Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 5391–5412.
DOI: 10.3934/math.2023271
Received: 21 September 2022
Revised: 21 November 2022
Accepted: 24 November 2022
Published: 15 December 2022

Research article

Transient scrutiny of $M^X/G(a, b)/1$ queueing system with feedback, balking and two phase of service subject to server failure under Bernoulli vacation

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Abstract: The transient scrutiny of a batch arrival feedback queueing system with balking and two stages of varying service with contrasting levels of service subjected to Bernoulli vacation has been examined in this study. Customers also have the option to decline services and leave the service area if the server is unable to fulfill their request when they arrive. The server may continue to serve the customers, if any, after each service with probability ω, or it may undergo a vacation with probability $(1 - \omega)$. The service channel may fail temporarily when the server is operating in any phase of service, which is then directed straight to the repair process. The model's steady state results and time-dependent probability generating functions in terms of their Laplace transforms have been derived. The mean queue length and the average time spent in the queue are explicitly determined as performance indicators in the various system states. A few unique cases and specific circumstances have also been presented. Finally, the effect of different parameters on the system's efficiency is then numerically analyzed.

Keywords: batch arrival; bulk service; feedback; balking; failure; Bernoulli vacation
Mathematics Subject Classification: 60K25, 60K30

1. Introduction

In real-world conditions, the idea of bulk arrivals and bulk service has taken on a new meaning. A comprehensive analysis of bulk queueing was provided by Chaudhry and Templeton [1]. Bulk arrival analysis has been a simple version of customer analysis, and it's a wonderful way to get started with customized modeling. Bailey [2] was the inventor of bulk service queueing models. He developed the technique called "fixed-batch service." In fixed-batch service queueing systems (QS), the server always serves a fixed batch of consumers in each group. Furthermore, a recent overview of bulk service queueing models was provided by Sasikala and Indhira [3]. There are several uses for bulk queuing models in traffic, transportation, production, and manufacturing systems.

Generally, several systems include more than one phase of service, so it is vital to examine two stages of service for many real-world models. We also expect a batch arrival and use a minimum batch size of 'a' and a maximal batch size of 'b' in addition to providing a bulk service. Sushil Ghimire et al. [4] examined a bulk QS with a fixed batch size 'b' with consumers entering the system in a Poisson process and being separated exponentially. Moreover, Sasikala et al. [5] studied the steady state (SS) behavior of the $M^{[X]}/G(a,b)/1$ queue in the presence of server downtime, multiple vacations, setup time, and N-policy. Suganya [6] also investigated a $M^{[X]}/G/1$ QS with a second optional service (SOS), multiple vacations, breakdowns, and repairs.

The server may be unavailable for a while in a vacation QS for a variety of reasons, such as being monitored for maintenance, attending to other queues, or just taking a break. A vacation is defined as a period of time during which the server is not accessible to customers. The idea of a single server queuing model with Bernoulli vacations (BV) was first suggested by Keilson and Servi (1986). A single server subjected to BV and a randomized breakdown was generalized by Ayyappan and Shyamala [7]. The SS and the Probability Generating Function (PGF) of transient solutions were both explicitly calculated. Sundar Rajan et al. [8] evaluated a feedback queueing model with Bernoulli server vacation, multistage unit service, and random server breakage. Ayyappan and Supraja [9] recently, explored an $M^{[X]}/G(a,b)/1$ queueing model with dual stages of service subjected to system breakages and BV.

Consumers in queueing models frequently exhibit the phenomenon of impatience (when servers are idle), off-duty, or undergoing maintenance, some consumers choose not to join the line at all). Queues with balking are widely studied in inventory and production systems, as well as in hospital emergency departments. Ayyappan and Karpagam [10] examined a bulk arrival, a bulk service QS with server breakage and a second optional repair, a standby server, balking, a variable arrival rate, and multiple vacations. Ayyappan and Supraja [11] explored a bulk arrival Non-Markovian QS with balking under BV, breakage, and repair. Recently, Lan and Tang [12] analyzed an unreliable discrete-time retrial queue with probabilistic preemptive priority, balking customers, and replacements of repair times.

In many situations in the real world, service disruptions are unavoidable. Most studies make the assumption that the server in the service station is always operational and that the station never malfunctions. These presumptions, however, are essentially irrational. In real life, it happens frequently that service stations break down and need to be fixed. QS $M^{[X]}/G(a,b)/1$ Haridass and Nithya [13] investigated generalized server failure and interrupted vacation. Ayyappan and Karpagam [14] examined a generic bulk service queueing approach with breakage and repair, standby servers, multiple vacations, and a re-service request control policy. It's also been noted that, as the main server's vacation rate increases, the estimated queue length also increases.

One of the fundamental presumptions made by many researchers who have studied queueing models with interruptions is that the service channel will promptly undergo repair after failure. The repair process for one of the two service stages begins with the termination of that stage. A group of customers who had received services prior to the server failure wait for the remaining services to be completed once the repair is complete. Ayyappan and Nirmala [15] evaluated the transient and SS behavior of the $M^{[X]}/G(a,b)/1$ queue with breakage and two stages of repair with a delay while using a multiple vacation policy. A $M^{[X]}/G(a,b)/1$ QS with multiple vacations, closedown essential and optional repairs was generalized by Ayyappan and Deepa [16]. Charan Jeet Singh et al. [17] investigated a unique recoverable server QS bulk input and state-dependent rates, taking into account general repair likelihood, time to repair, and service processes. Applications for these models include flexible advanced technologies and computer telecommunication systems, among others.

The phenomenon of feedback is yet another crucial tool for communication systems. Customers may be served frequently in numerous real-world circumstances for a particular cause. If a customer is dissatisfied with the service they received, they might keep trying until they are happy. This is initially investigated by Takacs [18], which enables the customer who has completed the service to give feedback to the tail of the queue. Zadeh [19] also investigated a bulk arrival multi-phase QS with random feedback in service and a single vacation policy. Bouchentouf and Guendouzi [20] in recent years analyzed a single server batch arrival Bernoulli feedback QS with waiting servers, K-variant vacations, and impatient customers. A Markovian bulk service queue with feedback and SOS was also recently analyzed by Laxmi et al. [21]

However, no work has been done in the transient analysis of a QS taking into account feedback, balking, and two phases of service subject to server failure and BV. Therefore, to fill up this gap, in this paper, we consider the transient scrutiny of a $M^X/G(a,b)/1$ QS with feedback, balking with two phases of service subjected to BV and server breakdown. Moreover, the queue size distribution at random epochs and the mean queue size have also been derived. A 3D graphical representation for various parameters has also been presented. Moreover, the QS under consideration finds application in diverse fields like LAN, telephone systems, electronic mail services on the internet, packet-switched networks, call centers, etc.

The following is the paper's outline: A comprehensive description of the mathematical model is provided in Section 2. Definitions and equations governing the system are discussed in subsections 2.1 and 2.2, along with the model's time-dependent solution. Subsection 2.3 explicitly presents the system's SS behavior as well as the PGF for the queue size at an arbitrary time. The relevant stability condition has been discovered in subsection 2.4. For each of the system's states, we explicitly determined the average queue length, the mean queue waiting time, and the performance metrics in subsection 2.5. Subsections 2.6 and 2.7 list a few special cases and particular cases. In subsection 2.8, a practical application of the model has been presented. In section 3, the numerical results along with the graphical representation are presented, and finally, section 4 provides the conclusion.

2. Model description and analysis

To describe the required queueing model, we assume the following:
The arrival process: Consumers follow a compound Poisson process when approaching the system in batches of varying sizes, and they are given bulk service on an $FCFS$ basis. Let $\vartheta c_i dt$ $(i \geq 1)$ be the first-order prob., that a batch of i consumers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\vartheta > 0$ is the mean arrival rate of batches.
The service process: A single server provides two distinct heterogeneous services to a group of customers in succession. At the First Phase of Service (FPS) and Second Phase of Service (SPS), which have rates of service η_1 and η_2, the service time follows general (arbitrary) dist., B_1 and B_2. Let the distance, function, and density function of the service time be $B_i(v)$ and $b_i(v)$ respectively.

Given an elapsed service time of φ, the conditional prob., the density function of service completion over the interval $(\varphi, \varphi + d\varphi]$ is

$$\eta_i(\varphi) = \frac{b_i(\varphi)}{1 - B_i(\varphi)}, \ i = 1, 2$$

and therefore,

$$b_i(u) = \eta_i(u) e^{-\int_0^u \eta_i(\varphi) d\varphi}, \ i = 1, 2.$$

Following the completion of SPS, the server may take the first type of BV with prob., ω, and prepares to serve the next group of users with prob., $(1 - \omega)$.

The vacation process: The server's vacation time is represented by a generic (arbitrary) dist. with dist., function $A_i(r)$ and density function $a_i(r)$. Given that φ represents the amount of vacation time already spent, let $\Upsilon_i(\varphi)d\varphi$ represent the conditional probability of ending a vacation during the interval $(\varphi, \varphi + d\varphi]$.

$$\Upsilon_i(\varphi) = \frac{a_i(\varphi)}{1 - A_i(\varphi)}, \ i = 1, 2$$

and therefore

$$a_i(r) = \Upsilon_i(r) e^{-\int_0^r \Upsilon_i(\varphi) d\varphi}, \ i = 1, 2.$$

The server may experience a breakdown while it is operating with any phase of service; this is assumed to happen according to a Poisson stream with mean breakdown rates of τ_1 for the FPS and τ_2 for the SPS.

