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Research article

Lag synchronization of complex-valued interval neural networks via distributed delayed impulsive control

Zhifeng Lu[1], **Fei Wang**[1], **Yujuan Tian**[1,*] **and Yaping Li**[2]

[1] School of Mathematics and Statistics, Shandong Normal University, Ji'nan, 250014, China
[2] School of Information Engineering, Shandong Management University, Ji'nan, 250357, China

* **Correspondence:** Email: yjtian@sdnu.edu.cn.

Abstract: This paper investigates the lag synchronization problem of complex-valued interval neural networks with both discrete and distributed time-varying delays under delayed impulsive control. A distributed delayed impulsive controller that depends on the accumulation of the states over a history time period is designed to guarantee the exponential lag synchronization between the drive and the response systems. By employing the complex Lyapunov method and a novel impulsive differential inequality technique, some delay-dependent synchronization criteria are established in terms of complex-valued linear matrix inequalities (LMIs). Finally, a numerical example is given to illustrate the effectiveness of the theoretical results.

Keywords: complex-valued interval neural networks; exponential lag synchronization; distributed delayed impulsive control; mixed time-delays; impulsive differential inequality
Mathematics Subject Classification: 34D06, 92B20, 93C27

1. Introduction

In recent years, neural networks (NNs) have been extensively investigated due to their wide applications in many areas, such as signal processing, fault diagnosis, combinatorial optimization, pattern recognition, and so on [1–3]. Compared with real-valued neural networks (RVNNs), complex-valued neural networks (CVNNs) have more complex characteristics, which can simulate more practical situations and deal with some problems that RVNNs cannot solve, such as XOR problem, symmetry detection, electromagnetic wave imaging, etc. [4–6]. Moreover, due to the inherent information transmission between neurons and the finite switching speed of actuators in NNs, time delays are always inevitable, which may lead to undesirable dynamical behaviors and seriously affect the performance of the systems [7,8]. Consequently, the study of CVNNs with time delays has become an active research topic and some significative results have been reported in [9, 10].

Synchronization, as a significant dynamic index of NNs, has received great attention because of its potential applications. Up to now, a wide variety of synchronization phenomena have been investigated, including finite-time synchronization [11], cluster synchronization [12], projective synchronization [13], lag synchronization [14] and others. In particular, lag synchronization is characterized as the coincidence of the states of two coupled systems in which one of the systems is delayed by a given finite time, and it has been proved to be a reasonable strategy from the viewpoint of engineering applications in secure communication and parallel image processing. However, it is worth noting that the existing results on lag synchronization mainly focus on RVNNs. There are few reports on the lag synchronization problem of CVNNs. On the other hand, because of the complex topology of NNs, appropriate external controllers are needed to realize synchronization, such as feedback control [15], adaptive control [16], intermittent control [17], impulsive control [18, 19], and so on. Among them, impulsive control has been widely used in control areas due to its lower control cost, higher confidentiality and stronger robustness [20]. Some interesting works on the impulsive synchronization problem of delayed CVNNs have been proposed in recent years. The global exponential synchronization of complex-valued memristor-based NNs with time-varying delays was investigated via impulsive control in [21]. Under hybrid impulsive controllers, [22] gave the results of the exponential synchronization for drive-response time-varying delayed CVNNs. In [23], some sufficient conditions were obtained for the exponential synchronization of CVNNs with time-varying delays by an impulsive controller. In fact, it is hard to require the process of sampling, processing, and transferring impulse information to achieve immediately [24, 25]. Hence, delayed impulsive control is perceived as a better way to model many practical problems. In [26], authors investigated the synchronization problem of CVNNs via a delayed pinning impulsive controller. By designing a delayed impulsive control scheme, some new sufficient conditions for the exponential synchronization of complex-valued complex dynamical networks with multiple time-varying delays were obtained in [27]. However, the delayed impulsive controllers in the above mentioned literatures only considered discrete delays, not distributed ones. The distributed delayed impulsive control, as another type of delayed impulsive control, can stabilize a system based on the accumulation of the states over a history time period, which is more applicable for many industrial and biological systems such as network connection, the spread of disease and epidemic model [28–30]. To our best knowledge, such impulsive control has not been studied for the synchronization problem of CVNNs.

As we know, since NNs have many parallel pathways with different axon sizes and lengths, delays may be distributed during a certain time period [31–33]. Therefore, besides discrete delays, distributed delays should be also incorporated in the models. In addition, model errors and parameter fluctuations are unavoidable in the modeling process of CVNNs, which will lead to poor performance and asynchronization of the systems [34]. It is very significant to study interval NNs whose parameters values are unknown but bounded in given compact sets [35,36]. However, there are very few developed achievements on the synchronization problem of complex-valued interval NNs with mixed delays, especially with distributed delayed impulsive control, which motivates our present study. Moreover, to deal with complex variables in CVNNs, the method of separating real-imaginary parts was extensively used in most of the existing literatures. Nevertheless, an explicit separation of complex-valued activation functions of CVNNs into their real and imaginary parts is needed when applying this kind of method, which may cause the theoretical results to be more conservative and disheveled. Therefore, how to take the model as a whole to develop theoretical results in complex domain for the synchronization of CVNNs with mixed delays and uncertain parameters is a very interesting and challenging work.

Based on the above discussions, this paper aims to investigate the exponential lag synchronization for complex-valued interval neural networks with both discrete and distributed delays via delayed impulsive control. The main contributions in our work can be summarized as follows: (i) The model of CVNNs under discussion is quite comprehensive, which takes mixed time-delays and uncertain parameters into simultaneous consideration. (ii) To realize lag synchronization of the drive and the response systems, a distributed delayed impulsive controller that depends on the accumulation of the states over a history time period is designed. (iii) A novel impulsive differential inequality is employed to deal with the mathematical difficulty caused by mixed time-delays and distributed delayed impulses, which makes our results less conservative than the existing ones. (iv) Instead of the separation approach, a uniform research framework in complex domain is developed and some synchronization criteria are derived in the form of complex-valued LMIs, which are easily checked by using the Matlab LMI toolbox. The rest of this paper is organized as follows. Model description and preliminaries are introduced in Section 2. The main results are obtained in Section 3. Section 4 provides an example to verify the validity of the derived results. The summary of this paper is given in Section 5.

Notations. Let $\mathbb{C}$, $\mathbb{C}^n$ and $\mathbb{C}^{n\times n}$ be the set of complex numbers, n-dimensional complex space and $n\times n$-dimensional complex space, respectively. Correspondingly, let $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n^2\times n^2}$ denote the set of real numbers, n-dimensional real space, $n\times n$-dimensional real space and $n^2\times n^2$-dimensional real space, respectively. The set of positive integers is denoted by $\mathbb{Z}_+$. For a matrix A, $A > 0$ ($A < 0$, $A \geq 0$, $A \leq 0$) denotes that A is a positive definite (negative definite, positive semi-definite, negative semi-definite) matrix, $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) stands for the maximum (minimum) eigenvalue of matrix A, A^{-1} represents the inverse of A and A^* means the conjugate transpose of A. Unless otherwise specified, I is the identity matrix with appropriate dimensions. The maximum value of α and β is defined as $\alpha \vee \beta$. Let i be the imaginary, i.e., $i = \sqrt{-1}$. $Re(x)$ and $Im(x)$ denote the real part and the imaginary part of a number $x \in \mathbb{C}$. For $x, y \in \mathbb{C}$, $x \preceq y$ means that $Re(x) \leq Re(y)$ and $Im(x) \leq Im(y)$. $\|x\| = \sqrt{xx^*}$, where x^* is the conjugate transpose of x. $\overline{x}$ represents the conjugate of x. For any $J \subseteq \mathbb{R}$ and $S \subseteq \mathbb{C}^k (1 \leq k \leq n, k \in \mathbb{Z}_+)$, set $PC(J, S) = \{\varphi : J \to S$ is continuous everywhere except at a finite number of points t, at which $\varphi(t^+), \varphi(t^-)$ exist and $\varphi(t) = \varphi(t^+)\}$. $\|\varphi\|_\rho = \sup_{t\in[-\rho,0]} \|\varphi(t)\|$. $\star$ is the conjugate transpose of a suitable block in a Hermitian matrix and $\Lambda = \{1, 2, \cdots, n\}, n \in \mathbb{Z}_+$ is an index set.

