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Research article

Results on multiple nontrivial solutions to partial difference equations

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Abstract: In this paper, we consider the existence and multiplicity of nontrivial solutions to second order partial difference equation with Dirichlet boundary conditions by Morse theory. Given suitable conditions, we establish multiple results that the problem admits at least two nontrivial solutions. Moreover, we provide five examples to illustrate applications of our theorems.

Keywords: partial difference equation; local linking; Morse theory; nontrivial solution
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1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be natural number set and integer set, respectively. For integers a, b, define the discrete interval $\mathbb{Z}(a,b) := \{a, a+1, \cdots, b\}$ for $a \le b$. Write $\Omega := \mathbb{Z}(1,T_1)\times\mathbb{Z}(1,T_2)$, where $T_1, T_2 \ge 2$ are given integers. We consider the existence and multiplicity of nontrivial solutions to the following nonlinear second order partial difference equation

$$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) = -f((i,j),u(i,j)), \qquad (i,j)\in\Omega, \tag{1.1}$$

with Dirichlet boundary conditions

$$u(i,0) = u(i,T_2+1) = 0 \quad i\in\mathbb{Z}(1,T_1), \qquad u(0,j) = u(T_1+1,j) = 0 \quad j\in\mathbb{Z}(1,T_2), \tag{1.2}$$

where Δ_1, Δ_2 are the forward difference operators defined by $\Delta_1 u(i,j) = u(i+1,j) - u(i,j)$ and $\Delta_2 u(i,j) = u(i,j+1) - u(i,j)$. $\Delta^2 u(i,j) = \Delta(\Delta u(i,j))$. Here $f((i,j),\cdot) \in C^1(\Omega\times\mathbb{R},\mathbb{R})$ satisfies $f((i,j),0) = 0$, which means that (1.1) and (1.2) possesses a trivial solution $u = 0$. Meanwhile, We are interested in nontrivial solutions and intend to seek nontrivial solutions to (1.1) and (1.2).

Due to wide applications in many fields such as computer science, economics and mechanical engineering, the theory of nonlinear discrete problems has been widely studied and many results are obtained [1–6]. With the rapid development of modern computer technology, more and more mathematical models involve functions with two or more variables. Partial difference equations, involving two or more discrete variables, are regarded as discrete analogs of partial differential equations. Therefore, the study of difference equations has gradually shifted to the study of partial difference equations and attracted much attentions, for example, [7–14].

As known to all, with the rapid development of critical point theory, the Morse theory becomes a more and more powerful tool to study the multiplicity and existence of nontrivial solutions to both differential equations and difference equations having variational structure [15–18]. Very recently, [19–21] studied partial difference equations via the Morse theory and obtained rich results on the existence and multiplicity of nontrivial solutions. Thus those reasons are encouraging us to consider the existence and multiplicity of nontrivial solutions to (1.1) and (1.2) by the Morse theory.

We organize this paper as follows. In Section 2, the variational structure and the corresponding functional are established. Moreover, we also recall some related definitions and propositions, which are necessary to our main results. Section 3 states our main results and their detailed proofs. Finally, five examples and numerical simulations are provided to demonstrate applications of our main results in Section 4.

2. Variational structure and some auxiliary results

Let E be a T_1T_2-dimensional Euclidean space equipped with the usual inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Let

$$\begin{aligned} S =&\{u:\mathbb{Z}(0,T_1+1)\times\mathbb{Z}(0,T_2+1)\to\mathbb{R} \quad \text{such that} \quad u(i,0)=u(i,T_2+1)=0,\\ & i\in\mathbb{Z}(0,T_1+1) \quad \text{and} \quad u(0,j)=u(T_1+1,j)=0, \quad j\in\mathbb{Z}(0,T_2+1)\}. \end{aligned}$$

Define the inner product $\langle\cdot,\cdot\rangle$ on S as

$$\langle u,v\rangle=\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2}\Delta_1u(i-1,j)\Delta_1v(i-1,j)+\sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1}\Delta_2u(i,j-1)\Delta_2v(i,j-1), \qquad \forall u,v\in S.$$

Then the induced norm is

$$\|u\|=\sqrt{\langle u,u\rangle}=\left(\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2}|\Delta_1u(i-1,j)|^2+\sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1}|\Delta_2u(i,j-1)|^2\right)^{\frac{1}{2}}, \quad \forall u\in S.$$

Thus S is a Hilbert space and isomorphic to E. Here and hereafter, we take $u\in S$ an extension of $u\in E$ if necessary.

Consider the functional $J:S\to\mathbb{R}$ as

$$\begin{aligned} J(u)=&\frac{1}{2}\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2}|\Delta_1u(i-1,j)|^2+\frac{1}{2}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1}|\Delta_2u(i,j-1)|^2-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}F((i,j),u(i,j))\\ =&\frac{1}{2}\|u\|^2-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}F((i,j),u(i,j)), \qquad \forall u\in S, \end{aligned} \tag{2.1}$$

where $F((i,j),u) = \int_0^u f((i,j),\tau)d\tau$ for each $(i,j) \in \Omega$. Notice that $f((i,j),u)$ is continuously differentiable with respect to u. Therefore, the expression of J means that $J \in C^2(S,\mathbb{R})$ and solutions of the problems (1.1) and (1.2) are precisely critical points of $J(u)$. Moreover, for any u, $v \in S$, applying Dirichlet boundary conditions, direct computations gives that the Fréchet derivative of $J(u)$ is

$$\begin{aligned}\langle J'(u), v\rangle = &\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2} \Delta_1 u(i-1,j)\Delta_1 v(i-1,j) + \sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1} \Delta_2 u(i,j-1)\Delta_2 v(i,j-1)\\ &- \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} f((i,j),u(i,j))v(i,j) \\ = &- \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left[\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + f((i,j),u(i,j))\right] v(i,j).\end{aligned} \tag{2.2}$$

Let Ξ be the discrete Laplacian, which is defined by $\Xi u(i,j) = \Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1)$. Owe to [11], $-\Xi$ is invertible and the distinct eigenvalues of $-\Xi$ with zero Dirichlet boundary conditions on Ω can be denoted by $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_{T_1T_2}$. Let $\phi_k = (\phi_k(1), \phi_k(2), \cdots, \phi_k(T_1T_2))^{tr}$, $k \in [1, T_1T_2]$ be an eigenvector corresponding to the eigenvalue λ_k. Write

$$W^- = \text{span}\{\phi_1, \cdots, \phi_{k-1}\}, \qquad W^0 = \text{span}\{\phi_k\}, \qquad W^+ = (W^- \oplus W^0)^\perp.$$

Then S can be expressed in the form as

$$S = W^- \oplus W^0 \oplus W^+.$$

For later use, define another norm as

$$\|u\|_2 = \left(\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} |u(i,j)|^2\right)^{\frac{1}{2}}, \qquad u \in S.$$

Then for any $u \in S$, we have

$$\lambda_1\|u\|_2^2 \le \|u\|^2 \le \lambda_{T_1T_2}\|u\|_2^2. \tag{2.3}$$

In particular, we have

$$\begin{aligned} &\lambda_{k+1}\|u\|_2^2 \le \|u\|^2 \le \lambda_{T_1T_2}\|u\|_2^2, \qquad u \in W^+,\\ &\lambda_1\|u\|_2^2 \le \|u\|^2 \le \lambda_{k-1}\|u\|_2^2, \qquad u \in W^-. \end{aligned} \tag{2.4}$$

Now we recall some basic results on the Morse theory.