The repair process: The FPS and SPS start their respective repair processes if the service is interrupted. Following a general (arbitrary) distribution with dist., function $M_i(t)$ and density function $m_i(t)$, the repair time is distributed in both phases. Given that the elapsed repair time is ϖ, let $\pi_i(\varpi)d\varpi$ be the conditional prob., of a repair time completion during the range $(\varpi, \varpi + d\varpi)$, so that

$$\pi_i(\varpi) = \frac{m_i(\varpi)}{1 - M_i(\varpi)}, \ i = 1, 2$$

and therfore,

$$m_i(t) = \pi_i(t) e^{-\int_0^t \pi_i(\varpi) d\varpi}, \ i = 1, 2.$$

We assume that $(1 - s)$ $(0 \leq s \leq 1)$ i the prob., that an arriving consumer balks during periods when the server is busy (idle), on vacation and under repair.

The consumer is served according to the FCFS discipline. The varied stochastic processes involved in the system are assumed to be independent of each other.

Feedback rule: Customers who are dissatisfied with their services may re-join the line after receiving them as a feedback customer to receive another service with prob., p $(0 \leq p \leq 1)$ or exit the system with complement prob., $q = (1 - p)$

2.1. Definitions

(1) $P_{i,n}(\varphi, t)$ = Prob., that at time t, the server is active providing i^{th} phase of service $(i = 1, 2)$ and there are n $(n \geq 0)$ customers in the queue excluding the batch being served and the elapsed service time on a batch of customers undergoing service is φ. As a result, $P_{i,n}(t) = \int_0^\infty P_{i,n}(\varphi, t)d\varphi$ signifies the likelihood that there will be n clients in the queue at time t, excluding the batch of clients in the i^{th} stage of service, without taking into account the amount of service time that has already passed.

(2) $A_{i,n}(\varphi, t)$ = Prob., that at a time t, the server is on i^{th} type of vacation with elapsed vacation time is φ and there are n $(n \geq 0)$ clients in the queue. Accordingly, $V_{i,n}(t) = \int_0^\infty V_{i,n}(\varphi, t)d\varphi$ denotes the prob., that at time t, there are n clients in the queue and the server is on i^{th} type of vacation without regard to the elapsed vacation time is φ.

(3) $M_{i,n}(\varphi, \varpi, t)$ = Prob., that at time t, the server is under i^{th} phase of repair $(i = 1, 2)$ (breakdown during i^{th} phase of service time) with the elapsed service time on a batch of customers undergoing service is φ, the elapsed repair time of the server is ϖ and there are n $(n \geq 0)$ customers in the queue. Accordingly, $M_{i,n}(\varphi, t) = \int_0^\infty M_{i,n}(\varphi, \varpi, t)d\varpi$ signifies the likelihood that there will be n clients in the queue at time t with an elapsed service time of φ and without consideration of an elapsed repair time of ϖ for a group of clients receiving service.

(4) $Q_r(t)$ is the prob., that there will be r $(0 \leq r \leq a - 1)$ clients in the system at time t and that the server will be idle yet accessible.

2.2. Equations governing the system

The model's governing Kolmogorov forward equations for the conditions of $i = 1, 2$; where sub index $i = 1, 2$ specifies the FPS and SPS, respectively, can be written as follows:

$$\frac{\partial}{\partial\varphi}P_{i,0}(\varphi, t) + \frac{\partial}{\partial t}P_{i,0}(\varphi, t) + (\vartheta + \eta_i(\varphi) + \tau_i)P_{i,0}(\varphi, t) = \vartheta q(1 - d)P_{i,0}(\varphi, t) + \int_0^\infty M_{i,0}(\varphi, \varpi, t)\pi_i(\varpi)d\varpi, i = 1, 2; \tag{2.1}$$

$$\frac{\partial}{\partial\varphi}P_{i,n}(\varphi, t) + \frac{\partial}{\partial t}P_{i,n}(\varphi, t) + (\vartheta + \eta_i(\varphi) + \tau_i)P_{i,n}(\varphi, t) = \vartheta q(1 - d)P_{i,n}(\varphi, t) + \vartheta d\sum_{k=1}^{n} C_k P_{i,n-k}(\varphi, t) + \int_0^\infty M_{i,n}(\varphi, \varpi, t)\pi_i(\varpi)d\varpi, (n \geq 1)\ i = 1,2; \tag{2.2}$$

$$\frac{\partial}{\partial\varphi}A_{1,0}(\varphi, t) + \frac{\partial}{\partial t}A_{1,0}(\varphi, t) + (\vartheta + \Upsilon_1(\varphi))A_{1,0}(\varphi, t) = \vartheta(1 - d)A_{1,0}(\varphi, t); \tag{2.3}$$

$$\frac{\partial}{\partial\varphi}A_{1,n}(\varphi, t) + \frac{\partial}{\partial t}A_{1,n}(\varphi, t) + (\vartheta + \Upsilon_1(\varphi))A_{1,n}(\varphi, t) = \vartheta(1 - d)A_{1,n}(\varphi, t) + \vartheta d\sum_{k=1}^{n} C_k A_{1,n-k}(\varphi, t), n \geq 1; \tag{2.4}$$

$$\frac{\partial}{\partial\varphi}A_{2,0}(\varphi, t) + \frac{\partial}{\partial t}A_{2,0}(\varphi, t) + (\vartheta + \Upsilon_2(\varphi))A_{2,0}(\varphi, t) = \vartheta(1 - d)A_{2,0}(\varphi, t); \tag{2.5}$$

$$\frac{\partial}{\partial\varphi}A_{2,n}(\varphi, t) + \frac{\partial}{\partial t}A_{2,n}(\varphi, t) + (\vartheta + \Upsilon_2(\varphi))A_{2,n}(\varphi, t) = \vartheta(1 - d)A_{2,n}(\varphi, t)$$

$$+\vartheta d\sum_{k=1}^{n}C_kA_{2,n-k}(\varphi,t), n\geq 1; \tag{2.6}$$

$$\frac{\partial}{\partial\varpi}M_{i,0}(\varphi,\varpi,t)+\frac{\partial}{\partial t}M_{i,0}(\varphi,\varpi,t)+(\vartheta+\pi_i(\varpi))M_{i,0}(\varphi,\varpi,t)=\vartheta(1-d)M_{i,0}(\varphi,\varpi,t), i=1,2; \tag{2.7}$$

$$\frac{\partial}{\partial\varpi}M_{i,n}(\varphi,\varpi,t)+\frac{\partial}{\partial t}M_{i,n}(\varphi,\varpi,t)+(\vartheta+\pi_i(\varpi))M_{i,n}(\varphi,\varpi,t)=\vartheta(1-d)M_{i,n}(\varphi,\varpi,t)$$
$$+\vartheta d\sum_{k=1}^{n}C_kM_{i,n-k}(\varphi,\varpi,t), n\geq 1; \tag{2.8}$$

$$\frac{d}{dt}Q_0(t)=-\vartheta Q_0(t)+\vartheta(1-d)Q_0(t)+(1-\omega)\int_0^\infty P_{2,0}(\varphi,t)\eta_2(\varphi)d\varphi+\int_0^\infty A_{1,0}(\varphi,t)\Upsilon_1(\varphi)d\varphi \tag{2.9}$$
$$+\int_0^\infty A_{2,0}(x,t)\Upsilon_2(\varphi)d\varphi;$$

$$\frac{d}{dt}Q_r(t)=-\vartheta Q_r(t)+\vartheta(1-d)Q_r(t)+\vartheta d\sum_{k=1}^{r}C_kQ_{r-k}(t)+(1-\omega)\int_0^\infty P_{2,r}(x,t)\eta_2(\varphi)d\varphi \tag{2.10}$$
$$+\int_0^\infty A_{1,r}(\varphi,t)\Upsilon_1(\varphi)d\varphi+\int_0^\infty A_{2,r}(\varphi,t)\Upsilon_2(\varphi)d\varphi, (1\leq r\leq a-1).$$

To solve the Eqs (2.1) to (2.10) the following boundary conditions at φ=0 and ϖ=0 are considered,

$$P_{1,0}(0,t)=p\vartheta d\sum_{r=a}^{b}\sum_{k=0}^{a-1}C_{r-k}Q_k(t)+(1-\omega)\sum_{r=a}^{b}\int_0^\infty P_{2,r}(\varphi,t)\eta_2(\varphi)d\varphi$$
$$+p\sum_{r=a}^{b}\int_0^\infty A_{1,r}(\varphi,t)\Upsilon_1(\varphi)d\varphi+q\sum_{r=a}^{b}\int_0^\infty A_{2,r}(\varphi,t)\Upsilon_2(\varphi)d\varphi; \tag{2.11}$$

$$P_{1,n}(0,t)=p\vartheta d\sum_{k=0}^{a-1}C_{b+n-k}Q_k(t)+(1-\omega)\int_0^\infty P_{2,n+b}(\varphi,t)\eta_2(\varphi)d\varphi$$
$$+p\int_0^\infty A_{1,n+b}(\varphi,t)\Upsilon_1(\varphi)d\varphi+q\int_0^\infty A_{2,n+b}(\varphi,t)\Upsilon_2(\varphi)d\varphi, n\geq 1; \tag{2.12}$$

$$P_{2,n}(0,t)=\int_0^\infty P_{1,n}(\varphi,t)\eta_1(\varphi)d\varphi, n\geq 0; \tag{2.13}$$

$$A_{1,n}(0,t)=\omega\int_0^\infty P_{2,n}(\varphi,t)\eta_2(\varphi)d\varphi, n\geq 0; \tag{2.14}$$

$$A_{2,n}(0,t)=\int_0^\infty A_{1,n}(\varphi,t)\Upsilon_1(\varphi)d\varphi, n\geq 0; \tag{2.15}$$

$$M_{i,n}(\varphi,0,t)=\tau_iP_{i,n}(\varphi,t), n\geq 0, i=1,2. \tag{2.16}$$

We assume that the server is idle initially because there aren't enough users in the system. Thus, the starting circumstances are

$$Q_0(0)=1, Q_r(0)=0 \text{ for } 1\leq r\leq a-1,$$

$$P_{i,n}(0) = R_{i,n}(0) = V_{1,n}(0) = V_{2,n}(0) = 0 \text{ for } n \geq 0, i = 1,2. \tag{2.17}$$