2. Preliminaries

In this paper, we consider the following complex-valued interval neural networks :

$$\begin{cases} \dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) + D\int_{t-\beta(t)}^{t} f(x(s))\,ds + J(t), t > 0, \\ x(t) = \phi(t), t \in [-\rho_1, 0], \end{cases} \tag{2.1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{C}^n$ is the state vector of the complex-valued interval neural networks; $f(x(\cdot)) = (f_1(x_1(\cdot)), \ldots, f_n(x_n(\cdot)))^T \in \mathbb{C}^n$ is the neuron activation function; $C = \mathrm{diag}\{c_1, c_2, \cdots, c_n\} \in \mathbb{R}^{n\times n}$ denotes the self-feedback connection weight matrix with $c_j > 0, j \in \Lambda$; A, B and $D \in \mathbb{C}^{n\times n}$ are the connection weight matrices; $J(t) \in \mathbb{C}^n$ is an external input vector; $\tau(t)$ is discrete delay and $\beta(t)$ is distributed delay satisfying $0 \leq \tau(t) \leq \tau$ and $0 \leq \beta(t) \leq \beta$, where τ and β are constants; $\phi(\cdot) \in PC([-\rho_1, 0], \mathbb{C}^n)$ is the initial condition for $\rho_1 = \tau \vee \beta$.

Throughout this paper, we make the following assumptions.

Assumption 1. *For all u, $v \in \mathbb{C}$, there exists a positive diagonal matrix $L = \mathrm{diag}\{L_1, L_2, \cdots, L_n\}$, $L_j \in \mathbb{R}$, $j \in \Lambda$, such that*

$$\| f_j(u) - f_j(v) \| \leq L_j \| u - v \| .$$

Assumption 2. *Matrices C, A, B and D in the model (2.1) are bounded in the following intervals*

$$C \in C_I,\ A \in A_I,\ B \in B_I,\ D \in D_I,$$

where

$$\begin{aligned}
C_I &= [\check{C}, \hat{C}] = \{\mathrm{diag}(c_j) \mid 0 < \check{c}_j \leq c_j \leq \hat{c}_j; j \in \Lambda\},\\
A_I &= [\check{A}, \hat{A}] = \{(a_{pq})_{n\times n} \mid \check{a}_{pq} \leq a_{pq} \leq \hat{a}_{pq}; p, q \in \Lambda\},\\
B_I &= [\check{B}, \hat{B}] = \{(b_{pq})_{n\times n} \mid \check{b}_{pq} \leq b_{pq} \leq \hat{b}_{pq}; p, q \in \Lambda\},\\
D_I &= [\check{D}, \hat{D}] = \{(d_{pq})_{n\times n} \mid \check{d}_{pq} \leq d_{pq} \leq \hat{d}_{pq}; p, q \in \Lambda\},
\end{aligned}$$

with $\check{C} = \mathrm{diag}(\check{c}_1, \dots, \check{c}_n)$, $\hat{C} = \mathrm{diag}(\hat{c}_1, \dots, \hat{c}_n)$, $\check{A} = (\check{a}_{pq})_{n\times n}$, $\hat{A} = (\hat{a}_{pq})_{n\times n}$, $\check{B} = (\check{b}_{pq})_{n\times n}$, $\hat{B} = (\hat{b}_{pq})_{n\times n}$, $\check{D} = (\check{d}_{pq})_{n\times n}$ and $\hat{D} = (\hat{d}_{pq})_{n\times n}$.

We refer to system (2.1) as the drive system, and the corresponding response system is characterized by

$$\begin{cases} \dot{y}(t) = -Cy(t) + Af(y(t)) + Bf(y(t-\tau(t))) + D\int_{t-\beta(t)}^{t} f(y(s))\,ds + u(t) + J(t), t > \alpha, \\ y(t) = \varphi_\alpha(t), t \in [-\rho_1, 0], \end{cases} \tag{2.2}$$

where $\varphi_\alpha(t) = \varphi(\alpha + t)$, $\forall t \in [-\rho_1, 0]$, $\varphi_\alpha(t) \in PC([-\rho_1, 0], \mathbb{C}^n)$, where $\alpha > 0$ is response time delay between the drive system (2.1) and the response system (2.2), and $u(t)$ is the control input that needs to be designed.

To achieve lag synchronization between the drive system (2.1) and the response system (2.2), a class of distributed delayed impulsive controller can be given as follows

$$u(t) = \sum_{n=1}^{\infty}(K\int_{t-r_k}^{t} e(s)\,ds - e(t))\delta(t - t_k),\ \ k \in \mathbb{Z}_+, \tag{2.3}$$

where r_k are the distributed delays satisfying $0 \leq r_k \leq r$ with r is a constant; $e(t) = y(t) - x(t-\alpha)$ is the lag synchronization error for $\alpha < \rho$ and $\rho = \tau \vee \beta \vee r$; $\delta(\cdot)$ is the Dirac delta function; $K \in \mathbb{C}^{n\times n}$ is a gain matrix to be designed. Then, one can obtain the following error system

$$\begin{cases} \dot{e}(t) = -Ce(t) + Ah(e(t)) + Bh(e(t-\tau(t))) + D\int_{t-\beta(t)}^{t} h(e(s))\,ds, t > \alpha, \\ e(t) = K\int_{t-r_k}^{t} e(s)\,ds, t = t_k, \\ e(t) = \varphi_\alpha(t) - \phi(t), t \in [-\rho, 0], \end{cases} \tag{2.4}$$

where $h(e(t)) = f(y(t)) - f(x(t-\alpha))$, $h(e(t-\tau(t))) = f(y(t-\tau(t))) - f(x(t-\alpha-\tau(t)))$.

Furthermore, let

$$\mathcal{L} \doteq \{\mathrm{diag}(\delta_{11}, \dots, \delta_{1n}, \dots, \delta_{n1}, \dots, \delta_{nn}) \in \mathbb{R}^{n^2\times n^2}; |\delta_{pq}| \leq 1, p, q \in \Lambda\},$$

$$C_0 = \frac{\hat{C}+\check{C}}{2}, A_0 = \frac{\hat{A}+\check{A}}{2}, B_0 = \frac{\hat{B}+\check{B}}{2}, D_0 = \frac{\hat{D}+\check{D}}{2},$$

$$H_C = \frac{\hat{C}-\check{C}}{2} = \mathrm{diag}(\gamma_1,\ldots,\gamma_n), H_A = \frac{\hat{A}-\check{A}}{2} = (\alpha_{pq})_{n\times n}, H_B = \frac{\hat{B}-\check{B}}{2} = (\beta_{pq})_{n\times n}, H_D = \frac{\hat{D}-\check{D}}{2} = (\eta_{pq})_{n\times n},$$

where C_0, A_0, B_0 and D_0 are reference matrices of interval matrices C, A, B and D, respectively. Define