We say that the functional J satisfies the Palais-Smale condition ((PS) in short) if any sequence $\{u_n\} \subseteq S$, there is a constant $M > 0$ such that $|J(u_n)| \le M$, $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. From [22, 23], if (PS) is satisfied, then both the weaker Cerami condition ((C) for short) and the deformation condition ((D) in short) are also fulfilled.

Definition 2.1. *[16, 24] Let u_0 be an isolated critical group of J with $J(u_0) = c \in \mathbb{R}$, and U be a neighborhood of u_0, the group*

$$C_q(J, u_0) := H_q(J^c \cap U, J^c \cap U \backslash u_0), \quad q \in \mathbb{Z},$$

is called the q-th critical group of J at u_0. Let $\kappa = \{u \in S | J'(u) = 0\}$. For all $a \in \mathbb{R}$ each critical point of J is greater than a and $J \in C^2(S, \mathbb{R})$ satisfies (D), the group

$$C_q(J, \infty) := H_q(S, J^a), \quad q \in \mathbb{Z},$$

is called the q-th critical group of J at infinity.

To compute critical groups of J at both an isolated critical point and infinity, the following auxiliary propositions are needed.

Proposition 2.1. *[16, 24] Suppose that u_0 is an isolated critical point of J with finite Morse index $\mu(u_0)$ and zero nullity $\nu(u_0)$. Then*
(Q_1) $\quad C_q(J, u_0) \cong 0$ *for* $q \notin [\mu(u_0), \mu(u_0) + \nu(u_0)]$*;*
(Q_2) $\quad C_q(J, u_0) \cong \delta_{q,u_0}\mathbb{Z}$, $q \in \mathbb{Z}$, *if u_0 is nondegenerate;*
(Q_3) $\quad C_q(J, u_0) \cong \delta_{q,k}\mathbb{Z}$ *for* $k = \mu(u_0)$ *or* $k = \mu(u_0) + \nu(u_0)$*, if* $C_k(J, u_0) \ncong 0$.

Proposition 2.2. *[17] Let $J \in C^2(S, \mathbb{R})$ satisfy (D). We have*
(Q_4) $\quad$ *if $C_q(J, \infty) \ncong 0$ holds for some q, then J possesses a critical point u such that $C_q(J, u) \ncong 0$;*
(Q_5) $\quad$ *if 0 is the isolated critical point of J and $C_q(J, \infty) \ncong C_q(J, 0)$ holds for some q, then J has a non-zero critical point.*

Proposition 2.3. *[25] Suppose that S is a Hilbert space. For all $t \in [0, 1]$, $J_t \in C^2(S, \mathbb{R})$ is a functional satisfying J_t' and $\partial_t J_t$ are locally continuous. If J_0 and J_1 satisfy (C), and there exist $a \in \mathbb{R}$ and $\delta > 0$ such that*

$$J_t(u) \leq a \Rightarrow (1 + \|u\|)\|J_t(u)\| \geq \delta, \qquad t \in [0, 1],$$

then

$$C_q(J_0, \infty) = C_q(J_1, \infty), \qquad q \in \mathbb{Z}. \tag{2.5}$$

In particular, if there is $R > 0$ such that

$$\inf_{t\in[0,1],\|u\|>R} (1 + \|u\|)\|J_t'(u)\| > 0, \tag{2.6}$$

and

$$\inf_{t\in[0,1],\|u\|\leq R} (1 + \|u\|)\|J_t'(u)\| > -\infty, \tag{2.7}$$

then (2.5) is satisfied.

Proposition 2.4. *[16] Let S be a real Hilbert space. $J \in C^1(S, \mathbb{R})$ satisfies*

$$J(u) = \frac{1}{2}\langle Tu, u\rangle + Q(u), \tag{2.8}$$

where $T : S \to S$ is a self-adjoint linear operator, and 0 is the isolated spectral point of T. Suppose $Q \in C^1(S, \mathbb{R})$ satisfies

$$\lim_{\|u\|\to\infty} \frac{\|Q'(u)\|}{\|u\|} = 0. \tag{2.9}$$

Let $W^+(W^-)$ be an invariant subspace corresponding to the positive (negative) of spectrum of T, which has a bounded inverse. Assume that $k = \dim W^-$ is finite, then J satisfies (PS) and

$$C_q(J,\infty) \cong \delta_{q,k}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

For the purpose to obtain the critical group at origin, the following proposition about local linking is important.

Proposition 2.5. *[26] Let 0 be an isolated critical point of J with Morse index μ_0 and nullity ν_0. Assume that J has a local linking at 0 subject to $S = S^- \oplus S^+$, $m = \dim S^- < \infty$, namely, there exists $\rho > 0$ such that*

$$\begin{aligned} J(u) &\le 0, \qquad u \in S^-, \qquad \|u\| \le \rho,\\ J(u) &\ge 0, \qquad u \in S^+, \qquad \|u\| \le \rho. \end{aligned}$$

Then if $m = \mu_0$ or $m = \mu_0 + \nu_0$, we get

$$C_q(J,0) \cong \delta_{q,m}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

3. Main results and proofs

In this section, we state our main results and provide detailed proofs. For convenience, we give some notations subject to our main results.

$$\alpha_\infty := f'((i,j),\infty) = \lim_{|u|\to\infty} \frac{f((i,j),u)}{u} \in \mathbb{R}, \qquad (i,j) \in \Omega, \tag{3.1}$$

and

$$\alpha_0 := f'((i,j),0) = \lim_{|u|\to 0} \frac{f((i,j),u)}{u} \in \mathbb{R}, \qquad (i,j) \in \Omega. \tag{3.2}$$

Moreover, for all $(i,j) \in \Omega$, we make the following assumptions:
($\mathbf{I_1}$) $\alpha_0 < \lambda_1$;
($\mathbf{I_2}$) $\alpha_\infty > \lambda_{T_1T_2}$;
($\mathbf{I_3}$) $\lambda_p < \alpha_0 < \lambda_{p+1}$, $p \in \mathbb{Z}(1, T_1T_2 - 1)$;
($\mathbf{I_4}$) $\alpha_0 > \lambda_{T_1T_2}$;
($\mathbf{I_5}$) $\alpha_\infty < \lambda_1$;
($\mathbf{I_6}$) $\lambda_p < \alpha_\infty < \lambda_{p+1}$, $p \in \mathbb{Z}(1, T_1T_2 - 1)$;
($\mathbf{F_\infty^\pm}$) For $\forall (i,j) \in \Omega$, there exists $k \in \mathbb{Z}(2, T_1T_2 - 1)$ such that

$$\lim_{u\to+\infty} (f((i,j),u) - \lambda_k u) = \pm\infty, \qquad \lim_{u\to-\infty} (f((i,j),u) - \lambda_k u) = \mp\infty.$$

We are now in a position to state our main results as the following:

Theorem 3.1. *If one of the following conditions is satisfied:*
(i) **($\mathbf{I_1}$)**, **($\mathbf{I_2}$)** *or* **($\mathbf{I_6}$)** **(ii)** **($\mathbf{I_3}$)**, **($\mathbf{I_2}$)** *or* **($\mathbf{I_5}$)** **(iii)** **($\mathbf{I_4}$)**, **($\mathbf{I_5}$)** *or* **($\mathbf{I_6}$)**,
then (1.1) and (1.2) possesses at least two nontrivial solutions.