To address the aforementioned problems, let us introduce the PGF for $i = 1,2$ for $|\xi| \leq 1$.

$$\begin{aligned}
P_i(\varphi,\xi,t) &= \sum_{n=0}^{\infty} \xi^n P_{i,n}(\varphi,t);\ P_i(\xi,t) = \sum_{n=0}^{\infty} \xi^n P_{i,n}(t),\\
A_1(\varphi,\xi,t) &= \sum_{n=0}^{\infty} \xi^n A_{1,n}(\varphi,t);\ A_1(\xi,t) = \sum_{n=0}^{\infty} \xi^n A_{1,n}(t),\\
A_2(\varphi,\xi,t) &= \sum_{n=0}^{\infty} \xi^n A_{2,n}(\varphi,t);\ A_2(\xi,t) = \sum_{n=0}^{\infty} \xi^n A_{2,n}(t),\\
M_i(\varphi,\varpi,\xi,t) &= \sum_{n=0}^{\infty} \xi^n M_{i,n}(\varphi,\varpi,t);\ M_i(\varphi,\xi,t) = \sum_{n=0}^{\infty} \xi^n M_{i,n}(\varphi,t),\\
C(\xi) &= \sum_{n=1}^{\infty} C_n \xi^n;\ Q(\xi) = \sum_{r=0}^{a-1} Q_r \xi^r.
\end{aligned} \tag{2.18}$$

Define a function $f(t)$ Laplace Transforms (LT) as

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \mathcal{R}(s) \geq 0. \tag{2.19}$$

Considering the LT of Eqs (2.1) to (2.16) and using (2.17) yields

$$\frac{\partial}{\partial\varphi}\bar{P}_{i,0}(\varphi,s) + (s + \vartheta d + \eta_i(\varphi) + \tau_i)\bar{P}_{i,0}(\varphi,s) = \int_0^{\infty} \bar{M}_{i,0}(\varphi,\varpi,s)\pi_i(\varpi)d\varpi, i = 1,2; \tag{2.20}$$

$$\begin{aligned}
\frac{\partial}{\partial\varphi}\bar{P}_{i,n}(\varphi,s) + (s + \vartheta d + \eta_i(\varphi) + \tau_i)\bar{P}_{i,n}(\varphi,s) = \vartheta d \sum_{k=1}^{n} C_k \bar{P}_{i,n-k}(\varphi,s)+\\
\int_0^{\infty} \bar{M}_{i,n}(\varphi,\varpi,s)\pi_i(\varpi)d\varpi, n \geq 1, i = 1,2;
\end{aligned} \tag{2.21}$$

$$\frac{\partial}{\partial\varphi}\bar{A}_{1,0}(\varphi,s) + (s + \vartheta d + \Upsilon_1(\varphi))\bar{A}_{1,0}(\varphi,s) = 0; \tag{2.22}$$

$$\frac{\partial}{\partial\varphi}\bar{A}_{1,n}(\varphi,s) + (s + \vartheta d + \Upsilon_1(\varphi))\bar{A}_{1,n}(\varphi,s) = \vartheta d \sum_{k=1}^{n} C_k \bar{A}_{1,n-k}(\varphi,s), n \geq 1; \tag{2.23}$$

$$\frac{\partial}{\partial\varphi}\bar{A}_{2,0}(\varphi,s) + (s + \vartheta d + \Upsilon_2(\varphi))\bar{A}_{2,0}(\varphi,s) = 0; \tag{2.24}$$

$$\frac{\partial}{\partial\varphi}\bar{A}_{2,n}(\varphi,s) + (s + \vartheta d + \Upsilon_2(\varphi))\bar{A}_{2,n}(\varphi,s) = \vartheta d \sum_{k=1}^{n} C_k \bar{A}_{2,n-k}(\varphi,s), n \geq 1; \tag{2.25}$$

$$\frac{\partial}{\partial\varpi}\bar{M}_{i,0}(\varphi,\varpi,s) + (s + \vartheta d + \pi_i(\varpi))\bar{M}_{i,0}(\varphi,\varpi,s) = 0, i = 1,2; \tag{2.26}$$

$$\frac{\partial}{\partial\varphi}\bar{M}_{i,n}(\varphi,\varpi,s) + (s + \vartheta d + \pi_i(\varpi))\bar{M}_{i,n}(\varphi,\varpi,s) = \vartheta d \sum_{k=1}^{n} C_k \bar{M}_{i,n-k}(\varphi,\varpi,s), n \geq 1, i = 1,2; \tag{2.27}$$

$$(s+\vartheta d)\bar{Q}_0(s) = 1 + (1-\omega)\int_0^\infty \bar{P}_{2,0}(\varphi,s)\eta_2(\varphi)d\varphi + \int_0^\infty \bar{A}_{1,0}(\varphi,s)\Upsilon_1(\varphi)d\varphi + \int_0^\infty \bar{A}_{2,0}(\varphi,s)\Upsilon_2(\varphi)d\varphi; \tag{2.28}$$

$$(s+\vartheta d)\bar{Q}_r(s) = \vartheta d\sum_{k=1}^{r} C_k\bar{Q}_{r-k}(s) + (1-\omega)\int_0^\infty \bar{P}_{2,r}(\varphi,s)\eta_2(\varphi)d\varphi + \int_0^\infty \bar{A}_{1,r}(\varphi,s)\Upsilon_1(\varphi)d\varphi \tag{2.29}$$
$$+ \int_0^\infty \bar{A}_{2,r}(\varphi,s)\Upsilon_2(\varphi)d\varphi, 1 \le r \le a-1;$$

$$\bar{P}_{1,0}(0,s) = p\vartheta d\sum_{r=a}^{b}\sum_{k=0}^{a-1} C_{r-k}\bar{Q}_k(s) + (1-\omega)\sum_{r=a}^{b}\int_0^\infty \bar{P}_{2,r}(\varphi,s)\eta_2(\varphi)d\varphi + p\sum_{r=a}^{b}\int_0^\infty \bar{A}_{1,r}(\varphi,s)\Upsilon_1(\varphi)d\varphi \tag{2.30}$$
$$+ q\sum_{r=a}^{b}\int_0^\infty \bar{A}_{2,r}(\varphi,s)\Upsilon_2(\varphi)d\varphi;$$

$$\bar{P}_{1,n}(0,s) = p\vartheta d\sum_{k=0}^{a-1} C_{b+n-k}\bar{Q}_k(s) + (1-\omega)\int_0^\infty \bar{P}_{2,n+b}(\varphi,s)\eta_2(\varphi)d\varphi + p\int_0^\infty \bar{A}_{1,n+b}(\varphi,s)\Upsilon_1(\varphi)d\varphi \tag{2.31}$$
$$+ q\int_0^\infty \bar{A}_{2,n+b}(\varphi,s)\Upsilon_2(\varphi)d\varphi, n \ge 1;$$

$$\bar{P}_{2,n}(0,s) = \int_0^\infty \bar{P}_{1,n}(\varphi,s)\eta_1(\varphi)d\varphi, n \ge 0; \tag{2.32}$$

$$\bar{A}_{1,n}(0,s) = \omega\int_0^\infty \bar{P}_{2,n}(\varphi,s)\eta_2(\varphi)d\varphi, n \ge 0; \tag{2.33}$$

$$\bar{A}_{2,n}(0,s) = \int_0^\infty \bar{A}_{1,n}(\varphi,s)\Upsilon_1(\varphi)d\varphi, n \ge 0; \tag{2.34}$$

$$\bar{M}_{i,n}(\varphi,0,s) = \tau_i\bar{P}_{i,n}(\varphi,s), n \ge 0, i = 1,2. \tag{2.35}$$