$$\begin{aligned}
\Theta_1 &= [\sqrt{\gamma_1}\epsilon_1,\ldots,0,0,\ \sqrt{\gamma_2}\epsilon_2,\ldots,0,\ldots,0,\ldots,\ \sqrt{\gamma_n}\epsilon_n]_{n\times n^2},\\
\Theta_2 &= [\sqrt{\alpha_{11}}\epsilon_1,\ldots,\ \sqrt{\alpha_{1n}}\epsilon_1,\ldots,\ \sqrt{\alpha_{n1}}\epsilon_n,\ldots,\ \sqrt{\alpha_{nn}}\epsilon_n]_{n\times n^2},\\
\Theta_3 &= [\sqrt{\beta_{11}}\epsilon_1,\ldots,\ \sqrt{\beta_{1n}}\epsilon_1,\ldots,\ \sqrt{\beta_{n1}}\epsilon_n,\ldots,\ \sqrt{\beta_{nn}}\epsilon_n]_{n\times n^2},\\
\Theta_4 &= [\sqrt{\eta_{11}}\epsilon_1,\ldots,\ \sqrt{\eta_{1n}}\epsilon_1,\ldots,\ \sqrt{\eta_{n1}}\epsilon_n,\ldots,\ \sqrt{\eta_{nn}}\epsilon_n]_{n\times n^2},\\
\Phi_1 &= [\sqrt{\gamma_1}\epsilon_1,\ldots,0,0,\ \sqrt{\gamma_2}\epsilon_2,\ldots,0,\ldots,0,\ldots,\ \sqrt{\gamma_n}\epsilon_n]^T_{n^2\times n},\\
\Phi_2 &= [\overline{\sqrt{\alpha_{11}}}\epsilon_1,\ldots,\ \overline{\sqrt{\alpha_{1n}}}\epsilon_n,\ldots,\ \overline{\sqrt{\alpha_{n1}}}\epsilon_1,\ldots,\ \overline{\sqrt{\alpha_{nn}}}\epsilon_n]^*_{n^2\times n},\\
\Phi_3 &= [\overline{\sqrt{\beta_{11}}}\epsilon_1,\ldots,\ \overline{\sqrt{\beta_{1n}}}\epsilon_n,\ldots,\ \overline{\sqrt{\beta_{n1}}}\epsilon_1,\ldots,\ \overline{\sqrt{\beta_{nn}}}\epsilon_n]^*_{n^2\times n},\\
\Phi_4 &= [\overline{\sqrt{\eta_{11}}}\epsilon_1,\ldots,\ \overline{\sqrt{\eta_{1n}}}\epsilon_n,\ldots,\ \overline{\sqrt{\eta_{n1}}}\epsilon_1,\ldots,\ \overline{\sqrt{\eta_{nn}}}\epsilon_n]^*_{n^2\times n},
\end{aligned}$$

where $\epsilon_j \in \mathbb{R}^n$, $j \in \Lambda$ represents the jth unit column vector. Then the system (2.4) can be rewritten as

$$\begin{cases}
\dot{e}(t) = -[C_0 + \Theta_1\Delta_1\Phi_1]e(t) + [A_0 + \Theta_2\Delta_2\Phi_2]h(e(t)) + [B_0 + \Theta_3\Delta_3\Phi_3]h(e(t-\tau(t)))\\
\qquad\quad + [D_0 + \Theta_4\Delta_4\Phi_4]\int_{t-\beta(t)}^{t} h(e(s))\,ds, t > 0,\\
e(t) = K\int_{t-r_k}^{t} e(s)\,ds, t = t_k,\\
e(t) = \varphi_\alpha(t) - \phi(t), t \in [-\rho, 0],
\end{cases}$$

which is equivalent to

$$\begin{cases}
\dot{e}(t) = -C_0 e(t) + A_0 h(e(t)) + B_0 h(e(t-\tau(t))) + D_0\int_{t-\beta(t)}^{t} h(e(s))\,ds + \Theta\Psi_e(t), t \neq t_k,\\
e(t) = K\int_{t-r_k}^{t} e(s)\,ds, t = t_k,\\
e(t) = \varphi_\alpha(t) - \phi(t), t \in [-\rho, 0],
\end{cases} \tag{2.5}$$

where $\Theta = [\Theta_1, \Theta_2, \Theta_3, \Theta_4]_{n\times 4n^2}$, $\Delta_j \in \mathcal{L}$, $j = 1, 2, 3, 4$,

$$\Psi_e(t) = \begin{pmatrix} \Delta_1\Phi_1 & 0 & 0 & 0\\ 0 & \Delta_2\Phi_2 & 0 & 0\\ 0 & 0 & \Delta_3\Phi_3 & 0\\ 0 & 0 & 0 & \Delta_4\Phi_4 \end{pmatrix}_{4n^2\times 4n} \times \begin{pmatrix} e(t)\\ h(e(t))\\ h(e(t-\tau(t)))\\ \int_{t-\beta(t)}^{t} h(e(s))\,ds \end{pmatrix}_{4n\times 1}.$$

Remark 1. *The robust stability of complex-valued interval neural networks has been studied in [9], where the interval parameters are processed by dividing them into real and imaginary parts. Here, without the separation method, we directly use matrix transformations in the complex field to deal with the interval parameter uncertainties.*

Based on the above analysis and description, the following definitions and lemmas are needed.

Definition 1 ([37]). *If sequence $\{t_k,\ k \in \mathbb{Z}_+\}$ satisfies*

$$0 \le t_0 < t_1 < \cdots < t_k \text{ with } t_k \to +\infty \text{ as } k \to +\infty,$$

then $\mathcal{A}_0 = \{t_k\}$ *is called impulse time sequence. Furthermore, for any constant* $\zeta > 0$*, define* $\mathcal{A}_\zeta$ *as the set of impulse time sequences in* $\mathcal{A}_0$ *such that* $t_k - t_{k-1} \le \zeta$, $\forall k \in \mathbb{Z}_+$.

Definition 2 ([38]). *The drive-response system (2.1) and (2.2) are said to be exponentially lag synchronized under the impulse set* $\mathcal{A}_\zeta$*, if there exist constants* $M \ge 1$ *and* $\varepsilon > 0$ *such that*

$$\|y(t) - x(t-\alpha)\| \le M\|\varphi_\alpha - \phi\|_\rho \exp(-\varepsilon(t-\alpha)), \forall t \ge \alpha,$$

where $\varphi_\alpha, \phi \in PC([-\rho, 0], \mathbb{C}^n)$*, and* ε *is called the degree of exponential lag synchronization.*

Lemma 1 ([39]). *Consider the following impulsive differential inequality,*

$$\begin{cases} D^+ g(t) \le \sigma_1 g(t) + \sigma_2 \bar{g}(t), t \ne t_k, \\ g(t) \le \mu_k \int_{t-r_k}^{t} g(s)\,ds, t = t_k, \end{cases} \tag{2.6}$$

where D^+ *is the right-upper Dini derivative,* $g \in PC(\mathbb{R}, \mathbb{R}_+)$, $\bar{g}(t) = \sup\limits_{s\in[t-\rho,t]} g(s)$*. If there exist constants* $\sigma_1 \in \mathbb{R}$, $\sigma_2 > 0$, $\mu_k > 0$, $k \in \mathbb{Z}_+$, $\zeta > 0$ *and* $\delta > 1$*, such that the following conditions hold:*

$$\sigma_1 + \sigma_2 \delta < \frac{\ln \delta}{\zeta}, \tag{2.7}$$

$$\inf_{k\in\mathbb{Z}_+} \Big\{ \frac{-\ln(\mu_k r_k \delta)}{r_k} \Big\} = \eta > 0, \tag{2.8}$$

then the solution of the inequality (2.6) over the set $\mathcal{A}_\zeta$ *satisfies*

$$g(t) \le \delta \bar{g}(0) \exp(-\lambda t), t \ge 0,$$

where $\lambda \in (0, \eta)$ *is a positive constant satisfying* $\theta(\lambda) > 0$ *for* $\theta(\lambda) = \frac{\ln\delta}{\zeta} - \sigma_1 - \sigma_2\delta \exp(\rho\lambda) - \lambda$.