Theorem 3.2. *Suppose that $\alpha_\infty = \lambda_k$. If T_1T_2 is odd, then (1.1) and (1.2) has at least two nontrivial solutions provided one of the following conditions is fulfilled:*
(**i**) $(\mathbf{I_1})$ (**ii**) $(\mathbf{I_4})$ (**iii**) $(\mathbf{I_3})$ *with* $p \neq \frac{T_1T_2}{2}$.

Theorem 3.3. *Let $(\mathbf{F}^+_\infty)[(\mathbf{F}^-_\infty)]$ hold and $\alpha_\infty = \lambda_k$. Then (1.1) and (1.2) admits at least two nontrivial solutions provided one of the following conditions is met:*
(**i**) $(\mathbf{I_1})$ (**ii**) $(\mathbf{I_4})$ (**iii**) $(\mathbf{I_3})$ *with* $p \neq k[p \neq k-1]$.

Given the following sign conditions:
$(\mathbf{F}^+_\mathbf{0})$ there exist $m \in \mathbb{Z}(1, T_1T_2 - 1)$ and $\delta > 0$ such that

$$2F((i, j), u) - \lambda_m u^2 > 0, \qquad |u(i, j)| \leq \delta, \qquad (i, j) \in \Omega,$$

$(\mathbf{F}^-_\mathbf{0})$ there exist $m \in \mathbb{Z}(2, T_1T_2)$ and $\delta > 0$ such that

$$2F((i, j), u) - \lambda_m u^2 < 0, \qquad |u(i, j)| \leq \delta, \qquad (i, j) \in \Omega.$$

Then we have

Theorem 3.4. *Assume that $(\mathbf{F}^+_\mathbf{0})[(\mathbf{F}^-_\mathbf{0})]$ holds and $\alpha_0 = \lambda_m$. Then (1.1) and (1.2) possesses at least two nontrivial solutions if one of the following conditions is fulfilled:*
(**i**) $(\mathbf{I_5})$ (**ii**) $(\mathbf{I_2})$ (**iii**) $(\mathbf{I_6})$ *with* $p \neq m[p \neq m-1]$.

Theorem 3.5. *Let $\alpha_\infty = \lambda_k$ and $\alpha_0 = \lambda_m$. If either*
(**i**) $(\mathbf{F}^-_\mathbf{0})$, $(\mathbf{F}^+_\infty)$ *and* $m + 1 \neq k$, *or*
(**ii**) $(\mathbf{F}^+_\mathbf{0})$, $(\mathbf{F}^-_\infty)$ *and* $k + 1 \neq m$,
then (1.1) and (1.2) admits at least two nontrivial solutions.

To calculate the critical group at infinity under conditions of Theorems 3.1 and 3.4, we have the following lemma.

Lemma 3.1. *If $(\mathbf{I_5})$ or $(\mathbf{I_2})$ or $(\mathbf{I_6})$ is satisfied, then $C_q(J, \infty) \cong \delta_{q,k}\mathbb{Z}$, $q \in \mathbb{Z}$.*

Proof. Let α_s be a constant for $s \in \mathbb{Z}(1, T_1T_2)$ and denote

$$\lim_{|u|\to\infty} \frac{f((i, j), u)}{u} = \alpha_s, \qquad (i, j) \in \Omega. \tag{3.3}$$

Set

$$J(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i, j), u(i, j)) = \frac{1}{2}\langle Tu, u\rangle + Q(u),$$

where $Q(u) = \frac{1}{2}\langle \Lambda u, u\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i, j), u(i, j))$. Then $Q'(u)$ is compact and $T : S \to S$ is a self-adjoint bounded linear operator such that 0 is not in the spectrum of T. Thus $T^\pm = T|_{W^\pm}$ has bounded inverse on $W^\pm$. Moreover, $k = \dim W^- = 0$ if $(\mathbf{I_5})$ is satisfied, $k = T_1T_2$ if $(\mathbf{I_2})$ is satisfied and $k = p$ if $(\mathbf{I_6})$ is satisfied. Together with (3.3), it yields that (2.9) is fulfilled. As a matter of fact, using (3.3), we obtain

$$\lim_{|u|\to\infty} \frac{f((i, j), u) - \alpha_s u}{u} := \lim_{|u|\to\infty} \frac{\widetilde{f}((i, j), u)}{u} = 0, \qquad \forall (i, j) \in \Omega.$$

Thus for any $\varepsilon > 0$, there exists $R > 0$ such that

$$|\widetilde{f}((i,j),u)| < \frac{\sqrt{\lambda_1}\varepsilon}{\sqrt{2}}|u(i,j)|, \qquad (i,j)\in\Omega, \qquad |u(i,j)| > R. \tag{3.4}$$

Thanks to the continuity of $\widetilde{f}((i,j),u)$, there exists some $M_\varepsilon > 0$ such that

$$|\widetilde{f}((i,j),u)| \le M_\varepsilon := \max_{(i,j)\in\Omega,|u(i,j)|\le R}\{|\widetilde{f}((i,j),u)|\}. \tag{3.5}$$

If $\|u\| > \max\{\sqrt{T_1T_2\lambda_{T_1T_2}}R, \frac{\sqrt{2T_1T_2}M_\varepsilon}{\varepsilon}\}$, (2.3) implies that $|u(i,j)| > R$ for any $(i,j)\in\Omega$. Consequently,

$$\begin{aligned}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\widetilde{f}^2((i,j),u(i,j)) =& \sum_{|u(i,j)|\le R}\widetilde{f}^2((i,j),u(i,j)) + \sum_{|u(i,j)|>R}\widetilde{f}^2((i,j),u(i,j))\\ <& T_1T_2M_\varepsilon^2 + \frac{\varepsilon^2\lambda_1}{2}\|u\|_2^2 \le T_1T_2M_\varepsilon^2 + \frac{\varepsilon^2}{2}\|u\|^2 \le \varepsilon^2\|u\|^2,\end{aligned}$$

which ensures that (2.9) is valid. By Proposition 2.4, we conclude that $C_q(J,\infty)\cong\delta_{q,k}\mathbb{Z}$, $q\in\mathbb{Z}$.

In the following Lemmas 3.2 and 3.3, we calculate critical groups at both infinity and origin to make preparations for the proof of Theorem 3.3.

Lemma 3.2. *Assume* $\alpha_\infty = \lambda_k$. *Then*
(1) $C_q(J,\infty)\cong\delta_{q,k-1}\mathbb{Z}$, $q\in\mathbb{Z}$ *if* $(\mathbf{F}_\infty^-)$ *holds;*
(2) $C_q(-J,\infty)\cong\delta_{q,T_1T_2-k}\mathbb{Z}$, $q\in\mathbb{Z}$ *if* $(\mathbf{F}_\infty^+)$ *is valid.*

Proof. We prove the case **(1)** at length. The proof of **(2)** is similar and is omitted for brevity.