With Eqs (2.21), (2.23), (2.25) and (2.27) multiplied by ξ^n and summing over all the possible values of n from 0 to ∞ and adding Eqs (2.20), (2.22), (2.24) and (2.26) respectively, and finally using the PGF specified in Eq (2.18), we obtain

$$\frac{\partial}{\partial\varphi}\bar{P}_i(\varphi,\xi,s) + (s+\vartheta d(1-C(\xi)) + \eta_i(\varphi) + \tau_i)\bar{P}_i(\varphi,\xi,s) = \int_0^\infty \bar{M}_i(\varphi,\varpi,\xi,s)\pi_i(\varpi)d\varpi, i = 1,2; \tag{2.36}$$

$$\frac{\partial}{\partial\varphi}\bar{A}_1(\varphi,\xi,s) + (s+\vartheta d(1-C(\xi)) + \Upsilon_1(\varphi))\bar{A}_1(\varphi,\xi,s) = 0; \tag{2.37}$$

$$\frac{\partial}{\partial\varphi}\bar{A}_2(\varphi,\xi,s) + (s+\vartheta d(1-C(\xi)) + \Upsilon_2(\varphi))\bar{A}_2(\varphi,\xi,s) = 0; \tag{2.38}$$

$$\frac{\partial}{\partial\varpi}\bar{M}_i(\varphi,\varpi,\xi,s) + (s+\vartheta d(1-C(\xi)) + \pi_i(\varpi))\bar{M}_i(\varphi,\varpi,\xi,s) = 0, i = 1,2. \tag{2.39}$$

By multiplying both sides of Eq (2.31) by ξ^n and summing the results for n ranging from 0 to ∞, and using the Eq (2.30) we get

$$\xi^b \bar{P}_1(0,\xi,s) = p\vartheta d \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} C_n \bar{Q}_r(s)(\xi^b - \xi^{n+r}) - \xi^b \sum_{r=0}^{a-1} (s+\vartheta d)\bar{Q_r}(s) + p\vartheta d \sum_{r=0}^{a-1} C(\xi)\bar{Q}_r(s)\xi^r + \xi^b$$
$$+ (1-\omega)\int_0^\infty \bar{P}_2(\varphi,\xi,s)\eta_2(\varphi)d\varphi + p\int_0^\infty \bar{A}_1(\varphi,\xi,s)\Upsilon_1(\varphi)d\varphi + q\int_0^\infty \bar{A}_2(\varphi,\xi,s)\Upsilon_2(\varphi)d\varphi$$
$$+ (1-\omega)\sum_{r=0}^{b-1}(\xi^b - \xi^r)\int_0^\infty \bar{P}_{2,r}(\varphi,s)\eta_2(\varphi)d\varphi + p\sum_{r=0}^{b-1}(\xi^b - \xi^r)\int_0^\infty \bar{A}_{1,r}(\varphi,s)\Upsilon_1(\varphi)d\varphi$$
$$+ q\sum_{r=0}^{b-1}(\xi^b - \xi^r)\int_0^\infty \bar{A}_{2,r}(\varphi,s)\Upsilon_2(\varphi)d\varphi. \tag{2.40}$$

Similarly from Eqs (2.32) to (2.35) we get

$$\bar{P}_2(0,\xi,s) = \bar{P}_1(0,\xi,s)\bar{B}_1(\chi_1(\xi,s)), \tag{2.41}$$
$$\bar{A}_1(0,\xi,s) = \omega\bar{P}_1(0,\xi,s)\bar{B}_1(\chi_1(\xi,s))\bar{B}_2(\chi_2(\xi,s)), \tag{2.42}$$
$$\bar{A}_2(0,\xi,s) = \omega\bar{P}_1(0,\xi,s)\bar{B}_1(\chi_1(\xi,s))\bar{B}_2(\chi_2(\xi,s))\bar{A}_1(\Phi(\xi,s)), \tag{2.43}$$
$$\bar{M}_i(\varphi,0,\xi,s) = \tau_i\bar{P}_i(\varphi,\xi,s), i = 1,2. \tag{2.44}$$

By solving partial differential equations (2.36) to (2.39), it follows that

$$\bar{P}_i(\varphi,\xi,s) = \bar{P}_i(0,\xi,s)e^{-\chi_i(\xi,s)\varphi - \int_0^\varphi \eta_i(t)dt}, i = 1,2; \tag{2.45}$$
$$\bar{A}_1(\varphi,\xi,s) = \bar{A}_1(0,\xi,s)e^{-\Phi(\xi,s)\varphi - \int_0^\varphi \Upsilon_1(t)dt}; \tag{2.46}$$
$$\bar{A}_2(\varphi,\xi,s) = \bar{A}_2(0,\xi,s)e^{-\Phi(\xi,s)\varphi - \int_0^\varphi \Upsilon_2(t)dt}; \tag{2.47}$$
$$\bar{M}_i(\varphi,\varpi,\xi,s) = \bar{M}_i(\varphi,0,\xi,s)e^{-\Phi(\xi,s)\varpi - \int_0^\varpi \pi_i(t)dt}, i = 1,2. \tag{2.48}$$

Integrating Eq (2.48) from 0 to ∞ with respect to ϖ, we get for $i = 1,2$

$$\bar{M}_i(\varphi,\xi,s) = \int_0^\infty \bar{M}_i(\varphi,\varpi,\xi,s)d\varpi,$$
$$\bar{M}_i(\varphi,\xi,s) = \bar{M}_i(\varphi,0,\xi,s)\left[\frac{1-\bar{M}_i(\Phi(\xi,s))}{\Phi(\xi,s)}\right]. \tag{2.49}$$

Now, multiplying both side of Eqs (2.45) to (2.48) by $\eta_i(\varphi), \Upsilon_1(\varphi), \Upsilon_2(\varphi), \pi_i(\varpi)$ respectively, and integrating, we obtain

$$\int_0^\infty \bar{P}_i(\varphi,\xi,s)\eta_i(\varphi)d\varphi = \bar{P}_i(0,\xi,s)\bar{B}_i(\chi_i(\xi,s)), \tag{2.50}$$
$$\int_0^\infty \bar{A}_1(\varphi,\xi,s)\Upsilon_1(\varphi)d\varphi = \bar{A}_1(0,\xi,s)\bar{A}_1(\Phi(\xi,s)), \tag{2.51}$$
$$\int_0^\infty \bar{A}_2(\varphi,\xi,s)\Upsilon_2(\varphi)d\varphi = \bar{A}_2(0,\xi,s)\bar{A}_2(\Phi(\xi,s)), \tag{2.52}$$

$$\int_0^\infty \bar{M}_i(\varphi,\varpi,\xi,s)\pi_i(\varpi)d\varpi = \bar{M}_i(\varphi,0,\xi,s)\bar{M}_i(\Phi(\xi,s)), \tag{2.53}$$

where,

$$\begin{aligned}
\bar{B}_i(\chi_i(\xi,s)) &= \int_0^\infty e^{-\chi_i(\xi,s)\varphi}dB_i(\varphi),\\
\bar{A}_1(\Phi(\xi,s)) &= \int_0^\infty e^{-\Phi(\xi,s)\varphi}dA_1(\varphi),\\
\bar{A}_2(\Phi(\xi,s)) &= \int_0^\infty e^{-\Phi(\xi,s)\varphi}dA_2(\varphi),\\
\bar{M}_i(\Phi(\xi,s)) &= \int_0^\infty e^{-\Phi(\xi,s)\varpi}dM_i(\varpi),
\end{aligned}$$

are, respectively, the Laplace-Stieltjes transforms(LST) of the service time $B_i(\varphi)$, the vacation time $A_1(\varphi)$ and $A_2(\varphi)$, and the repair time $M_i(\varpi)$. Using Eqs (2.41) to (2.44), we may obtain for $i = 1,2$ by again integrating Eqs (2.45) to (2.47) and (2.49) by parts with respect to φ.

$$\bar{P}_1(\xi,s) = \bar{P}_1(0,\xi,s)\left[\frac{1-\bar{B}_1(\chi_1(\xi,s))}{\chi_1(\xi,s)}\right], \tag{2.54}$$

$$\bar{P}_2(\xi,s) = \bar{P}_1(0,\xi,s)\bar{B}_1(\chi_1(\xi,s))\left[\frac{1-\bar{B}_2(\chi_2(\xi,s))}{\chi_2(\xi,s)}\right], \tag{2.55}$$

$$\bar{A}_1(\xi,s) = \omega\bar{P}_1(0,\xi,s)\bar{B}_1(\chi_1(\xi,s))\bar{B}_2(\chi_2(\xi,s))\left[\frac{1-\bar{A}_1(\Phi(\xi,s))}{\Phi(\xi,s)}\right], \tag{2.56}$$

$$\bar{A}_2(\xi,s) = \omega\bar{P}_1(0,\xi,s)\bar{B}_1(\chi_1(\xi,s))\bar{B}_2(\chi_2(\xi,s))\bar{A}_1(\Phi(\xi,s))\left[\frac{1-\bar{A}_2(\Phi(\xi,s))}{\Phi(\xi,s)}\right], \tag{2.57}$$

$$\bar{M}_1(\xi,s) = \tau_1\bar{P}_1(0,\xi,s)\left[\frac{1-\bar{B}_1(\chi_1(\xi,s))}{\chi_1(\xi,s)}\right]\left[\frac{1-\bar{M}_1(\Phi(\xi,s))}{\Phi(\xi,s)}\right], \tag{2.58}$$

$$\bar{M}_2(\xi,s) = \tau_2\bar{P}_1(0,\xi,s)\bar{B}_1(\chi_1(\xi,s))\left[\frac{1-\bar{B}_2(\chi_2(\xi,s))}{\chi_2(\xi,s)}\right]\left[\frac{1-\bar{M}_2(\Phi(\xi,s))}{\Phi(\xi,s)}\right]. \tag{2.59}$$