Lemma 2 ([40]). *For any positive definite Hermitian matrix* $E \in \mathbb{C}^{n\times n}$*, any function* $u(s) : [a, b] \to \mathbb{C}^n$ *with scalars* $a < b$ *such that the integrals concerned are well defined, then*

$$\left(\int_a^b u(s)\,ds\right)^* E \int_a^b u(s)\,ds \le (b-a) \int_a^b u^*(s) E u(s)\,ds.$$

Lemma 3 ([26]). *For any vector* x*,* $y \in \mathbb{C}^n$ *and positive definite Hermitian matrix* $R \in \mathbb{C}^{n\times n}$*, it holds that*

$$x^* y + y^* x \le x^* R x + y^* R^{-1} y.$$

3. Main results

In this section, with the help of the Lyapunov method and the delayed impulsive differential inequality technique, we get some criteria of exponential lag synchronization of complex-valued interval neural networks by designing the distributed delayed impulsive controller.

Theorem 1. *Under Assumption 1 and Assumption 2, if there exist constants $\sigma_1 \in \mathbb{R}$, $\sigma_2 > 0$, $\zeta > 0$, $\delta > 1$, $0 < \mu < 1$ and positive definite Hermitian matrix P, positive diagonal matrices $Q_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2, 3)$, $Q_j \in \mathbb{R}^{n^2\times n^2}$ $(j = 4, 5, 6, 7)$, matrix $Z \in \mathbb{C}^{n\times n}$ such that (2.7) and the following inequalities hold*

$$\begin{pmatrix} \Omega_{11} & PA_0 & PB_0 & PD_0 & P\Theta \\ \star & -Q_1 & 0 & 0 & 0 \\ \star & \star & -Q_2 & 0 & 0 \\ \star & \star & \star & -Q_3 & 0 \\ \star & \star & \star & \star & \begin{pmatrix} -Q_4 & 0 & 0 & 0 \\ 0 & -Q_5 & 0 & 0 \\ 0 & 0 & -Q_6 & 0 \\ 0 & 0 & 0 & -Q_7 \end{pmatrix} \end{pmatrix} \le 0, \tag{3.1}$$

$$\begin{pmatrix} -\frac{P}{\delta} & rZ \\ \star & -P \end{pmatrix} < 0, \tag{3.2}$$

$$L(Q_2 - \Phi_3^* Q_6 \Phi_3)L \le \mu\sigma_2 P, \tag{3.3}$$

$$\beta^2 L(Q_3 - \Phi_4^* Q_7 \Phi_4)L \le (1-\mu)\sigma_2 P, \tag{3.4}$$

where $\Omega_{11} = -PC_0 - C_0^* P - \sigma_1 P + LQ_1 L + \Phi_1^* Q_4 \Phi_1 + L\Phi_2^* Q_5 \Phi_2 L$, then the drive system (2.1) and the response system (2.2) are exponentially lag synchronized over the set $\mathcal{A}(\zeta)$ with control gain

$$K = P^{-1} Z^*.$$

Proof. Choose the Lyapunov function

$$V(t) = e^*(t)Pe(t). \tag{3.5}$$

Taking the derivative of $V(t)$ along the trajectory of the system (2.5) on $t \in [t_{k-1}, t_k)$, $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} D^+V(t) &= \dot{e}^*(t)Pe(t) + e^*(t)P\dot{e}(t) \\ &= e^*(t)(PC_0 + C_0^* P)e(t) + e^*(t)PA_0 h(e(t)) + h^*(e(t))A_0^* Pe(t) \\ &\quad + e^*(t)PB_0 h(e(t-\tau(t))) + h^*(e(t-\tau(t)))B_0^* Pe(t) \\ &\quad + e^*(t)PD_0 \int_{t-\beta(t)}^{t} h(e(s))\,ds + \int_{t-\beta(t)}^{t} h^*(e(s))\,ds D_0^* Pe(t) \\ &\quad + e^*(t)P\Theta\Psi_e(t) + \Psi_e^*(t)\Theta^* Pe(t) \end{aligned}$$

By Lemma 3 and Assumption 1,

$$D^+V(t) \le -e^*(t)(PC_0 + C_0^* P)e(t) + e^*(t)PA_0 Q_1^{-1} A_0^* Pe(t) + h^*(e(t))Q_1 h(e(t)) + e^*(t)PB_0 Q_2^{-1} B_0^* Pe(t)$$

$$
\begin{aligned}
&+ h^*(e(t-\tau(t)))Q_2 h(e(t-\tau(t))) + e^*(t)PD_0Q_3^{-1}D_0^*Pe(t) \\
&+ \int_{t-\beta(t)}^{t} h^*(e(s))\,ds Q_3 \int_{t-\beta(t)}^{t} h(e(s))\,ds + e^*(t)P\Theta\Psi_e(t) + \Psi_e^*(t)\Theta^*Pe(t) \\
\leq & -e^*(t)(PC_0 + C_0^*P)e(t) + e^*(t)PA_0Q_1^{-1}A_0^*Pe(t) + e^*(t)LQ_1Le(t) + e^*(t)PB_0Q_2^{-1}B_0^*Pe(t) \\
&+ e^*(t-\tau(t))LQ_2Le(t-\tau(t)) + e^*(t)PD_0Q_3^{-1}D_0^*Pe(t) \\
&+ \int_{t-\beta(t)}^{t} h^*(e(s))\,ds Q_3 \int_{t-\beta(t)}^{t} h(e(s))\,ds + e^*(t)P\Theta\Psi_e(t) + \Psi_e^*(t)\Theta^*Pe(t).
\end{aligned} \tag{3.6}
$$

Applying Lemma 3 again to the last term of inequality (3.6), it can be seen that

$$
\begin{aligned}
e^*(t)P\Theta\Psi_e(t) + \Psi_e^*(t)\Theta^*Pe(t) \leq e^*(t)P\Theta & \begin{pmatrix} Q_4^{-1} & 0 & 0 & 0 \\ 0 & Q_5^{-1} & 0 & 0 \\ 0 & 0 & Q_6^{-1} & 0 \\ 0 & 0 & 0 & Q_7^{-1} \end{pmatrix} \Theta^*Pe(t) \\
&+ \Psi_e^*(t) \begin{pmatrix} Q_4 & 0 & 0 & 0 \\ 0 & Q_5 & 0 & 0 \\ 0 & 0 & Q_6 & 0 \\ 0 & 0 & 0 & Q_7 \end{pmatrix} \Psi_e(t)
\end{aligned} \tag{3.7}
$$

Moreover, according to the definition of $\Psi_e(t)$, we can derive that

$$
\begin{aligned}
&\Psi_e^*(t) \begin{pmatrix} Q_4 & 0 & 0 & 0 \\ 0 & Q_5 & 0 & 0 \\ 0 & 0 & Q_6 & 0 \\ 0 & 0 & 0 & Q_7 \end{pmatrix} \Psi_e(t) \\
&= \begin{pmatrix} e(t) \\ h(e(t)) \\ h(e(t-\tau(t))) \\ \int_{t-\beta(t)}^{t} h(e(s))\,ds \end{pmatrix}^* \\
&\quad \times \begin{pmatrix} \Phi_1^*\Delta_1Q_4\Delta_1\Phi_1 & 0 & 0 & 0 \\ 0 & \Phi_2^*\Delta_2Q_5\Delta_2\Phi_2 & 0 & 0 \\ 0 & 0 & \Phi_3^*\Delta_3Q_6\Delta_3\Phi_3 & 0 \\ 0 & 0 & 0 & \Phi_4^*\Delta_4Q_7\Delta_4\Phi_4 \end{pmatrix} \\
&\quad \times \begin{pmatrix} e(t) \\ h(e(t)) \\ h(e(t-\tau(t))) \\ \int_{t-\beta(t)}^{t} h(e(s))\,ds \end{pmatrix} \\
&\leq e^*(t)\Phi_1^*Q_4\Phi_1e(t) + h^*(e(t))\Phi_2^*Q_5\Phi_2h(e(t)) + h^*(e(t-\tau(t)))\Phi_3^*Q_6\Phi_3h(e(t-\tau(t))) \\
&\quad + \left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right)^* \Phi_4^*Q_7\Phi_4 \left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right) \\
&\leq e^*(t)\Phi_1^*Q_4\Phi_1e(t) + e^*(t)L\Phi_2^*Q_5\Phi_2Le(t) + e^*(t-\tau(t))L\Phi_3^*Q_6\Phi_3Le(t-\tau(t))
\end{aligned}
$$

$$+\left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right)^{*} \Phi_4^{*} Q_7 \Phi_4 \left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right). \tag{3.8}$$