For $t\in[0,1]$, consider

$$\widehat{J}(u) = \|u^+\|^2 + \|u^0\|^2 - \|u^-\|^2, \qquad u^+\in W^+, \quad u^-\in W^-, \quad u^0\in W^0.$$

Define $J_t : S\to\mathbb{R}$ as

$$J_t(u) = (1-t)J(u) + t\widehat{J}(u), \qquad u\in S. \tag{3.6}$$

In order to apply Proposition 2.3, we need to prove that there exist $a\in\mathbb{R}$ and $\delta > 0$ such that

$$J_t(u)\le a \Rightarrow \|J_t'(u)\|\ge\delta, \qquad t\in[0,1]. \tag{3.7}$$

Otherwise, there exist $\{u_n\}\subseteq S$, $t_n\in[0,1]$ such that

$$J_{t_n}(u_n)\le -n, \qquad \|J_{t_n}'(u_n)\| < \frac{1}{n},$$

that is,

$$-J_{t_n}(u_n)\to+\infty, \qquad J_{t_n}'(u_n)\to 0. \tag{3.8}$$

Note

$$\begin{aligned}|-J_{t_n}(u_n)| =& |(t_n-1)J(u_n) - t_n\widehat{J}(u_n)| \le |(t_n-1)J(u_n)| + |t_n\widehat{J}(u_n)|\\ \le& |t_nJ(u_n)| + |J(u_n)| + |t_n\widehat{J}(u_n)| \le 2|J(u_n)| + |\widehat{J}(u_n)| \le 2|J(u_n)| + \|u_n\|^2.\end{aligned}$$

If $\{u_n\}$ is bounded, for J is continuous, then there exists $M > 0$ such that $\|J(u_n)\| \leq M\|u_n\|$, which leads to

$$\| - J_{t_n}(u_n)\| \leq (2M + 1)\|u_n\|. \tag{3.9}$$

Obviously, (3.9) is inconsistent with (3.8). Thus, $\|u_n\| \to \infty$.

Define a bilinear function

$$\sigma(u, v) = \lambda_k \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (u(i, j), v(i, j)), \qquad u, v \in S.$$

Since $|\sigma(u, v)| \leq \frac{\lambda_k}{\lambda_1}\|u\|\|v\|$, there exists an unique continuous linear operator $K : S \to S$ such that

$$\langle Ku, v\rangle = \lambda_k \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (u(i, j), v(i, j)).$$

Let $g((i, j), u) = f((i, j), u) - \lambda_k u$, where $G((i, j), u) = \int_0^u g((i, j), \tau)d\tau = F((i, j), u) - \frac{1}{2}\lambda_k u^2$. Then for any u, v in S,

$$\begin{aligned} \langle J'(u), v\rangle =&\langle u, v\rangle - \lambda_k\langle u, v\rangle - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} g((i, j), u(i, j))v(i, j) \\ =&\langle (I - K)u, v\rangle - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} g((i, j), u(i, j))v(i, j), \end{aligned} \tag{3.10}$$

and $\partial_t J_t = -J(u) + \widehat{J}(u)$ is locally continuous. Denoted by $\widehat{u} = u^+ + u^0 - u^-$, then (3.10) can be rewritten as

$$\langle J'(u), \widehat{u}\rangle = \langle (I - K)u, \widehat{u}\rangle - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} g((i, j), u(i, j))\widehat{u}(i, j). \tag{3.11}$$

By the definition of $\widehat{u}$, we have

$$\begin{aligned} \langle (I - K)u, \widehat{u}\rangle =&\langle (I - K)u^+ + u^0 + u^-, u^+ + u^0 - u^-\rangle \\ =&\|u^+\|^2 - \lambda_k\|u^+\|_2^2 + \|u^0\|^2 - \lambda_k\|u^0\|_2^2 - \|u^-\|^2 + \lambda_k\|u^-\|_2^2 \\ \geq&\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|u^+\|^2 + \left(\frac{\lambda_k}{\lambda_{k-1}} - 1\right)\|u^-\|^2. \end{aligned}$$

In view of $\alpha_\infty = \lambda_k$ and $(\mathbf{F}_\infty^-)$, there exist $0 < \varepsilon < \frac{\lambda_k}{\lambda_{k-1}} - 1$, $R_1 > 0$ such that

$$-\lambda_1 \cdot \varepsilon < \frac{g((i, j), u)}{u} \leq 0, \qquad |u(i, j)| > R_1, \quad (i, j) \in \Omega.$$

Moreover,

$$g((i, j), u)\widehat{u} = \frac{g((i, j), u)}{u} u\widehat{u} = \frac{g((i, j), u)}{u}[(u^+ + u^0)^2 - (u^-)^2] < \lambda_1\varepsilon(u^-)^2.$$

Consequently,

$$\begin{aligned}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} g((i,j),u(i,j)\widehat{u}(i,j) =& \sum_{|u(i,j)|>R_1} g((i,j),u(i,j)\widehat{u}(i,j) + \sum_{|u(i,j)|\le R_1} g((i,j),u(i,j)\widehat{u}(i,j)\\ <&\lambda_1\varepsilon\sum_{|u(i,j)|>R_1}(u^-(i,j))^2 + C_1\sum_{|u(i,j)|\le R_1}|\widehat{u}(i,j)|\\ \le&\lambda_1\varepsilon\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}(u^-(i,j))^2 + C_1\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|\widehat{u}(i,j)|\\ \le&\lambda_1\varepsilon\|u^-\|_2^2 + C_1\|\widehat{u}\|_2 = \lambda_1\varepsilon\|u^-\|_2^2 + C_1\|u\|_2\\ \le&\varepsilon\|u^-\|^2 + \frac{C_1}{\sqrt{\lambda_1}}\|u\|,\end{aligned}$$

where $C_1 := \max\limits_{(i,j)\in\Omega,|u(i,j)|\le R_1}\{|g((i,j),u(i,j))|\}$. Denoted by $C_2 := \frac{C_1}{\sqrt{\lambda_1}}$ and $C_3 := \min\{1-\frac{\lambda_k}{\lambda_{k+1}}, \frac{\lambda_k}{\lambda_{k-1}}-1-\varepsilon\}$, from (3.11), we obtain

$$\begin{aligned}\langle J'(u),\widehat{u}\rangle \ge& \left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u^+\|^2 + \left(\frac{\lambda_k}{\lambda_{k-1}}-1-\varepsilon\right)\|u^-\|^2 - C_2\|u\|\\ \ge& C_3(\|u^+\|^2+\|u^-\|^2) - C_2\|u\|.\end{aligned}$$

Define $P^-: S \to W^-$ as $P^-u = u^-$. Then

$$\begin{aligned}&\widehat{J}(u) = \langle u,u\rangle - 2\langle P^-u,u\rangle = \langle (I-2P^-)u,u\rangle,\\ &\widehat{J}'(u) = 2(I-2P^-)u, \qquad \widehat{J}''(u) = 2(I-2P^-).\end{aligned}$$

In the following, our aim is to show $t_n \to 0$ as $n\to\infty$. Or else, there exists $t_0>0$ such that $t_n \ge t_0$ and $-C_2(1-t_n) \ge -C_2(1-t_0)$. Define $C_4 := C_2(1-t_0)$, since $J'_{t_n}(u_n)\to 0$ as $\|u_n\|\to\infty$, there exists some $R_2>0$ such that

$$\begin{aligned}\|u_n\| \ge& \langle J'_{t_n}(u_n),\widehat{u_n}\rangle = (1-t_n)\langle J'(u_n),\widehat{u_n}\rangle + 2t_n\|u_n\|^2\\ \ge&(1-t_n)C_3(\|u^+\|^2+\|u^-\|^2) - C_4\|u_n\| + 2t_n\|u_n\|^2 \qquad \text{as} \quad |u(i,j)|>R_2,\end{aligned}$$

which implies

$$(1+C_4)\|u_n\| \ge (1-t_n)C_3(\|u^+\|^2+\|u^-\|^2) + 2t_n\|u_n\|^2 \ge 2t_n\|u_n\|^2.$$