If we combine Eqs (2.40), (2.50) to (2.52), we have

$$\bar{P}_1(0,\xi,s) = \frac{\begin{array}{r} p\vartheta d\sum_{r=0}^{a-1}C(\xi)\bar{Q}_r(s)\xi^r - \xi^b(s+\vartheta d)\sum_{r=0}^{a-1}\bar{Q}_r(s) + \xi^b \\ +p\vartheta d\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}C_n\bar{Q}_r(s)(\xi^b-\xi^{n+r}) + \sum_{r=0}^{b-1}(\xi^b-\xi^r)\{q\int_0^\infty \bar{A}_{2,r}(\varphi,s)\Upsilon_2(\varphi)d\varphi \\ +(1-\omega)\int_0^\infty \bar{P}_{2,r}(\varphi,s)\eta_2(\varphi)d\varphi + p\int_0^\infty \bar{A}_{1,r}(\varphi,s)\Upsilon_1(\varphi)d\varphi\} \end{array}}{\left[\xi^b - \{(1-\omega)+\omega K(\xi,s)\}\bar{B}_1(\chi_1(\xi,s)\bar{B}_2(\chi_2(\xi,s))\right]} \tag{2.60}$$

where,

$$\begin{aligned}
\Phi(\xi,s) &= s+\vartheta d(1-C(\xi)),\\
\chi_i(\xi,s) &= \Phi(\xi,s)+\tau_i(1-\bar{M}_i(\Phi(\xi,s))), i=1,2,\\
K(\xi,s) &= p\bar{A}_1(\Phi(\xi,s)) + q\bar{A}_1(\Phi(\xi,s))\bar{A}_2(\Phi(\xi,s)).
\end{aligned}$$

We obtain the PGF of various states of the system defined during the transient state by inserting the Eq (2.60) into the Eqs (2.54) to (2.59) and applying the inverse LT of these equations.

2.3. Results from the steady state

We will calculate the SS prob., distribution for our QS in this section. We suppress the argument 't' everywhere it occurs in the time-dependent analysis in order to determine the SS probability. This can be obtained by applying the Taubarian property .

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t).$$

In SS conditions, the server's state queue length dist., is represented by the PGF provided by

$$P_1(\xi) = P_1(0,\xi)\left[\frac{1-\bar{B}_1(\chi_1(\xi))}{\chi_1(\xi)}\right], \tag{2.61}$$

$$P_2(\xi) = P_1(0,\xi)\bar{B}_1(\chi_1(\xi))\left[\frac{1-\bar{B}_2(\chi_2(\xi))}{\chi_2(\xi)}\right], \tag{2.62}$$

$$A_1(\xi) = \omega P_1(0,\xi)\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi))\left[\frac{1-\bar{A}_1(\Phi(\xi))}{\Phi(\xi)}\right], \tag{2.63}$$

$$A_2(\xi) = \omega P_1(0,\xi)\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi))A_1(\Phi(\xi))\left[\frac{1-A_2(\Phi(\xi))}{\Phi(\xi)}\right], \tag{2.64}$$

$$M_1(\xi) = \tau_1 P_1(0,\xi)\left[\frac{1-\bar{B}_1(\chi_1(\xi))}{\chi_1(\xi)}\right]\left[\frac{1-\bar{M}_1(\Phi(\xi))}{\Phi(\xi)}\right], \tag{2.65}$$

$$M_2(\xi) = \tau_2 P_1(0,\xi)\bar{B}_1(\chi_1(\xi))\left[\frac{1-\bar{B}_2(\chi_2(\xi))}{\chi_2(\xi)}\right]\left[\frac{1-\bar{M}_2(\Phi(\xi))}{\Phi(\xi)}\right], \tag{2.66}$$

where,

$$P_1(0,\xi) = \frac{\begin{array}{r} p\vartheta d\sum_{r=0}^{a-1} Q_r(C(\xi)\xi^r-\xi^b) + p\vartheta d\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1} C_n Q_r(\xi^b-\xi^{n+r}) \\ +\sum_{r=0}^{b-1}(\xi^b-\xi^r)[q\int_0^\infty \bar{A}_{2,r}(\varphi)\Upsilon_2(\varphi)d\varphi \\ +(1-\omega)\int_0^\infty \bar{P}_{2,r}(\varphi)\eta_2(\varphi)d\varphi + p\int_0^\infty \bar{A}_{1,r}(\varphi)\Upsilon_1(\varphi)d\varphi]\end{array}}{[\xi^b-\{(1-\omega)+\omega K(\xi)\}\bar{B}_1(\chi_1(\xi)\bar{B}_2(\chi_2(\xi))]} \tag{2.67}$$

Distribution of queue size at arbitrary epoch

We can obtain the PGF of the queue length dist., at a arbitrary time by summing (2.62) to (2.67) with idle term

$$P(\xi) = P_1(\xi) + P_2(\xi) + A_1(\xi) + A_2(\xi) + M_1(\xi) + M_2(\xi) + Q(\xi),$$

$$P(\xi) = \frac{Nr(\xi)}{Dr(\xi)}, \tag{2.68}$$

$$\begin{aligned} Nr(\xi) = \{&p\vartheta d\sum_{r=0}^{a-1} Q_r(C(\xi)\xi^r-\xi^b)\Phi(\xi) + p\vartheta d\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1} C_n Q_r(\xi^b-\xi^{n+r})\Phi(\xi) \\ &+\sum_{r=0}^{b-1}(\xi^b-\xi^r)U_r\Phi(\xi)\}\{V\Phi(\xi)\left(\frac{1-\bar{B}_1(\chi_1(\xi))}{\chi_1(\xi)}\right) + V\Phi(\xi)\bar{B}_1(\chi_1(\xi))\left(\frac{1-\bar{B}_2(\chi_2(\xi))}{\chi_2(\xi)}\right)\end{aligned}$$

$$+ V\omega\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi))(1-\bar{A}_1(\Phi(\xi))) + V\omega\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi))A_1(\Phi(\xi))(1-\bar{A}_2(\Phi(\xi)))$$
$$+ V\tau_1\left(\frac{1-\bar{B}_1(\chi_1)(\xi)}{\chi_1(\xi)}\right)\{1-\bar{M}_1(\Phi(\xi))\} + V\tau_2\bar{B}_1(\chi_1(\xi))\left(\frac{1-\bar{B}_2(\chi_2)(\xi)}{\chi_2(\xi)}\right)(1-\bar{M}_2(\Phi(\xi)))\}$$
$$+ \Phi(\xi)VQ(\xi),$$
$$Dr(\xi) = [\xi^b - \{(1-\omega)+\omega K(\xi)\}\bar{B}_1(\chi_1(\xi)\bar{B}_2(\chi_2(\xi))]\Phi(\xi).$$

Where,

$$U_r = q\int_0^\infty \bar{A}_{2,r}(\varphi)\Upsilon_2(\varphi)d\varphi + (1-\omega)\int_0^\infty \bar{P}_{2,r}(\varphi)\eta_2(\varphi)d\varphi + p\int_0^\infty \bar{A}_{1,r}(\varphi)\Upsilon_1(\varphi)d\varphi,$$
$$V = \xi^b - [(1-\omega)+\omega K(\xi)]\bar{B}_1(\chi_1(\xi)\bar{B}_2(\chi_2(\xi)),$$
$$\Phi(\xi) = \vartheta d(1-C(\xi)),$$
$$\chi_1(\xi) = \Phi(\xi) + \tau_1(1-\bar{M}_1(\Phi(\xi))),$$
$$\chi_2(\xi) = \Phi(\xi) + \tau_2(1-\bar{M}_2(\Phi(\xi))),$$
$$K(\xi) = p\bar{A}_1(\Phi(\xi)) + q\bar{A}_1(\Phi(\xi))\bar{A}_2(\Phi(\xi)).$$

2.4. Stability condition

The PGF has to satisfy $P(1) = 1$. In order to satisfy this condition, here we are applying L'Hopital rules and equating the expression to 1, we get

$$Z_1[D_1 + \omega E(\bar{A}_1) + \omega qE(\bar{A}_2)] + [b - \vartheta dE(I)(D_1 + \omega E(\bar{A}_1) + \omega qE(\bar{A}_2)]\sum_{r=0}^{a-1} Q_r \tag{2.69}$$
$$= [b - \vartheta dE(I)(D_1 + \omega E(\bar{A}_1) + \omega qE(\bar{A}_2))].$$

Next we calculate the unknown prob., U_r, $r = 0, 1, 2, \ldots, b-1$ which are related to the idle-server prob., Q_r, $r = 0, 1, 2, \ldots, a-1$, consequently, the left half of the statement above must be true. $P(1) = 1$ is therefore satisfied if

$$[\xi^b - ((1-\omega)+\omega K(\xi))\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi))] > 0$$
$$\text{If } \rho = \frac{\vartheta dE(I)(D_1 + \omega E(\bar{A}_1) + \omega qE(\bar{A}_2))}{b}. \tag{2.70}$$

The existence of the SS for the model in discussion is then required to satisfy criterion $\rho < 1$. $b + a$ are unknowns in Eq (2.69). Using the following result, we can express U_r in terms of Q_r such that numerator only contains 'b' constants: The PGF of the number of clients using 'b' unknowns is now given by Eq (2.69).

According to Rouche's theorem, the statement $[\xi^b - ((1-\omega)+\omega K(\xi))\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi))]$ has $(b-1)$ zeros inside and one on the unit circle $|\xi| = 1$.