It follows from (3.6)-(3.8) that

$$\begin{aligned}
D^{+}V(t) \le e^{*}(t)\Big\{ &- PC_0 - C_0^{*}P - \sigma_1 P + LQ_1L + \Phi_1^{*}Q_4\Phi_1 + L\Phi_2^{*}Q_5\Phi_2L \\
&+ PA_0Q_1^{-1}A_0^{*}P + PB_0Q_2^{-1}B_0^{*}P + PD_0Q_3^{-1}D_0^{*}P \\
&+ P\Theta \begin{pmatrix} Q_4^{-1} & 0 & 0 & 0 \\ 0 & Q_5^{-1} & 0 & 0 \\ 0 & 0 & Q_6^{-1} & 0 \\ 0 & 0 & 0 & Q_7^{-1} \end{pmatrix} \Theta^{*}P \Big\} e(t) \\
&+ e^{*}(t-\tau(t))[LQ_2L + L\Phi_3^{*}Q_6\Phi_3L]e(t-\tau(t)) + \sigma_1 e^{*}(t)Pe(t) \\
&+ \left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right)^{*} (\Phi_4^{*}Q_7\Phi_4 + Q_3) \left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right).
\end{aligned} \tag{3.9}$$

Next, we deal with the last two terms of the above inequality. From (3.3), we have

$$\begin{aligned}
&e^{*}(t-\tau(t))[LQ_2L + L\Phi_3^{*}Q_6\Phi_3L]e(t-\tau(t)) \\
&\le \mu\sigma_2 e^{*}(t-\tau(t))Pe(t-\tau(t)) \\
&\le \mu\sigma_2 \sup_{s\in[t-\rho,t]} e^{*}(s)Pe(s).
\end{aligned} \tag{3.10}$$

Based on Lemma 2, Assumption 1 and (3.4), we get

$$\begin{aligned}
&\left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right)^{*} (\Phi_4^{*}Q_7\Phi_4 + Q_3) \left(\int_{t-\beta(t)}^{t} h(e(s))\,ds\right) \\
&\le \beta(t) \int_{t-\beta(t)}^{t} h(e(s))^{*}(\Phi_4^{*}Q_7\Phi_4 + Q_3)h(e(s))\,ds \\
&\le \beta(t) \int_{t-\beta(t)}^{t} e^{*}(s)L(\Phi_4^{*}Q_7\Phi_4 + Q_3)Le(s)\,ds \\
&\le \beta^{2} \sup_{s\in[t-\rho,t]} e^{*}(s)[LQ_3L + L\Phi_4^{*}Q_7\Phi_4L]e(s) \\
&\le (1-\mu)\sigma_2 \sup_{s\in[t-\rho,t]} e^{*}(s)Pe(s).
\end{aligned} \tag{3.11}$$

Applying the Schur complement lemma to (3.1), together with (3.9) − (3.11), we can obtain

$$D^{+}V(t) \le \sigma_1 V(t) + \sigma_2 \bar{V}(t).$$

On the other hand, at $t = t_k$, $k \in \mathbb{Z}_{+}$, we have

$$\begin{aligned}
V(t_k) &= e^{*}(t_k)Pe(t_k) \\
&= \left(\int_{t_k-r_k}^{t_k} e(s)\,ds\right)^{*} K^{*}PK \left(\int_{t_k-r_k}^{t_k} e(s)\,ds\right)
\end{aligned}$$

$$\leq r_k \int_{t_k-r_k}^{t_k} e^*(s)K^*PKe(s)\,ds. \tag{3.12}$$

Meanwhile, it follows from (3.2) that there exists a constant $\tilde{\delta} > \delta$ such that $-\frac{P}{\tilde{\delta}} + r^2K^*PK \leq 0$. Then,

$$K^*PK \leq \frac{P}{r^2\tilde{\delta}} \leq \frac{P}{r_k^2\tilde{\delta}}.$$

It can be obtained that

$$V(t_k) \leq \mu_k \int_{t_k-r_k}^{t_k} e^*(s)Pe(s)\,ds = \mu_k \int_{t_k-r_k}^{t_k} V(s)\,ds,$$

where $\mu_k = \frac{1}{r_k\tilde{\delta}}$. Noting that $\mu_k r_k \delta = \frac{\delta}{\tilde{\delta}} < 1$, we have

$$\inf_{k\in\mathbb{Z}_+}\Big\{-\frac{\ln(\mu_k r_k \delta)}{r_k}\Big\} = \frac{\ln\frac{\tilde{\delta}}{\delta}}{r} > 0.$$

Therefore, the conditions of Lemma 1 are satisfied and we get

$$V(t) \leq \delta\overline{V}(0)\exp(-\lambda t) < \delta\lambda_{\max}(P)\|\varphi_\alpha - \phi\|_\rho^2 \exp(-\lambda(t-\alpha)),\, t \geq \alpha,$$

where $\lambda \in (0, \frac{\ln\frac{\tilde{\delta}}{\delta}}{r})$. From (3.5), one can derive that

$$V(t) \geq \lambda_{\min}(P)\|e(t)\|^2 = \lambda_{\min}(P)\|y(t) - x(t-\alpha)\|^2,\ t \geq \alpha.$$

Hence,

$$\|y(t) - x(t-\alpha)\| \leq M\|\varphi_\alpha - \phi\|_\rho \exp\Big(-\frac{\lambda}{2}(t-\alpha)\Big),\ t \geq \alpha,$$

where $M = (\frac{\delta\lambda_{\max}(P)}{\lambda_{\min}(P)})^{\frac{1}{2}}$. This means that the drive system (2.1) is exponentially lag synchronized with the response system (2.2) over the set $\mathcal{A}(\zeta)$, which completes the proof. □

Remark 2. *Recently, there have been a lot of interesting works on the impulsive control synchronization of CVNNs [21–23, 26]. However, these results are concerned with either delay-independent impulsive control or delayed impulsive control involving discrete delays. In this paper, the distributed delayed impulsive control is proposed to synchronize the driver and response systems. It is an important type of delayed impulsive control that can be employed to stabilize a system based on the accumulation of the states over a history time period, not just the states at certain history instant. This feature enables the distributed delayed impulsive control to be more applicable for many industrial and biological systems such as network connection [28], the spread of disease [29] and epidemic model [30].*

Remark 3. *It should be mentioned that three types of delays $\tau(t)$, $\beta(t)$ and r_k are involved in the closed-loop systems (2.5), which brings certain difficulties to the research. In Theorem 1, a novel impulsive differential inequality combined with Lyapunov function is adopted to overcome these difficulties. The obtained results require neither the differentiability of time-varying delays nor the restriction on the relationship between delays $\tau(t)$, $\beta(t)$ and r_k, which makes our results posses better applications.*