Making use of (3.11), we obtain $t_n\to 0$ and

$$(1+C_4)\|u_n\| \ge C_3(\|u_n^+\|^2+\|u_n^-\|^2).$$

Therefore,

$$\frac{\|u_n^+\|^2}{\|u_n\|} + \frac{\|u_n^-\|^2}{\|u_n\|} \le \frac{1+C_4}{C_3},$$

which means both $\{\frac{\|u_n^+\|^2}{\|u_n\|}\}$ and $\{\frac{\|u_n^-\|^2}{\|u_n\|}\}$ are bounded, and

$$\frac{\|u_n^+\|^2}{\|u_n\|^2} + \frac{\|u_n^-\|^2}{\|u_n\|^2} \le \frac{1+C_4}{C_3}\frac{1}{\|u_n\|}. \tag{3.12}$$

Recall $\|u_n\| \to \infty$, (3.12) implies that

$$\frac{\|u_n^+\|}{\|u_n\|} \to 0 \qquad and \qquad \frac{\|u_n^-\|}{\|u_n\|} \to 0.$$

Joint with $\|u_n\|^2 = \|u_n^+\|^2 + \|u_n^0\|^2 + \|u_n^-\|^2$, we obtain $\frac{\|u_n^0\|}{\|u_n\|} \to 1$. Therefore, $\|u_n^+\|^2 + \|u_n^0\|^2 - \|u_n^-\|^2 > 0$, namely, $\widehat{J}(u_n) > 0$.

Since

$$\begin{aligned} J(u_n) =& \frac{1}{2}\|u_n\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)) = \frac{1}{2}\langle (I-K)u_n, u_n\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),u(i,j)) \\ \geq& \frac{1}{2}\left[\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u_n^+\|^2 + \left(\frac{\lambda_k}{\lambda_{k-1}}-1\right)\|u_n^-\|^2\right] - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),u(i,j)) \\ \geq& \frac{1}{2}\left(\frac{\lambda_k}{\lambda_{k-1}}-1\right)\|u_n^-\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),u(i,j)), \end{aligned}$$

then

$$\begin{aligned} \frac{1}{\|u_n\|}J(u_n) \geq& \frac{1}{2}\left(\frac{\lambda_k}{\lambda_{k-1}}-1\right)\frac{\|u_n^-\|^2}{\|u_n\|} - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\frac{1}{\|u_n\|}G((i,j),u(i,j)) \\ \geq& -C_5 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\frac{1}{\|u_n\|}G((i,j),u(i,j)), \end{aligned} \tag{3.13}$$

where $C_5 := \frac{1}{2}(\frac{\lambda_k}{\lambda_{k-1}} - 1)\frac{1+C_4}{C_3}$. Denote $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Hence up to a convergent subsequence, without loss of generality, we set the subsequence to be the subsequence, which means that there exists some $v \in S$ satisfying $\|v\| = 1$ such that $v_n \to v$ as $n \to \infty$.

Setting

$$\Theta := \{(i,j)|(i,j) \in \Omega, v(i,j) \neq 0\},$$

then $\Theta \neq \emptyset$. If $(i,j) \in \Theta$, then $u_n(i,j) = v_n(i,j)\cdot\|u_n\| \to \infty$ and $\lim\limits_{|u|\to\infty}\frac{-G((i,j),u)}{|u(i,j)|} = +\infty$. Therefore, for any $M_1 > 0$, there exists $N_2 > 0$ such that $\frac{-G((i,j),u)}{|u(i,j)|} > M_1$ as $n > N_2$. If $(i,j) \notin \Theta$, then $v_n(i,j) \to 0$. Since $\|u_n\| \to \infty$, there exist $C_6, N_3 > 0$ such that $\frac{-G((i,j),u)}{\|u_n\|} \geq -C_6$. Consequently,

$$\lim_{n\to\infty}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\frac{-G((i,j),u_n(i,j))}{\|u_n\|} = +\infty.$$

Combining with (3.13), we can deduce that $\frac{1}{\|u_n\|}J(u_n) \to +\infty$. Further,

$$J_{t_n}(u_n) = (1-t_n)J(u_n) + t_n\widehat{J}(u_n) \geq (1-t_n)J(u_n) \geq \frac{\|u_n\|}{2}\left(\frac{1}{\|u_n\|}J(u_n)\right) \to +\infty,$$

which is a contradiction. Thus $\{J_t : t \in [0,1]\}$ satisfies (PS), that is, $J = J_0$ and J_1 satisfy (PS). By Proposition 2.3, we have

$$C_q(J,\infty) \cong C_q(J_0,\infty) \cong C_q(J_1,\infty). \tag{3.14}$$

If $\widehat{J'}(u) = J_1'(u) = 0$, then $u^+ + u^0 = u^-$, namely $u = 2u^-$. Therefore, $u = 0$ is the only critical point of J_1 such that

$$C_q(J_1, \infty) = C_q(J_1, 0). \tag{3.15}$$

Let $J_1''(0)u = 0$, it is easy to compute that $u = 0$, which means that $u = 0$ is a nondegenerate critical point of J_1 and its corresponding Morse index $\mu_0 = \dim W^- = k - 1$. Finally, combining (3.14) with (3.15), we achieve $C_q(J, \infty) \cong \delta_{q,k-1}\mathbb{Z}$, $q \in \mathbb{Z}$. And this completes the proof of Lemma 3.2.

Lemma 3.3. *Assume $\alpha_\infty = \lambda_k$ and ($\mathbf{F}_\infty^-$) holds. Then*
(1) *$C_q(J, 0) \cong \delta_{q,0}$ if ($\mathbf{I_1}$) is satisfied;*
(2) *$C_q(J, 0) \cong \delta_{q,T_1T_2}$ if ($\mathbf{I_4}$) is satisfied;*
(3) *$C_q(J, 0) \cong \delta_{q,p}$ if ($\mathbf{I_3}$) is satisfied, where $p \neq k - 1$.*

Proof. In case **(1)**, $u = 0$ is a local minimizer of J and $C_q(J, 0) \cong \delta_{q,0}\mathbb{Z}$. In case **(2)**, combing $u = 0$ is a local maximum of J with the Morse index on $u = 0$ is $\mu_0 = T_1T_2$, it follows that $C_q(J, 0) \cong \delta_{q,T_1T_2}\mathbb{Z}$. In case **(3)**, $\mu_0 = p \neq k - 1$, which means that $C_q(J, 0) \cong \delta_{q,p}\mathbb{Z}$.

To prove Theorem 3.4, the following lemma on local linking is needed.

Lemma 3.4. *Let $\alpha_0 = \lambda_m$ and ($\mathbf{F_0^+}$) (or ($\mathbf{F_0^-}$)) hold. Then J has a local linking at 0 with respect to*

$$S = S^- \oplus S^+,$$

where $S^- = span\{\phi_1, \cdots, \phi_m\}$ (or $S^- = span\{\phi_1, \cdots, \phi_{m-1}\}$).