Because $P(\xi)$ is analytic within and on the unit circle, the numerator must be zero at these points, resulting in 'b' equations in 'b' unknowns. With the aid of a suitable numerical approach, these equations can be solved.

Result: Let, in terms of Q_r, U_r can be expressed as follows.

$$\sum_{r=0}^{a-1} U_r = \vartheta d\sum_{r=0}^{a-1} Q_r - \vartheta d\sum_{r=0}^{a-1} Q_r\sum_{k=1}^{a-r-1} C_k.$$

Where Z_1, D_1, $E(I)$ are as defined in Section 2.5, and U_r is the probability that the 'r' clients will remain in the queue during the idle period.

2.5. *Performance evaluation*

In this part, we obtain some system prob., as well as the average length of time a client spends in line (W_q) and the average number of consumers in line (L_q) . We may deduce from (2.71) that $\rho < 1$ is a stability criterion.

2.5.1. Probabilities of a system state

The following outcomes are obtained by setting $\xi \to 1$ and using L'Hopital's rule whenever needed from Eqs (2.62) to (2.67)

- Let $P_q(1)$ be the SS Prob., that the server is busy.

$$P_q(1) = P_1(1) + P_2(1) = \frac{Z_1[E(B_1) + E(B_2)]}{[b - \vartheta dE(I)(D_1 + \omega E(\bar{A}_1) + \omega qE(\bar{A}_2))]}.$$

- Let $A_q(1)$ be the SS Prob., that the server is on vacation.

$$A_q(1) = A_1(1) + A_2(1) = \frac{Z_1[\omega E(A_1) + \omega qE(A_2)]}{[b - \vartheta dE(I)(D_1 + \omega E(\bar{A}_1) + \omega qE(\bar{A}_2))]}.$$

- Let $M_q(1)$ be the SS Prob., that the server is under repair.

$$M_q(1) = M_1(1) + M_2(1) = \frac{Z_1[\tau_1 E(M_1)E(B_1) + \tau_1 E(M_2)E(B_2)]}{[b - \vartheta dE(I)(D_1 + \omega E(\bar{A}_1) + \omega qE(\bar{A}_2))]}.$$

2.5.2. Average queue size

The average number of clients in the queue (L_q) in SS conditions is calculated by differentiating (2.69) with respect to 'ξ' and calculating at $\xi = 1$

$$L_q = \lim_{\xi\to 1}\frac{d}{d\xi}P(\xi),$$
$$P'(1) = \frac{N'''(1)D''(1) - D'''(1)N''(1)}{3(D''(1))^2},$$

$$\begin{aligned}
D''(1) &= -2[b\vartheta dE(I) + (\vartheta dE(I))^2(D_1 + \omega + \omega qE(\bar{A}_2))],\\
D'''(1) &= 3\{(-\vartheta dE(I(I-1)))(b + \vartheta dE(I)[D_1 + \omega(1+q)E(\bar{A}_2)])\\
&\quad + (-\vartheta dE(I))[b(b-1) + \vartheta d(\vartheta E(I))^2[D_1^2 + 2N_1 + S_1] + \vartheta E(I(I-1))D_1]\}\\
&\quad + \omega d\vartheta\{(\vartheta E(I))^2[F_1^2 + 2qH_1 + F_1] + \vartheta E(I(I-1))F_1\} - (\vartheta dE(I))^2\{2(1-\omega)N_1\\
&\quad + 2\omega p[E(\bar{A}_1)D_1 + N_1] + 2\omega q[H_1 + D_1K_1 + E(\bar{A}_2)N_1]\}\},
\end{aligned}$$

$$N''(1) = 2[(-\vartheta dE(I))(b + \vartheta dE(I))D_1 + \vartheta dE(I)[\omega(1+q)E(\bar{A}_2)](\sum_{r=0}^{a-1} Q_r \xi^r)],$$

$$\begin{aligned}N'''(1) =& 3\{[-\vartheta dE(I(I-1))][b + \vartheta dE(I)[D_1 + \omega(1+q)E(\bar{A}_2)]][\sum_{r=0}^{a-1} Q_r]\},\\ &+ 3\{[-\vartheta dE(I)][b(b-1) + \vartheta d\{(\vartheta E(I))^2[D_1^2 + 2N_1 + S_1] + \vartheta E(I(I-1))D_1\}\\ &+ \omega d\vartheta\{(\vartheta E(I))^2[F_1^2 + 2qH_1 + F_1] + \vartheta E(I(I-1))F_1\} - (\vartheta dE(I))^2\{2(1-\omega)N_1\\ &+ 2\omega p(E(\bar{A}_1)D_1 + N_1) + 2\omega q(H_1 + D_1K_1 + E(\bar{A}_2)N_1)\}][\sum_{r=0}^{a-1} Q_r]\}\\ &+ 6\{[-\vartheta dE(I)][b + \vartheta dE(I)[D_1 + \omega(1+q)E(\bar{A}_2)][\vartheta d\sum_{r=0}^{a-1} Q_r(E(I) + r - b)]\},\end{aligned}$$

where,

$$\begin{aligned}D_1 =& (1 + \tau_1 E(\bar{M}_1)E(\bar{B}_1)) + (1 + \tau_2 E(\bar{M}_2)E(\bar{B}_2)),\\ N_1 =& (1 + \tau_1 E(\bar{M}_1)E(\bar{B}_1))(1 + \tau_2 E(\bar{M}_2)E(\bar{B}_2)),\\ S_1 =& \tau_1 E(\bar{B}_1)E(\bar{M}_1{}^2)) + \tau_2 E(\bar{B}_2)E(\bar{M}_2{}^2)),\\ F_1 =& E(\bar{A}_1) + qE(\bar{A}_2)),\\ H_1 =& E(\bar{A}_1)E(\bar{A}_2)),\\ K_1 =& E(\bar{A}_1) + E(\bar{A}_2)),\\ E(I) =& C'(1); E(I(I-1)) = C''(1).\end{aligned}$$

- The Little's formula (W_q), is used to calculate the average time a client spends in line.

$$W_q = \frac{L_q}{\vartheta E(I)}$$

2.6. Special case

Case: 1.

If the vacation time dist., is Erlang-K with the parameter σ, then A is LST which is given by

$$\begin{aligned}\bar{A}(s) =& \left(\frac{\sigma q}{\sigma q + s}\right)^q,\\ E(\bar{A}_1) =& \frac{1}{\sigma_1}; E(\bar{A}_2) = \frac{1}{\sigma_2},\\ \bar{A}_1(\Phi_1(\xi)) =& \left(\frac{\sigma_1 q_1}{\sigma_1 q_1 + \Phi_1(\xi)}\right)^{q_1},\\ \bar{A}_2(\Phi_2(\xi)) =& \left(\frac{\sigma_2 q_2}{\sigma_2 q_2 + \Phi_2(\xi)}\right)^{q_2},\\ \Phi(\xi) =& \vartheta d(1 - C(\xi)),\end{aligned}$$

$$P(\xi) = \frac{J_1(\xi)}{J_2(\xi)},$$

where,

$$\begin{aligned}
J_1(\xi) = & \left[p\vartheta d \sum_{r=0}^{a-1} Q_r(C(\xi)\xi^r - \xi^b)\Phi(\xi) + p\vartheta d \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} C_n Q_r(\xi^b - \xi^{n+r})\Phi(\xi) + \sum_{r=0}^{b-1} (\xi^b - \xi^r) U_r \Phi(\xi) \right] \\
& \{ J\Phi(\xi) \left(\frac{1 - \bar{B}_1(\chi_1(\xi))}{\chi_1(\xi)} \right) + J\Phi(\xi)(\bar{B}_1(\chi_1(\xi))) \left(\frac{1 - \bar{B}_2(\chi_2(\xi))}{\chi_2(\xi)} \right) + J\omega \bar{B}_1(\chi_1(\xi)) \bar{B}_2(\chi_2(\xi)) \\
& \left(1 - \left(\frac{\sigma_1 q_1}{\sigma_1 q_1 + \Phi_1(\xi)} \right)^{q_1} \right) + J\omega \bar{B}_1(\chi_1(\xi)) \bar{B}_2(\chi_2(\xi)) \left(\frac{\sigma_1 q_1}{\sigma_1 q_1 + \Phi(\xi)} \right)^{q_1} \left(1 - \left(\frac{\sigma_2 q_2}{\sigma_2 q_2 + \Phi(\xi)} \right)^{q_2} \right) \\
& + J\tau_1 \left(\frac{1 - \bar{B}_1(\chi_1(\xi))}{\chi_1(\xi)} \right) (1 - \bar{M}_1(\Phi(\xi))) + J\tau_2 \bar{B}_1(\chi_1(\xi)) \left(\frac{1 - \bar{B}_2(\chi_2(\xi))}{\chi_2(\xi)} \right) (1 - \bar{M}_2(\Phi(\xi))) \} \\
& + \Phi(\xi) J Q(\xi), \\
J_2 = & J\Phi(\xi),
\end{aligned}$$

where,

$$\begin{aligned}
J = & \left\{ \xi^b - \left[(1-\omega) + \omega \left[p \left(\frac{\sigma_1 q_1}{\sigma_1 q_1 + \Phi(\xi)} \right)^{q_1} + q \left(\frac{\sigma_1 q_1}{\sigma_1 q_1 + \Phi(\xi)} \right)^{q_1} \left(\frac{\sigma_2 q_2}{\sigma_2 q_2 + \Phi(\xi)} \right)^{q_2} \right] \right] \right. \\
& \left. \bar{B}_1(\chi_1(\xi)) \bar{B}_2(\chi_2(\xi)) \right\}.
\end{aligned}$$

This is the PGF of the queue size dist., of $M^X/G(a,b)/1$ QS with feedback, balking and dual stage of service subject to server failure under BV.

