^2)dxdy/\pi$ is a Carleson measure, and the result follows. □

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Conflict of interest

The authors declare no conflicts of interest.

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