Remark 4. *By the separation approach, the impulsive control synchronization problem for CVNNs has been discussed in [10, 21, 22, 41]. However, this kind of approach is invalid when the activation functions cannot be divided into their real parts and imaginary parts in an analytical form. Here, instead of the separation approach, we retain the complex nature of NNs and explore the synchronization criteria by utilizing the Lyapunov function in the complex field. Only the Lipschitz condition on the activation functions is assumed, which broadens the applications of our results.*

Remark 5. *In [10, 26, 41, 42], the synchronization of delayed CVNNs have been investigated via impulsive control. While in this paper, using a different impulsive control scheme, we extend the previous results from the following aspects: Firstly, the mixed time-delays in the models and the distributed delays in the impulsive controller are considered simultaneously in our work. Secondly, by introducing a novel differential inequality with the distributed delayed impulsive effects, the synchronization conditions are obtained in terms of complex-valued LMIs, which are less conservative than the existing ones. Finally, to govern more general real-world applications, the influence of interval parameter uncertainties and the lag effects of signal transmission during synchronization are also considered in our models.*

Remark 6. *It can be observed from Theorem 1 that the size of delays r_k may affect the convergence rate of synchronization. As the delays r_k increase, the convergence speed of synchronization will become slower.*

If the system considered only contains discrete time-delays, i.e. $D = 0$ in the system (2.1), we have the following corollary.

Corollary 1. *Under Assumption 1 and Assumption 2, if there exist constants $\sigma_1 \in \mathbb{R}$, $\sigma_2 > 0$, $\zeta > 0$, $\delta > 1$, $0 < \mu < 1$ and positive definite Hermitian matrix P, positive diagonal matrices $Q_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2)$, $Q_j \in \mathbb{R}^{n^2\times n^2}$ $(j = 4, 5, 6)$, matrix $Z \in \mathbb{C}^{n\times n}$ such that (2.7), (3.2), (3.3) and the following inequalities hold*

$$\begin{pmatrix} \Omega_{11} & PA_0 & PB_0 & P\Theta \\ \star & -Q_1 & 0 & 0 \\ \star & \star & -Q_2 & 0 \\ \star & \star & \star & \begin{pmatrix} -Q_4 & 0 & 0 \\ 0 & -Q_5 & 0 \\ 0 & 0 & -Q_6 \end{pmatrix} \end{pmatrix} \leq 0,$$

then the drive system (2.1) and the response system (2.2) are exponentially lag synchronized over the set $\mathcal{A}(\zeta)$ with control gain

$$K = P^{-1}Z^*.$$

In particular, if C, A, B and D are known constant matrices in the system (2.1), then we can get the following corollary based on Theorem 1.

Corollary 2. *Under Assumption 1, if there exist constants $\sigma_1 \in \mathbb{R}$, $\sigma_2 > 0$, $\zeta > 0$, $\delta > 1$, $0 < \mu < 1$, and positive definite Hermitian matrix P, positive diagonal matrices $Q_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2, 3)$, matrix*

$Z \in \mathbb{C}^{n\times n}$ such that (2.7) and the following inequalities hold

$$\begin{pmatrix} \Omega_{11} & PA & PB & PD \\ \star & -Q_1 & 0 & 0 \\ \star & \star & -Q_2 & 0 \\ \star & \star & \star & -Q_3 \end{pmatrix} \le 0,$$

$$\begin{pmatrix} -\frac{P}{\delta} & rZ \\ \star & -P \end{pmatrix} < 0,$$

$$LQ_2L \le \mu\sigma_2 P,$$

$$\beta^2 LQ_3L \le (1-\mu)\sigma_2 P,$$

where $\Omega_{11} = -PC - C^*P - \sigma_1 P + LQ_1L$, then the drive system (2.1) and the response system (2.2) are exponentially lag synchronized over the set $\mathcal{A}(\zeta)$ with control gain

$$K = P^{-1}Z^*.$$

4. Numerical example

In this section, a numerical example is given to verify the effectiveness of our main results. Consider the complex-valued interval networks (2.1) with the following parameters:

$$C \in \begin{pmatrix} [1, 1.2] & 0 \\ 0 & [0.8, 1.2] \end{pmatrix},$$

$$A \in \begin{pmatrix} [0.9 - 0.01i, 1.1 + i] & [-0.2 - 0.01i, 0.01i] \\ [-6 - 5.2i, -4 - 0.03i] & [3 - 0.03i, 3.2 + 2.1i] \end{pmatrix},$$

$$B \in \begin{pmatrix} [1.2 - 0.01i, 1.5 + 1.3i] & [-0.2 - 0.1i, 0.04i] \\ [-0.2 - 0.1i, 0.04i] & [-4.2 - 3.8i, -3.3 + 0.14i] \end{pmatrix},$$

$$D \in \begin{pmatrix} [-0.5 - 0.45i, -0.4 + 0.17i] & [-1 - 0.5i, 0.1i] \\ [-4 - 3i, -2 + 1.3i] & [-3.5 - 3.41i, -3.3 + 0.22i] \end{pmatrix},$$

and the initial conditions $\phi_1(t) = 1 + i$, $\phi_2(t) = -1 - i$, $t \in [-0.4, 0]$. Select the activation functions $f_1(s) = f_2(s) = \tanh(Re(s)) + i\tanh(Im(s))$, time-delays $\tau(t) = 0.39 - 0.01\sin(t)$, $\beta(t) = 0.38 + 0.01\cos(t)$ and the external input $J = (0, 0)^T$. The initial conditions in the response system (2.2) are chosen as $\varphi_{\alpha 1}(t) = -1.5 - 1.5i$, $\varphi_{\alpha 2}(t) = 2 + 2i$, $t \in [-0.4, 0]$. Then it is easy to obtain that $L = I_{2\times 2}$,

$$C_0 = \begin{pmatrix} 1.1 & 0 \\ 0 & 1 \end{pmatrix},\ A_0 = \begin{pmatrix} 1 + 0.495i & -0.1 \\ -5 - 2.615i & 3.1 + 1.035i \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1.35 + 0.645i & -0.1 - 0.03i \\ -0.1 - 0.03i & -3.75 - 1.83i \end{pmatrix},\ D_0 = \begin{pmatrix} -0.45 - 0.14i & -0.5 - 0.2i \\ -3 - 0.85i & -3.4 - 1.595i \end{pmatrix},$$

$$\Theta_1 = \begin{pmatrix} \sqrt{0.1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{0.2} \end{pmatrix},$$

$$\Theta_2 = \begin{pmatrix} 0.5544+0.4554i & 0.3166+0.0158i & 0 & 0 \\ 0 & 0 & 1.3733+0.9412i & 0.7647+0.6963i \end{pmatrix},$$

$$\Theta_3 = \begin{pmatrix} 0.6411+0.5109i & 0.3332+0.105i & 0 & 0 \\ 0 & 0 & 0.3332+0.105i & 1.1115+0.8862i \end{pmatrix},$$

$$\Theta_4 = \begin{pmatrix} 0.4266+0.3633i & 0.7359+0.2038i & 0 & 0 \\ 0 & 0 & 1.2983+0.828i & 0.9792+0.9268i \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} \sqrt{0.1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{0.2} \end{pmatrix}^T,$$

$$\Phi_2 = \begin{pmatrix} 0.5544-0.4554i & 0 & 1.3733-0.9412i & 0 \\ 0 & 0.3166-0.0158i & 0 & 0.7647-0.6963i \end{pmatrix}^*,$$

$$\Phi_3 = \begin{pmatrix} 0.6411-0.5109i & 0 & 0.3332-0.105i & 0 \\ 0 & 0.3332-0.105i & 0 & 1.1115-0.8862i \end{pmatrix}^*,$$

$$\Phi_4 = \begin{pmatrix} 0.4266-0.3633i & 0 & 1.2983-0.828i & 0 \\ 0 & 0.7359-0.2038i & 0 & 0.9792-0.9268i \end{pmatrix}^*.$$