The stability condition for the queueing model is given by

$$\rho = \frac{\vartheta d E(I) \left[(1 + \tau_1 E(\bar{M}_1))(\frac{1}{\Omega_1}) + (1 + \tau_2 E(\bar{M}_2))(\frac{1}{\Omega_2}) + \frac{\omega}{\sigma_1} + \frac{\omega q}{\sigma_2} \right]}{b}.$$

Case: 2.

Suppose that the service time dist., is exponential with Ω, then the LST of B is given by

$$\begin{aligned}
\bar{B}(s) &= \left(\frac{\Omega}{\Omega + s} \right), \\
E(\bar{B}_1) &= \frac{1}{\Omega_1};\ E(\bar{B}_2) = \frac{1}{\Omega_2}, \\
\bar{B}_1(\chi_1(\xi)) &= \left(\frac{\Omega_1}{\Omega_1 + \chi_1(\xi)} \right), \\
\bar{B}_2(\chi_2(\xi)) &= \left(\frac{\Omega_2}{\Omega_2 + \chi_2(\xi)} \right), \\
\text{where, } \chi_1(\xi) &= \Phi(\xi) + \tau_1(1 - \bar{M}_1(\Phi(\xi))), \\
\chi_2(\xi) &= \Phi(\xi) + \tau_2(1 - \bar{M}_2(\Phi(\xi))),
\end{aligned}$$

$$P(\xi)=\frac{L_1(\xi)}{L_2(\xi)},$$

where

$$\begin{aligned}L_1(\xi)=&\Bigg\{\Bigg[p\vartheta d\sum_{r=0}^{a-1}Q_r(C(\xi)\xi^r-\xi^b)\Phi(\xi)+p\vartheta d\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}C_nQ_r(\xi^b-\xi^{n+r})\Phi(\xi)+\sum_{r=0}^{b-1}(\xi^b-\xi^r)U_r\Phi(\xi)\Bigg]\\&\Bigg[L\Phi(\xi)\left(\frac{1}{\Omega_1+\chi_1(\xi)}\right)+L\Phi(\xi)\left(\frac{\Omega_1}{\Omega_1+\chi_1(\xi)}\right)\left(\frac{1}{\Omega_2+\chi_2(\xi)}\right)+L\omega\left(\frac{\Omega_1}{\Omega_1+\chi_1(\xi)}\right)\left(\frac{\Omega_2}{\Omega_2+\chi_2(\xi)}\right)\\&(1-\bar{A}_1(\Phi(\xi)))+L\omega\left(\frac{\Omega_1}{\Omega_1+\chi_1(\xi)}\right)\left(\frac{\Omega_2}{\Omega_2+\chi_2(\xi)}\right)(\bar{A}_1(\Phi(\xi)))(1-\bar{A}_2(\Phi(\xi)))+L\tau_1\left(\frac{\Omega_1}{\Omega_1+\chi_1(\xi)}\right)\\&(1-\bar{M}_1(\Phi(\xi)))+L\tau_2\left(\frac{\Omega_1}{\Omega_1+\chi_1(\xi)}\right)\left(\frac{\Omega_1}{\Omega_2+\chi_2(\xi)}\right)(1-\bar{M}_2(\Phi(\xi)))\Bigg]+\Phi(\xi)LQ(\xi)\Bigg\},\\L_2(\xi)=&\,L\Phi(\xi),\end{aligned}$$

where,

$$L=\left[\xi^b-\{(1-\omega)+\omega K(\xi)\}\left(\frac{\Omega_1}{\Omega_1+\chi_1(\xi)}\right)\left(\frac{\Omega_2}{\Omega_2+\chi_2(\xi)}\right)\right].$$

This is the PGF of the queue size dist., of $M^X/G(a,b)/1$ QS with feedback, balking and dual stage of service subject to server failure under BV.

The stability condition for the QS is given by

$$\rho=\frac{\vartheta dE(I)[(1+\tau_1E(\bar{M}_1))(\frac{1}{\Omega_1})+(1+\tau_2E(\bar{M}_2))(\frac{1}{\Omega_2})+\omega E(\bar{A}_1)+\omega qE(\bar{A}_2)]}{b}.$$

2.7. Particular cases:

Case: 1.

The PGF of the queue length is obtained as follows if batch arrival, a single server providing a single stage of service, and no failure are taken into account:

$$P(\xi)=\frac{Q[(1-\bar{B}(\Phi(\xi)))+\omega\bar{B}(\Phi(\xi))(1-\bar{A}_1(\Phi(\xi)))+\omega\bar{B}(\Phi(\xi))\bar{A}_1(\Phi(\xi)))(1-\bar{A}_2(\Phi(\xi)))]}{[(1-\omega)+\omega K(\xi,s))\bar{B}(\Phi(\xi))-\xi]}$$

where

$$Q=1-\rho,\Phi(\xi)=\vartheta d(1-C(\xi)),$$
$$\rho=\vartheta dE(I)[D_1+\omega E(\bar{A}_1)+\omega qE(\bar{A}_2)].$$

These expressions closely reflect the findings by Ayyappan and Supraja [9].

Case: 2.

The PGF of the queue length is determined as follows if single arrival, single server, two stage service, no balking, and no feedback are taken into account:

$$P(\xi)=\frac{R_1(\xi)}{R_2(\xi)}$$

where,

$$
\begin{aligned}
R_1(\xi) = Q\Bigg[& R\Phi(\xi)\left(\frac{1-\bar{B}_1(\chi_1(\xi))}{\chi_1(\xi)}\right) + R\Phi(\xi)(\bar{B}_1(\chi_1(\xi)))\left(\frac{1-\bar{B}_2(\chi_2(\xi))}{\chi_2(\xi)}\right) + R\omega\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi)) \\
& (1-\bar{A}_1(\Phi(\xi))) + R\omega\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi))(\bar{A}_1(\Phi(\xi)))(1-\bar{A}_2(\Phi(\xi))) + R\tau_1\left(\frac{1-\bar{B}_1(\chi_1(\xi))}{\chi_1(\xi)}\right) \\
& (1-\bar{M}_1(\Phi(\xi))) + R\tau_2\bar{B}_1(\chi_1(\xi))\left(\frac{1-\bar{B}_2(\chi_2(\xi))}{\chi_2(\xi)}\right)(1-\bar{M}_2(\Phi(\xi)))\Bigg], \\
R_2(\xi) = R&\Phi(\xi),
\end{aligned}
$$

where,

$$
\begin{aligned}
R &= [(1-\omega) + \omega[\bar{A}_1(\Phi(\xi))\bar{A}_2(\Phi(\xi))]\bar{B}_1(\chi_1(\xi))\bar{B}_2(\chi_2(\xi)) - \xi], \\
Q &= 1-\rho; \Phi(\xi) = \vartheta(1-C(\xi)), \\
\rho &= \vartheta E(I)[D_1 + \omega K_1].
\end{aligned}
$$

2.8. Practical application of the model

In the area of telephonic consultant medical service systems, our model may be helpful (TCMSS). A TCMSS is a centralized system that handles a large number of customers tele-medical inquiries. A customer call is answered right away when a free medical service agent answers it. Staff members of the TCMSS are medical service agents (servers). Here, customers can also select remote monitoring services (an optional service). If a customer calls and all the agents are busy, he may decide to leave the system as well (balk). An agent may need to send medical inquiries or queries to other agents who are accessible or consult a specialist in order to get the answers because they are outside of their scope of knowledge or competence. This situation can be modeled as a service failure. Here repair rate can be defined as how quickly the agent can get responses from the expert. Additionally, the server might engage in various maintenance tasks that are referred to as vacation. Furthermore, unsatisfied customers may re-join the queue and be classified as feedback consumers after each customer's service is complete.

The proposed model also has a potential application in the field of e-commerce, a platform for purchasing and selling goods and services through an electronic network. Customers access an online store (server) to browse through and place orders for products or services via their own devices. The customer's web browser will communicate back and forth with the server hosting the e-commerce website as the order is placed. The order manager, a central computer, will receive information about the order. After that, databases that control inventory levels will receive it. This is done to ensure that there is enough stock in the store and money in the customer's account to fulfill the order. The order manager will alert the store's web server once the order has been confirmed. The order manager will then notify the warehouse or fulfillment department that the product or service can be delivered to the customer by sending order data to those departments. Customers can browse through and order any additional products they need if they want to order a few more (optional services). Also, if the server is busy, the customers may prefer to leave (balk). If the customers are unsatisfied with the delivered products, then they can reorder (feedback) as well. In addition, if the server is down or if there are no proper internet services, then the server may experience a breakdown and be sent for repair. Finally, if there are no customers to place orders, then the server may go on vacation or else wait for the customers to place orders. Thus, e-commerce has the potential to increase a company's clientele and enables businesses to provide a variety of products.