Proof. In view of ($\mathbf{F_0^+}$), there exists $\bar{\delta} > 0$ such that

$$F((i, j), u) \geq \frac{1}{2}\lambda_m u^2, \qquad |u(i, j)| \leq \bar{\delta}, \quad (i, j) \in \Omega.$$

For $u \in S^-$ with $|u(i, j)| \leq \bar{\delta}$, there has

$$J(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i, j), u(i, j)) \leq \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_m\|u\|_2^2 = 0. \tag{3.16}$$

Since $\alpha_0 = \lambda_m$, we have

$$\lim_{u\to 0}\frac{2F((i, j), u)}{u^2} = \lim_{u\to 0}\frac{f((i, j), u)}{u} = \lambda_m.$$

Then for any $\varepsilon > 0$, there exists $\tilde{\delta} > 0$ such that $\left|\frac{2F((i,j),u)}{u^2} - \lambda_m\right| < \varepsilon$ as $0 < |u(i, j)| < \tilde{\delta}$, that is, $\lambda_m - \varepsilon < \frac{2F((i,j),u)}{u^2} < \lambda_m + \varepsilon$. Thus,

$$\frac{1}{2}(\lambda_m - \varepsilon)u^2 < F((i, j), u) < \frac{1}{2}(\lambda_m + \varepsilon)u^2, \qquad 0 < |u(i, j)| < \tilde{\delta}, \quad (i, j) \in \Omega.$$

For $u \in S^-$ with $0 < |u(i, j)| < \tilde{\delta}$, we have

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\lambda_m + \varepsilon)\|u\|_2^2 \geq \frac{1}{2}\left(1 - \frac{\lambda_m + \varepsilon}{\lambda_{m+1}}\right)\|u\|^2. \tag{3.17}$$

Choose $\delta = \min\{\bar{\delta}, \tilde{\delta}\}$ and $0 < \varepsilon < \lambda_{m+1} - \lambda_m$. Denote $\rho = \delta\sqrt{T_1T_2\lambda_{T_1T_2}}$. Then (3.16) and (3.17) indicate that

$$J(u) \le 0, \quad u \in S^-, \quad \|u\| \le \rho,$$
$$J(u) \ge 0, \quad u \in S^+, \quad \|u\| \le \rho.$$

Moreover, $J(0) = 0$ is obvious. Consequently, J has a local linking at 0. And the proof is achieved.

As for Theorem 3.5, we consider the critical groups at infinity with respect to $-J$. In the same manner as Lemma 3.3, we have

Lemma 3.5. *Let* ($\mathbf{F}_\infty^+$) *hold and* $\alpha_\infty = \lambda_k$. *Then*
(1) $C_q(-J, 0) \cong \delta_{q,T_1T_2}$, *if* ($\mathbf{I_1}$) *is satisfied;*
(2) $C_q(-J, 0) \cong \delta_{q,0}$, *if* ($\mathbf{I_4}$) *is satisfied;*
(3) $C_q(-J, 0) \cong \delta_{q,T_1T_2-p}$, *if* ($\mathbf{I_3}$) *is satisfied and* $p \neq k$.

With above preparations, it is time for us to provide detailed proofs of Theorems 3.1–3.5.
Proof of Theorem 3.1 Since all three cases of Theorem 3.1 can be proved similarly, here we only prove the case **(i)** at length for brevity.

On account of ($\mathbf{I_1}$), $u = 0$ is a local minimizer of J and its Morse index $\mu_0 = 0$ and zero nullity $\nu_0 = 0$ and

$$C_q(J, 0) \cong \delta_{q,0}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

By Lemma 3.1, we get

$$C_q(J, \infty) \cong \delta_{q,k}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

Moreover, the process of proof of Lemma 3.1 indicates that J satisfies (PS). By Proposition 2.2, there exists $u_1 \in \kappa$ such that $u_1 \neq 0$ and $C_k(J, u_1) \not\cong 0$. Then u_1 is a non-zero critical point of J and

$$J''(u_1) = I - \mathrm{diag}\{f'((1,1), u_1(1,1)), \cdots, f'((T_1, T_2), u_1(T_1, T_2))\}.$$

Note that the rank of $J''(u_1)$ is greater than $T_1T_2 - 1$, which implies $\nu(u_1) = \dim\ker(J''(u_1)) \le 1$ and $C_q(J, u_1) \cong \delta_{q,k}\mathbb{Z}$, $q \in \mathbb{Z}$. Assume that $\kappa = \{0, u_1\}$, then the Morse equality is

$$(-1)^0 + (-1)^k = (-1)^k,$$

which is impossible. Hence, J has at least another nontrivial critical point, namely, (1.1) and (1.2) possesses at least two nontrivial solutions. And the proof of Theorem 3.1 is completed.
Proof of Theorem 3.2 Recall $G((i, j), u) = F((i, j), u) - \frac{1}{2}\lambda_k u^2$, there has

$$J(u) = \frac{1}{2}\langle (I - K)u, u\rangle - G(u) := \frac{1}{2}\langle Bu, u\rangle - G(u).$$

Then $B : S \to S$ is a self-adjoint bounded linear operator such that 0 is not in the spectrum of B and $B'(u)$ is compact. Write $B^\pm = B|_{W^\pm}$, then $B^\pm$ has a bounded inverse on $W^\pm$. Let $k = \dim W^- = \frac{T_1T_2}{2}$, then $\alpha_\infty = \lambda_k$ guarantees that (2.9) is valid. By Proposition 2.4, we obtain

$$C_q(J, \infty) \cong \delta_{q,\frac{T_1T_2}{2}}\mathbb{Z}, \qquad q \in \mathbb{Z},$$

and J satisfies (PS). Use $(\mathbf{I_1})$ once more, we have $u = 0$ is a local minimizer of J and

$$C_q(J, 0) \cong \delta_{q,0}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

According to Proposition 2.2 and $\nu(u_2) \le 1$, we draw a conclusion that there exists $u_2 \in \kappa$ with $u_2 \ne 0$ such that

$$C_{\frac{T_1T_2}{2}}(J, u_2) \ncong 0.$$

Assume $\kappa = \{0, u_2\}$, then the Morse equality expresses as

$$(-1)^0 + (-1)^{\frac{T_1T_2}{2}} = (-1)^{\frac{T_1T_2}{2}},$$

which is a contradiction. Therefore, J has at least another nontrivial critical point, (1.1) and (1.2) possesses at least two nontrivial solutions. And this completes the proof of Theorem 3.2.

Proof of Theorem 3.3 By Lemma 3.2, $C_q(J, \infty) = \delta_{q,k-1}\mathbb{Z}$, $q \in \mathbb{Z}$ and J satisfies (PS). Then Proposition 2.2 indicates there exists $u_3 \in \kappa$ such that $C_{k-1}(J, u_3) \ncong 0$, which means that u_3 is a non-zero critical point of J. Since

$$J''(u_3) = I - \operatorname{diag}\{f'((1, 1), u_3(1, 1)), \cdots, f'((T_1, T_2), u_3(T_1, T_2))\},$$

and the rank of $J''(u_3)$ is greater than $T_1T_2 - 1$. Then $\nu(u_3) = \dim\ker(J''(u_3)) \le 1$. If $q \notin [\mu(u_3), \mu(u_3) + \nu(u_3)]$ and $C_q(J, u_3) = 0$, then either $k - 1 = \mu(u_3) + \nu(u_3)$ or $k - 1 = \mu(u_3)$. Thus, $C_q(J, u_3) \cong \delta_{q,k-1}\mathbb{Z}$. If $\kappa = \{0, u_3\}$, by the Morse equality, we have

$$(-1)^* + (-1)^{k-1} = (-1)^{k-1}, \tag{3.18}$$

where $* = 0$, T_1T_2 or p corresponds to $(\mathbf{I_1})$, $(\mathbf{I_4})$ or $(\mathbf{I_3})$, respectively. Meanwhile, it is impossible for (3.18) to be true. Therefore, J at least has another non-zero critical point, and (1.1)–(1.2) possesses at least two nontrivial solutions and the proof is achieved.