We consider the control input $u(t)$ with distributed delay $r_k = 0.4$ and impulsive interval $t_{k+1} - t_k = 0.03$, $k \in \mathbb{Z}_+$. Choose $\sigma_1 = 70.3$, $\sigma_2 = 92$, $\delta = 2.65$, $\mu = 0.8$, $\beta = 0.39$, $\rho = 0.4$ and $\alpha = 0.2$. Based on Theorem 1 and Matlab LMI toolbox, the following feasible solutions can be obtained

$$P = \begin{pmatrix} 0.0557 & 0 \\ 0 & 0.0535 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1.0153 & 0 \\ 0 & 1.0153 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 1.1931 & 0 \\ 0 & 1.1931 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1.0285 & 0 \\ 0 & 1.0285 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 1.0802 & 0 & 0 & 0 \\ 0 & 1.0802 & 0 & 0 \\ 0 & 0 & 1.0802 & 0 \\ 0 & 0 & 0 & 1.0802 \end{pmatrix},$$

$$Q_5 = \begin{pmatrix} 0.8167 & 0 & 0 & 0 \\ 0 & 0.8167 & 0 & 0 \\ 0 & 0 & 0.8167 & 0 \\ 0 & 0 & 0 & 0.8167 \end{pmatrix},$$

$$Q_6 = \begin{pmatrix} 1.031 & 0 & 0 & 0 \\ 0 & 1.031 & 0 & 0 \\ 0 & 0 & 1.031 & 0 \\ 0 & 0 & 0 & 1.031 \end{pmatrix},$$

$$Q_7 = \begin{pmatrix} 1.0157 & 0 & 0 & 0 \\ 0 & 1.0157 & 0 & 0 \\ 0 & 0 & 1.0157 & 0 \\ 0 & 0 & 0 & 1.0157 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0.0167 - 0.0006i & 0.0013 - 0.0053i \\ 0.0073 - 0.0028i & 0.0214 - 0.0059i \end{pmatrix}.$$

Then impulsive control gain matrix K is designed as follows

$$K = P^{-1}Z^{*} = \begin{pmatrix} 0.3 + 0.01i & 0.13 + 0.05i \\ 0.025 + 0.1i & 0.4 + 0.11i \end{pmatrix}.$$

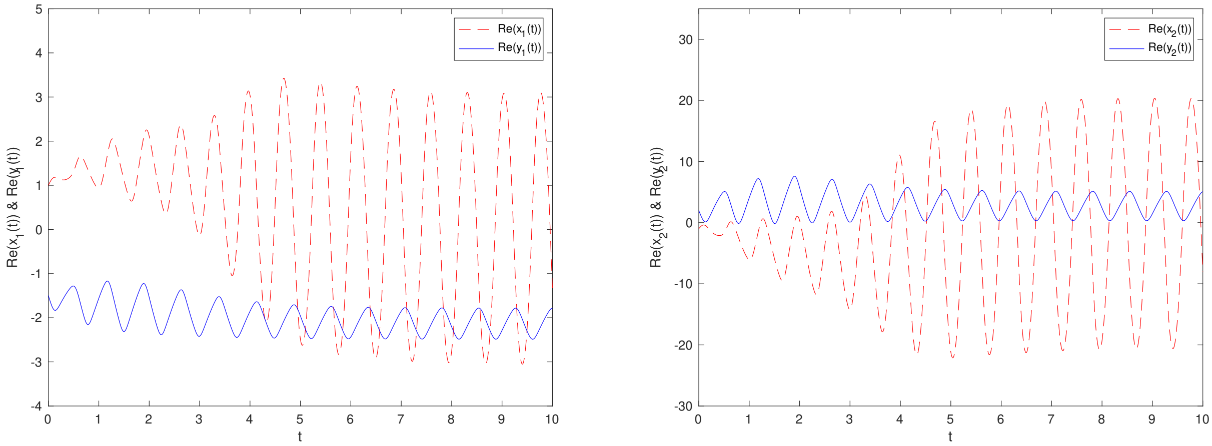

Figure 1. State trajectories of real parts of the drive system (2.1) and the response system (2.2) without impulsive control.

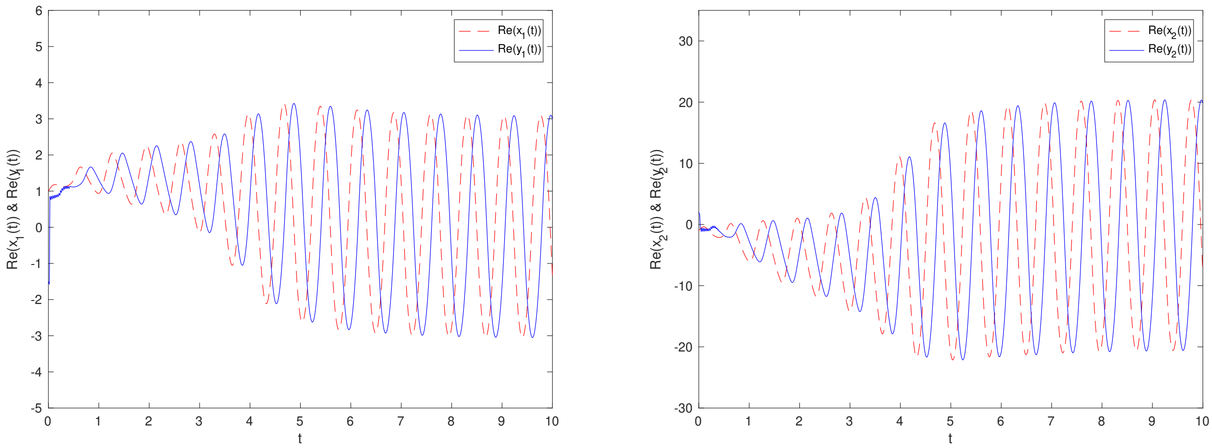

Figure 2. State trajectories of real parts of the drive system (2.1) and the response system (2.2) with impulsive control ($\alpha = 0.2$).

Take $C = C_0$, $A = A_0$, $B = B_0$, $D = D_0$. When the control input $u(t) = 0$, we can see that the drive system (2.1) and the response system (2.2) cannot achieve exponential lag synchronization from Figure 1, Figure 3 and Figure 5. Under the distributed delayed impulsive control, when $\alpha = 0.2$, the state trajectories of the real parts and the imaginary parts of vectors x and y are illustrated separately in Figure 2 and Figure 4, while the state trajectories of the error system (2.5) is shown in Figure 6. According to the above simulation results, it can be clearly seen that the drive system (2.1) and the response system (2.2) can be exponentially lag synchronized by an appropriate delayed impulsive controller over the set $\mathcal{A}_\xi$ ($\xi = 0.03$).