3. Numerical results

In this segment, we utilize MATLAB to show how various parameters affect system behavior measurements. In this section, the batch size dist., of the arrival is geometric, with a mean of 2. Here the dual service stages, vacation time, and repair stage follow the exponential distribution. We ensure that the stability condition is met by making arbitrary assumptions about the parameters. Our queueing model's utilization factor (ρ), average queue size (L_q), and average waiting time (W_q) have been calculated its values are shown in Tables 1–3.

Table 1 clearly depicts that as (ρ) escalates, L_q, W_q also escalates for the value of $a = 2.3$, $b = 0.2, \omega = 1, p = 0.1, D_1 = 0.5$, $q = 0.2, d = 0.2, r = 5$, $N_1 = 0.25$, $S_1 = 1$, $F_1 = 0.2, H_1 = 0.5$, $K_1 = 0.7$ $\tau_1 = 1.3$, $\tau_2 = 2.6$

Table 2 clearly depicts that as the FPS rate (η_1) escalates, L_q, W_q diminish for the value of $a = 2.6$, $b = 5, \vartheta = 5$, $\omega = 3$, $p = 0.1$, $D_1 = 0.5$, $q = 0.1, d = 0.1, r = 5.1$, $N_1 = 0.2$, $S_1 = 5.5$, $F_1 = 0.2, H_1 = 0.5$, $K_1 = 2$ $\tau_1 = 0.5$, $\tau_2 = 0.6$, $C = 1$

Table 3 clearly depicts that as the SPS rate (η_2) escalates, L_q, W_q diminish for the value of $a = 2.6$, $b = 5, \vartheta = 5$, $\omega = 3$, $p = 0.1$, $D_1 = 0.5$, $q = 0.1, d = 0.1, r = 5.1$, $N_1 = 0.2$, $S_1 = 5.5$, $F_1 = 0.2, H_1 = 0.5$, $K_1 = 2$ $\tau_1 = 0.5$, $\tau_2 = 0.6$, $B = 5.8$

Table 1. The influence of the arrival rate (ϑ) on ρ, L_q, W_q.

Arrival rate (ϑ)	ρ	L_q	W_q
2.4	0.6440	1.6303	0.3396
2.5	0.6500	7.8275	1.5655
2.6	0.6560	13.7621	2.6466
2.7	0.6620	19.4876	3.6088
2.8	0.6680	25.0484	4.4729
2.9	0.6740	30.4815	5.2554
3.0	0.6800	35.8181	5.9697

Table 2. The influence of the service rate (η_1) on ρ, L_q, W_q.

service rate (η_1)	ρ	L_q	W_q
5	0.7120	3.6590	0.3659
5.1	0.6770	3.6543	0.3654
5.2	0.6420	3.6496	0.3650
5.3	0.6070	3.6449	0.3645
5.4	0.5720	3.6402	0.3640
5.5	0.5370	3.6355	0.3636
5.6	0.5020	3.6309	0.3631

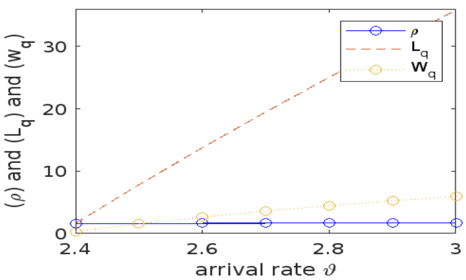
Table 3. The influence of the service rate (η_2) on ρ, L_q, W_q.

service rate (η_2)	ρ	L_q	W_q
1	0.4320	3.3335	0.3334
1.1	0.3970	3.2945	0.3295
1.2	0.3620	3.2561	0.3256
1.3	0.3270	3.2181	0.3218
1.4	0.2920	3.1807	0.3181
1.5	0.2570	3.1439	0.3144
1.6	0.2220	3.1075	0.3108

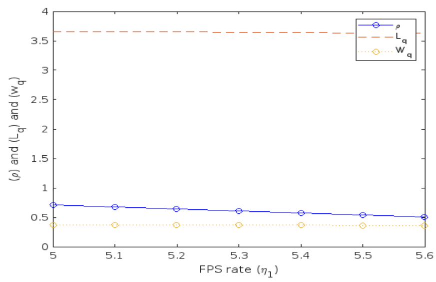
The Figure 1(a) shows that as arrival rate (ϑ) escalates, the utilization factor (ρ), expected queue length (L_q) and expected waiting time (W_q) also escalate. The Figure 1(b) shows that as FPS rate (η_1) escalates, the utilization factor (ρ), expected queue length (L_q) and expected waiting time (W_q) diminishes. The Figure 1(c) shows that as SPS rate (η_2) escalates, the utilization factor (ρ), expected queue length (L_q) and expected waiting time (W_q) diminishes.

Figure 2($d - f$) shows the three-dimensional graph that depicts the system performance measures. In Figure 2(d), the surface displays the escalation of the arrival rate (ϑ), expected queue length (L_q) and expected waiting time (W_q). In Figure 2(e), we found that expected queue length (L_q) and expected waiting time (W_q) diminishes while increasing the FPS rate η_1. Figure 2(f) we found that expected queue length (L_q) and expected waiting time (W_q) diminishes while increasing the SPS rate η_2.

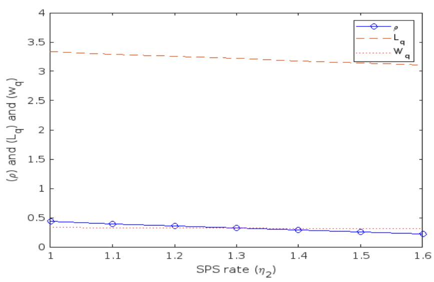
We can identify the effect of characteristics on the system's evaluation criteria using the numerical results above, and we can be certain that the results are comparable to real-world circumstances.



(a) ρ, L_q, W_q verses arrival rate ϑ

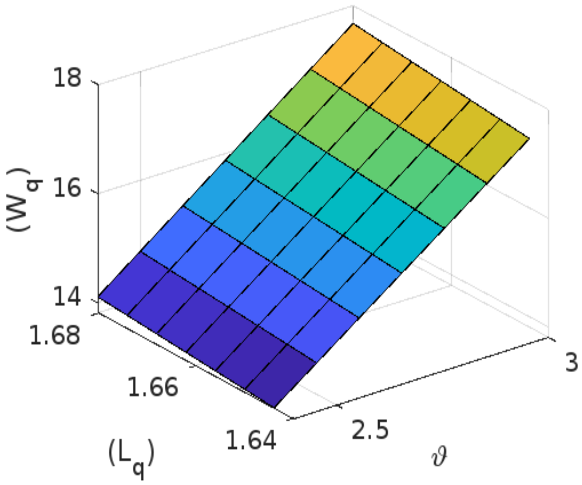


(b) ρ, L_q, W_q verses First phase of service rate η_1

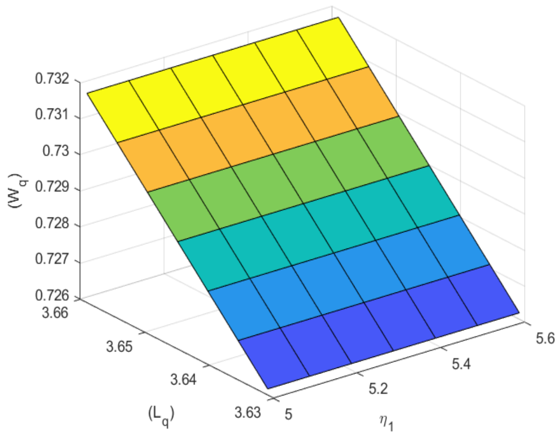


(c) ρ, L_q, W_q verses Second phase of service rate η_2

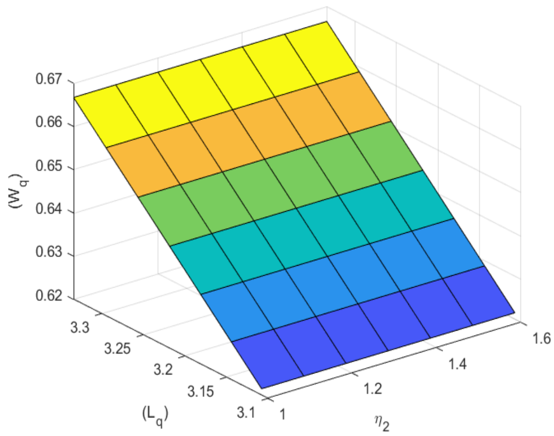
Figure 1. Effects of a few parameters on 2D representation.



(a) ρ,L_q, W_q verses arrival rate ϑ



(b) ρ,L_q, W_q verses First phase of service rate η_1



(c) ρ,L_q, W_q verses Second phase of service rate η_2

Figure 2. Effects of a few parameters on 3D representation.

4. Conclusions

This work investigates the transient scrutiny of a $M^X/G(a,b)/1$ QS with feedback, balking, and dual stages of service subject to server failure during a BV. We build the PGF for the batch of consumers standing in line at an arbitrary epoch under transient and SS conditions. The average size dist., at a departure epoch was also obtained. Performance measures like the average queue length and the average waiting time spent in the queue are calculated for the different system states. We also go through a few special and particular cases. The influence of various parameters on the efficiency of the system is then numerically analyzed.

Acknowledgments

The authors would like to thank Vellore Institute of Technology (VIT), for financial support.

Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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