Proof of Theorem 3.4 Lemma 3.1 yields $C_q(J, \infty) \cong \delta_{q,k}\mathbb{Z}$, $q \in \mathbb{Z}$, which means that there exists $u_4 \in \kappa$ such that $C_k(J, \infty) \ncong 0$. Since

$$J''(u_4) = I - \operatorname{diag}\{f'((1, 1), u_4(1, 1)), \cdots, f'((T_1, T_2), u_4(T_1, T_2))\},$$

and the rank of $J''(u_4)$ is greater than $T_1T_2 - 1$, then $\nu(u_4) = \dim\ker(J''(u_4)) \le 1$. If $q \notin [\mu(u_4), \mu(u_4) + \nu(u_4)]$ and $C_q(J, u_4) = 0$, then either $k = \mu(u_4)$ or $k = \mu(u_4) + \nu(u_4)$, which implies that $C_q(J, u_4) \cong \delta_{q,k}\mathbb{Z}$. Note that Lemma 3.4 shows that J has a local linking at 0 and $\dim S^- = m$. Further, 0 is the isolated critical point of J and $J''(0)$ is a Fredholm operator and $C_m(J, 0) \ncong 0$, then $C_q(J, 0) \cong \delta_{q,m}\mathbb{Z}$. If $\kappa = \{0, u_4\}$, the Morse equality is in the form as

$$(-1)^m + (-1)^k = (-1)^k. \tag{3.19}$$

However, (3.19) is impossible. Consequently, J at least has another non-zero critical point, (1.1) and (1.2) possesses at least two nontrivial solutions. The proof of Theorem 3.4 is finished.

Proof of Theorem 3.5 Let $\alpha_\infty = \lambda_k$ and $(\mathbf{F}_\infty^+)$ be satisfied, Lemma 3.2 gives

$$C_q(-J, \infty) \cong \delta_{q,T_1T_2-k}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

Furthermore, ($\mathbf{F_0^-}$) and $\alpha_0 = \lambda_m$ lead to

$$C_q(-J,0) \cong \delta_{q,T_1T_2-(m+1)}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

Notice that $m+1 \neq k$ and nondegenerate critical points are isolated, then there exists some critical point $u_5 \in \kappa$ with $u_5 \neq 0$ such that

$$C_{T_1T_2-k}(-J,u_5) \not\cong 0,$$

then

$$C_q(-J,u_5) \cong \delta_{q,T_1T_2-k}\mathbb{Z}.$$

If $\kappa = \{0, u_5\}$, then there holds the Morse equality

$$(-1)^{T_1T_2-(m+1)} + (-1)^{T_1T_2-k} = (-1)^{T_1T_2-k},$$

which is impossible. Then $-J$ at least has another non-zero critical point, (1.1) and (1.2) possesses at least two nontrivial solutions. The proof of Theorem 3.5 is completed.

4. Examples

Finally, we present five examples to verify the feasibility of our results.

Example 4.1. *Take $T_1 = 3$, $T_2 = 2$, consider*

$$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + \frac{(\frac{\lambda_1}{2} - 2\lambda_{T_1T_2})u}{1+u^2} + 2\lambda_{T_1T_2}u = 0, \tag{4.1}$$

with boundary value conditions (1.2).

Because $f((i,j),u) = \frac{(\frac{\lambda_1}{2}-2\lambda_{T_1T_2})u}{1+u^2} + 2\lambda_{T_1T_2}u$, it follows that $f((i,j),0) = 0$ and

$$f'((i,j),u) = \frac{(6\lambda_{T_1T_2} - \frac{\lambda_1}{2})u^2 + 2\lambda_{T_1T_2}u^4 + \frac{\lambda_1}{2}}{(1+u^2)^2}.$$

Then $f'((i,j),0) = \frac{\lambda_1}{2} < \lambda_1$ and $f'((i,j),\infty) = 2\lambda_{T_1T_2} > \lambda_{T_1T_2}$, which means that ($\mathbf{I_1}$) and ($\mathbf{I_2}$) are satisfied. Consequently, Theorem 3.1 guarantees that (4.1) and (1.2) admits at least two nontrivial solutions.

Example 4.2. *Take $T_1 = 3$, $T_2 = 5$, consider*

$$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + \frac{(\frac{\lambda_1}{2} - \lambda_k)u}{1+u^2} + \lambda_k u = 0, \tag{4.2}$$

with boundary value conditions (1.2).

Clear, $T_1T_2 = 15$ is odd and

$$f((i,j),u) = \frac{(\frac{\lambda_1}{2} - \lambda_k)u}{1+u^2} + \lambda_k u.$$

It is easy to calculate that $f((i, j), 0) = 0$ and

$$\alpha_\infty = \lim_{|u|\to\infty} \frac{f((i, j), u)}{u} = \lim_{|u|\to\infty} \left[\frac{\frac{\lambda_1}{2} - \lambda_k}{1 + u^2} + \lambda_k \right] = \lambda_k.$$

Moreover, direct computation gives

$$f'((i, j), u) = \frac{(3\lambda_k - \frac{\lambda_1}{2})u^2 + \lambda_k u^4 + \frac{\lambda_1}{2}}{(1 + u^2)^2},$$

which indicates $f'((i, j), 0) = \frac{\lambda_1}{2} < \lambda_1$. As a result, $(\mathbf{I_1})$ is valid and Theorem 3.2 ensures that (4.2) and (1.2) admits at least two nontrivial solutions.

Example 4.3. *Take* $T_1 = 3$*,* $T_2 = 2$*,* $r = e^{\frac{\lambda_1}{2} - \lambda_k} > 0$*. Consider*

$$\Delta_1^2 u(i - 1, j) + \Delta_2^2 u(i, j - 1) + \frac{(\frac{\lambda_1}{2} - \lambda_k)\sin u}{u + 1} + \lambda_k u + u^{\frac{1}{3}} \ln(r + |u|^3) = 0, \tag{4.3}$$

with boundary value conditions (1.2).

Since $f((i, j), u) = \frac{(\frac{\lambda_1}{2} - \lambda_k)\sin u}{u+1} + \lambda_k u + u^{\frac{1}{3}} \ln(r + |u|^3)$, it is easy to get that $f((i, j), 0) = 0$ and

$$\alpha_\infty = \lim_{|u|\to\infty} \frac{f((i, j), u)}{u} = \lim_{|u|\to\infty} \left[\frac{(\frac{\lambda_1}{2} - \lambda_k)\sin u}{u(u + 1)} + \lambda_k + \frac{\ln(r + |u|^3)}{u^{\frac{2}{3}}} \right] = \lambda_k.$$

Further,

$$f'((i, j), u) = \lambda_k + \frac{\cos u(\frac{\lambda_1}{2} - \lambda_k)(u + 1) - \sin u(\frac{\lambda_1}{2} - \lambda_k)}{(u + 1)^2} + u^{-\frac{2}{3}} \ln(r + |u|^3) + \frac{3u^{\frac{7}{3}}}{r + |u|^3}.$$

Thus $f'((i, j), 0) = \frac{\lambda_1}{2} < \lambda_1$ and $(\mathbf{I_1})$ is satisfied.

At last, by direct computation, we obtain

$$\lim_{u\to+\infty} (f((i, j), u) - \lambda_k u) = \lim_{u\to+\infty} \left(\frac{(\frac{\lambda_1}{2} - \lambda_k)\sin u}{u + 1} + u^{\frac{1}{3}} \ln(r + |u|^3) \right) = +\infty,$$
$$\lim_{u\to-\infty} (f((i, j), u) - \lambda_k u) = \lim_{u\to-\infty} \left(\frac{(\frac{\lambda_1}{2} - \lambda_k)\sin u}{u + 1} + u^{\frac{1}{3}} \ln(r + |u|^3) \right) = -\infty,$$

which means that $(\mathbf{F}_\infty^+)$ is met.