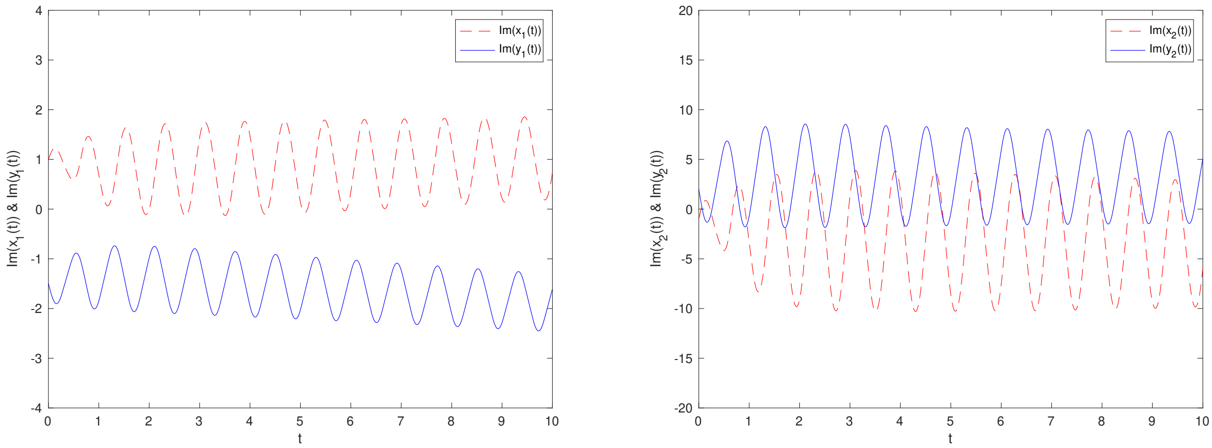

Figure 3. State trajectories of imaginary parts of the drive system (2.1) and the response system (2.2) without impulsive control.

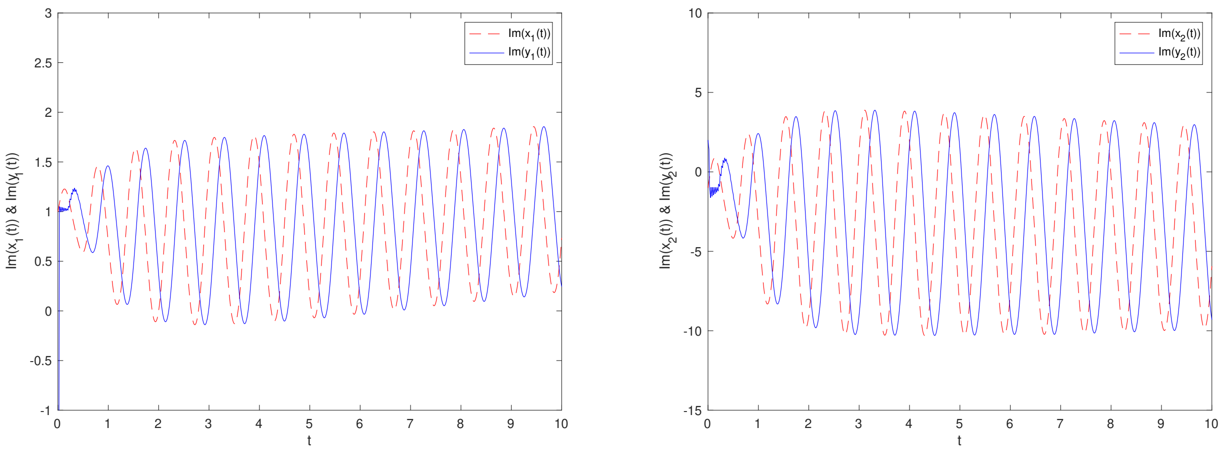

Figure 4. State trajectories of imaginary parts of the drive system (2.1) and the response system (2.2) with impulsive control ($\alpha = 0.2$).

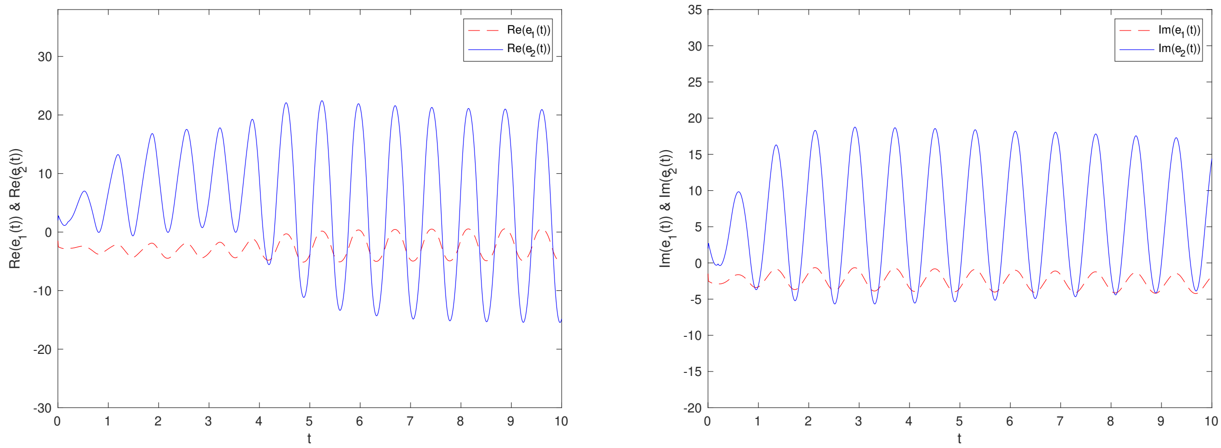

Figure 5. State trajectories of real parts and imaginary parts of the error system (2.5) without impulsive control.

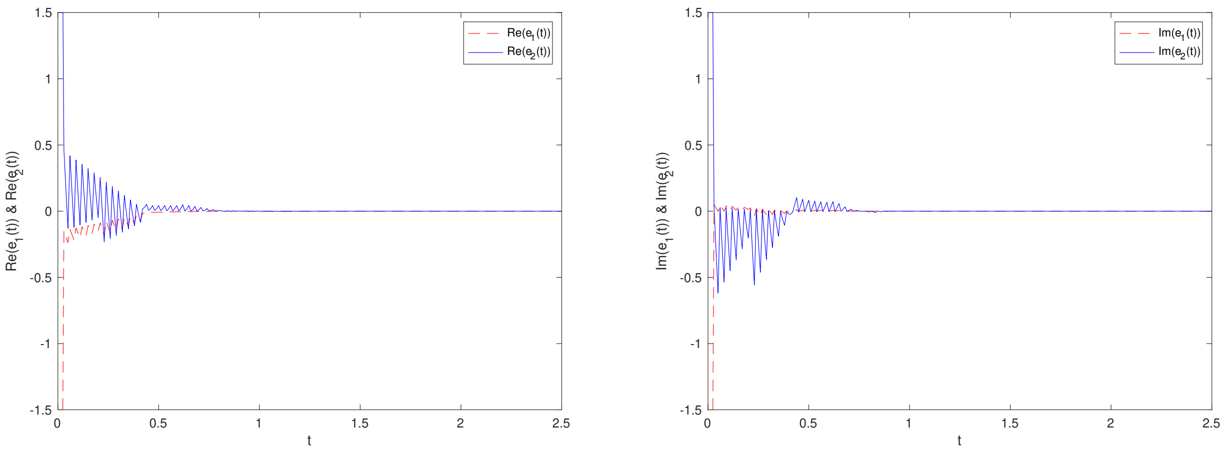

Figure 6. State trajectories of real parts and imaginary parts of the error system (2.5) with impulsive control ($\alpha = 0.2$).

5. Conclusions

In this paper, the exponential lag synchronization of complex-valued interval neural networks has been investigated under distributed delayed impulsive control. Both discrete and distributed time-varying delays were considered in the model, which is more general than the previous works. A new impulsive differential inequality was applied to resolve the difficulties caused by the mixed time-delays and distributed delayed impulse effects. Instead of the separation approach, some synchronization criteria were derived based on the complex Lyapunov method, where only the Lipschitz condition on the activation functions need to be assumed. All the synchronization criteria were formulated in the form of complex-valued LMIs, which were easily checked by using the Matlab LMI toolbox. Finally, a numerical example was given to illustrate the validity of the proposed results. In the future, we will consider finite-time synchronization problem of complex-valued interval neural networks via delayed impulsive control.

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Conflict of interest

The authors declare no conflict of interest.

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