Therefore, all conditions of Theorem 3.3 are fulfilled and (4.3) and (1.2) admits at least two nontrivial solutions.

Example 4.4. *Take* $T_1 = 3$*,* $T_2 = 2$*. Consider*

$$\Delta_1^2 u(i - 1, j) + \Delta_2^2 u(i, j - 1) + \frac{2(\lambda_m - \frac{\lambda_p + \lambda_{p+1}}{2})u}{2 - u^2} + \frac{\lambda_p + \lambda_{p+1}}{2} u = 0, \tag{4.4}$$

with boundary value conditions (1.2).

Owe to $f((i,j),u)=\frac{2(\lambda_m-\frac{\lambda_p+\lambda_{p+1}}{2})u}{2-u^2}+\frac{\lambda_p+\lambda_{p+1}}{2}u$, it follows that $f((i,j),0)=0$ and

$$F((i,j),u)=(\frac{\lambda_p+\lambda_{p+1}}{2}-\lambda_m)\ln(2-u^2)+\frac{\lambda_p+\lambda_{p+1}}{4}u^2.$$

Then

$$f'((i,j),u)=\frac{2(\lambda_m-\frac{\lambda_p+\lambda_{p+1}}{2})(2+u^2)}{(2-u^2)^2}+\frac{\lambda_p+\lambda_{p+1}}{2},$$

and

$$\alpha_\infty=\lim_{|u|\to 0}\frac{f((i,j),u)}{u}=\lambda_m.$$

Additionally, $\lambda_p<f'((i,j),\infty)=\frac{\lambda_p+\lambda_{p+1}}{2}<\lambda_{p+1}$, which implies that $(\mathbf{I_6})$ is met. In the following, we check $(\mathbf{F_0^+})$. Write

$$A=\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0\\ -1 & 4 & -1 & 0 & -1 & 0\\ 0 & -1 & 4 & 0 & 0 & -1\\ -1 & 0 & 0 & 4 & -1 & 0\\ 0 & -1 & 0 & -1 & 4 & -1\\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix},$$

then A is positive-define and the corresponding eigenvalues are

$$\lambda_1=3-\sqrt{2},\quad \lambda_2=3,\quad \lambda_3=5-\sqrt{2},\quad \lambda_4=3+\sqrt{2},\quad \lambda_5=5,\quad \lambda_6=5+\sqrt{2}.$$

Take $m=1$, $p=3$, then there exists $\delta>0$ such that $2F((i,j),u)-\lambda_m u^2>0$ when $|u(i,j)|\leq\delta$. In fact, for any $(i,j)\in\mathbb{Z}(1,3)\times\mathbb{Z}(1,2)$, we can choose $\delta=1>0$, then $0<|u(i,j)|^2\leq 1$ for $0<|u(i,j)|\leq 1$, which means that

$$(\sqrt{2}+1)\ln(2-u^2)+(\sqrt{2}-1)u^2>0.$$

Thus $(\mathbf{F_0^+})$ is fulfilled. Consequently, Theorem 3.4 ensures that (4.4) and (1.2) possesses at least two nontrivial solutions.

More clearly, using Matlab, we find that problem (4.4) and (1.2) has 63 solutions including 1 trivial solution and 62 nontrivial solutions. Here we list a few: (-2.1408, 1.8608, -2.1408, 2.1408, -1.8608, 2.1408), (2.1408, -1.8608, 2.1408, -2.1408, 1.8608, -2.1408), $(-8.3211\times 10^{-9}$, -1.1767×10^{-8}, -8.3211×10^{-9}, -8.3211×10^{-9}, -1.1767×10^{-8}, $-8.3211\times 10^{-9})$, $(8.3211\times 10^{-9}$, 1.1767×10^{-8}, 8.3211×10^{-9}, 8.3211×10^{-9}, 1.1767×10^{-8}, $8.3211\times 10^{-9})$.

Example 4.5. *Take $T_1=3$, $T_2=2$. Consider*

$$\Delta_1^2u(i-1,j)+\Delta_2^2u(i,j-1)+\frac{(\lambda_k-\lambda_m)u^3}{1+u^2}+\lambda_m u+u^{\frac{1}{3}}\ln(1+|u|^3)=0,\tag{4.5}$$

with boundary value conditions (1.2).

From (4.5), we find $f((i,j),u) = \frac{(\lambda_k-\lambda_m)u^3}{1+u^2} + \lambda_m u + u^{\frac{1}{3}}\ln(1+|u|^3)$, then

$$F((i,j),u) = \frac{1}{2}\lambda_m u^2 + \frac{\lambda_k-\lambda_m}{2}(u^2 - \ln(1+u^2)) + u\ln(1+u) - u + \ln(1+u) + C,$$

take $C = -1$ and $\delta = e - 1 > 0$, then when $m > k$ and $0 < |u(i,j)| < \delta$,

$$\begin{aligned}F((i,j),u) - \frac{1}{2}\lambda_m u^2 =& \frac{\lambda_k-\lambda_m}{2}(u^2 - \ln(1+u^2)) + u\ln(1+u) - u + \ln(1+u) + C\\ =&(u+1)(\ln(u+1)-1) + \frac{\lambda_k-\lambda_m}{2}(u^2-\ln(1+u^2))\\ =&(u+1)(\ln(u+1)-\ln e) + \frac{\lambda_k-\lambda_m}{2}(u^2-\ln(1+u^2)) < 0,\end{aligned}$$

which means that $(\mathbf{F}_{\mathbf{0}}^-)$ is fulfilled.

On the other side, it is easy to get $f((i,j),0) = 0$ and

$$\begin{aligned}\alpha_\infty &= \lim_{|u|\to\infty}\frac{f((i,j),u)}{u} = \lim_{|u|\to\infty}\left[\frac{(\lambda_k-\lambda_m)u^2}{1+u^2} + \lambda_m + \frac{\ln(1+|u|^3)}{u^{\frac{2}{3}}}\right] = \lambda_k,\\ \alpha_0 &= \lim_{|u|\to 0}\frac{f((i,j),u)}{u} = \lim_{|u|\to 0}\left[\frac{(\lambda_k-\lambda_m)u^2}{1+u^2} + \lambda_m + \frac{\ln(1+|u|^3)}{u^{\frac{2}{3}}}\right] = \lambda_m.\end{aligned}$$

Furthermore, there hold

$$\begin{aligned}\lim_{u\to+\infty}(f((i,j),u) - \lambda_k u) &= \lim_{u\to+\infty}\left(u^{\frac{1}{3}}\ln(1+|u|^3) + (\lambda_m-\lambda_k)u + \frac{(\lambda_k-\lambda_m)u^3}{1+u^2}\right) = +\infty,\\ \lim_{u\to-\infty}(f((i,j),u) - \lambda_k u) &= \lim_{u\to-\infty}\left(u^{\frac{1}{3}}\ln(1+|u|^3) + (\lambda_m-\lambda_k)u + \frac{(\lambda_k-\lambda_m)u^3}{1+u^2}\right) = -\infty,\end{aligned}$$

which guarantees that $(\mathbf{F}_\infty^+)$ is satisfied.

Therefore, all conditions of Theorem 3.5 are valid and (4.5) and (1.2) has at least two nontrivial solutions.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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